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SUPPLEMENT TO “SPOT VOLATILITY ESTIMATION FOR HIGH-FREQUENCY DATA: ADAPTIVE ESTIMATION IN PRACTICE”

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In this supplement, we recall the proof of Lemma 3.1 in “Spot volatility estimation for high-frequency data: adaptive estimation in practice” as it is given in Schmidt-Hieber (2010), Lemma 6, p. 65.

APPENDIX A: PROOF OF LEMMA 3.1

To keep notation simple, we use the following quantities in the spirit of the definitions of Section 2.3: For any process $(A_{i,n}) \in \{(Y_{i,n}), (\epsilon_{i,n}), (X)_{i,n}\}$, define

$$\overline{A}_{i,m} = \overline{A}_{i,m}(\lambda) := \frac{m}{n} \sum_{\frac{j}{n} \in [\frac{i-2}{m}, \frac{i}{m}]} \lambda\big(m\tfrac{j}{n} - (i-2)\big) A_{j,n}.$$

$$\mathfrak{b}(A)_{i,m} = \mathfrak{b}(\lambda, A.)_{i,m} := \frac{m^2}{2n^2} \sum_{\frac{j}{n} \in [\frac{i-2}{m}, \frac{i}{m}]} \lambda^2\big(m\tfrac{j}{n} - (i-2)\big)\big(A_{j,n} - A_{j-1,n}\big)^2.$$

Further, recall that our estimator for the integrated volatility is given by $\widehat{\langle 1, \sigma^2 \rangle} = \sum_{i=2}^m \overline{Y}_{i,m}^2 - \mathfrak{b}(Y)_{i,m}$.

To prove the lemma, let us first show that the bias is of smaller order than $n^{-1/4}$. In fact, note that $\mathbb{E}\left[\,\overline{Y}_{i,m}^2\right] = \mathbb{E}\left[\,\overline{X}_{i,m}^2\right] + \mathbb{E}\left[\overline{\epsilon}_{i,m}^2\right]$. Clearly, one can bound

$$\Big|\mathbb{E}\left[\overline{\epsilon}_{i,m}^2\right] - \mathbb{E}\left[\mathfrak{b}(\lambda, Y)_{i,m}\right]\Big| = O(\tfrac{1}{n}).$$

Further, Lipschitz continuity of λ together with a Riemann approximation argument gives us

$$\big|\,\mathbb{E}\left[\,\overline{X}_{i,m}^2\right] - \tfrac{\sigma^2}{m}\big| = \Big|\tfrac{\sigma^2}{m}\int_0^2\int_0^2 \lambda(s)\lambda(t)(s\wedge t)dtds - \tfrac{\sigma^2}{m}\Big| + O(\tfrac{1}{n}) = O(\tfrac{1}{n}).$$

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Here, the last equation is due to partial integration and the definition of a pre-average function (cf. Definition 2.1). Since both approximations are uniformly in i, this shows that the bias is of order $O(n^{-1/2})$.

For the asymptotic variance, first observe that $\mathrm{Var}(\sum_{i=2}^m \mathfrak{b}(\lambda, Y)_{i,m}) = o(n^{-1/2})$. Hence,

$$\mathrm{Var}(\widehat{\langle 1, \sigma^2 \rangle}) = \mathrm{Var}(\sum_{i=2}^m \overline{Y}_{i,m}^2) + o\Big(n^{-1/4}\big(\mathrm{Var}(\sum_{i=2}^m \overline{Y}_{i,m}^2)\big)^{1/2} + n^{-1/2}\Big),$$

by Cauchy-Schwarz inequality. Recall that for centered Gaussian random variables U, V, $\mathrm{Cov}(U^2, V^2) = 2(\mathrm{Cov}(U, V))^2$. Therefore, it suffices to compute $\mathrm{Cov}(\overline{Y}_{i,m}, \overline{Y}_{k,m}) = \mathbb{E}[\overline{Y}_{i,m}\overline{Y}_{k,m}]$.

By the same arguments as above, that is Riemann summation and partial integration, we find

$$\mathbb{E}\left[\left|\overline{X}_{i,m}\overline{X}_{k,m} - \int_0^1 \Lambda(ms-(i-2))dX_s \int_0^1 \Lambda(ms-(k-2))dX_s\right|\right] \lesssim n^{-1}.$$

Therefore,

$$\mathbb{E}\left[\overline{X}_{i,m}\overline{X}_{k,m}\right] = \sigma^2 \int_0^1 \Lambda(ms-(i-2))\Lambda(ms-(k-2))ds + O(n^{-1}),$$

where the last two arguments hold uniformly in i, k.

In order to calculate $\mathbb{E}[\overline{Y}_{i,m}\overline{Y}_{k,m}]$, we must treat three different cases, $|i-k| \geq 2$, $|i-k| = 1$ and $i = k$, denoted by I, II and III.

I:. In this case $(\frac{i-2}{m}, \frac{i}{m}]$ and $(\frac{k-2}{m}, \frac{k}{m}]$ do not overlap. By the equalities above, it follows $\mathrm{Cov}(\overline{Y}_{i,m}, \overline{Y}_{k,m}) = O(n^{-1})$.

II:. Without loss of generality, we set $k = i+1$. Then, we obtain

$$\begin{aligned}
&\mathrm{Cov}(\overline{Y}_{i,m}, \overline{Y}_{i+1,m}) = \mathbb{E}\left[\overline{X}_{i,m}\overline{X}_{i+1,m}\right] + \mathbb{E}\left[\overline{\epsilon}_{i,m}\overline{\epsilon}_{i+1,m}\right] \\
=&\sigma^2 \int_0^1 \Lambda(ms-(i-2))\Lambda(ms-(i-1))ds + O(n^{-1}) \\
&+ \tau^2 \frac{m^2}{n^2} \sum_{\frac{j}{n} \in (\frac{i-2}{m}, \frac{i}{m}]} \lambda(m\tfrac{j}{n} - (i-2))\lambda(m\tfrac{j}{n} - (i-1)) \\
=&\frac{\sigma^2}{m} \int_0^1 \Lambda(u)\Lambda(1+u)du + \tau^2 \frac{m}{n} \int_0^1 \lambda(u)\lambda(1+u)du + O(n^{-1}),
\end{aligned}$$

where the last inequality can be verified by Riemann summation. Noting that λ is a pre-average function, we obtain $\lambda(1+u) = -\lambda(1-u)$ and

$$\begin{aligned}&\operatorname{Cov}(\overline{Y}_{i,m}, \overline{Y}_{i+1,m})\\ =&\frac{\sigma^2}{m}\int_0^1 \Lambda(u)\Lambda(1-u)du - \frac{\tau^2 m}{n}\int_0^1 \lambda(u)\lambda(1-u)du + O(n^{-1}).\end{aligned}$$

III:. It can be shown by redoing the arguments in II that

$$\begin{aligned}&\operatorname{Var}(\overline{Y}_{i,m}) = \operatorname{Var}(\overline{X}_{i,m}) + \operatorname{Var}(\overline{\epsilon}_{i,m})\\ =&\frac{\sigma^2}{m}\int_0^2 \Lambda^2(u)du + \tau^2\frac{m}{n}\int_0^2 \lambda^2(u)du + O(n^{-1}).\end{aligned}$$

Note that $\|\Lambda\|_{L^2[0,2]} = 1$. Since the above results hold uniformly in i, k, it follows directly that

$$\begin{aligned}&\operatorname{Var}(\sum_{i=2}^m \overline{Y}_{i,m}^2)\\ =&\sum_{i,k=2,\ |i-k|\geq 2}^m 2\Big(\operatorname{Cov}(\overline{Y}_{i,m}, \overline{Y}_{k,m})\Big)^2\\ &+ 2\sum_{i=2}^{m-1} 2\Big(\operatorname{Cov}(\overline{Y}_{i,m}, \overline{Y}_{i+1,m})\Big)^2 + \sum_{i=2}^m 2\Big(\operatorname{Var}(\overline{Y}_{i,m})\Big)^2\\ =&O(n^{-1}) + 4\Big(\frac{\sigma^2}{\sqrt{c}}\int_0^1 \Lambda(u)\Lambda(1-u)du - \tau^2 c^{3/2}\int_0^1 \lambda(u)\lambda(1-u)du\Big)^2 n^{-1/2}\\ &+ 2\Big(\frac{\sigma^2}{\sqrt{c}} + 2\tau^2 c^{3/2}\|\lambda\|_{L^2[0,1]}^2\Big)^2 n^{-1/2}. \qquad \square\end{aligned}$$

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