Solutions to Stochastic Partial Differential Equations as Elements of Tensor Product Spaces

Dissertation

zur Erlangung des Doktorgrades der Mathematisch-NaturwissenschftlichenFakultäten der Georg-August-Universität zu Göttingen

vorgelegt von

Paul Besold

aus

Paducah, U.S.A.

Göttingen 2000

D 7
Referent: Prof. Dr. H.Hering
Koreferent: Prof. Dr. M. Denker
Tag der mündlichen Prüfung:

Contents

1 Introduction

In their book as well as a number of papers, Øksendal, Holden, Uboe and Zhang study the question of solving stochastic partial differential equations. These methods involve the use of various transformations which transform a stochastic partial differential equation into a classical one. The equation is then solved in the classical sense. The only question remaining is whether an inverse transformation is possible. This method has the advantage that only very few assumptions have to be verified in order to justify an inverse transformation. However, each differential operator has to be considered seperately, i.e. it does not provide any general results on existence and uniqueness on a broader class of operators. A further disadvantage is that the solutions found lie in classical spaces, i.e. in spaces of continous functions in the spatial variable. This disadvantage restricts us to the considerations of special cases of problems in which the data available is continuous in the spatial variable. In what follows, an approach will be provided which addresses both of these problems. The solutions to stochastic partial differential equations will turn out to lie in spaces which correspond to analogs to Hilbert space solutions in the classical setup. The approach which will be taken is to consider the data as elements of certain tensor products of spaces of distributions, one space being a classical distribution space, the other being the space of generalized stochastic Hida functionals. As a result, we will find solutions which lie in much more general spaces than those described by Øksendal and others. The details of the approach are the following: In the first chapter we will review the topological requirements needed for the construction of these generalized distribution spaces. The most important of these requirements is that spaces of stochastic distributions be nuclear. Further we will discuss certain properties of tensor products of these spaces with spaces usually used in solving, in combination with Hilbert space methods, broader classes of partial differential equations. In addition, we will discuss properties of tensor products of linear operators on the respective spaces, in particular when these operators are homomorphisms respectively isomorphisms of topological vector spaces. In the third chapter, we recall some well-known techniques in obtaining Hilbert space solutions to elliptic partial differential equations. These will be particularly useful when we discuss stochastic partial differential operators, i.e. differential operators with coefficients which are random variables. This will be the main topic under discussion in the last chapter. In chapters four and five, the construction and topological properties of Gaussian spaces, and the spaces of stochastic distributions and test functions will be discussed after a short introduction of various topologies on tensor product spaces. The subsequent chapter will be devoted to finding stochastic solutions of elliptic partial differential equations with deterministic coefficients. As was mentioned above, the final chapter deals with the case of stochastic coefficients. The spaces $(\mathcal{E})_{p,\infty}$ and $(\mathcal{E})_\infty$, which are in a sense analogous to the spaces $C^\infty(D)$ for some open subset D of R^d are defined and it is shown that a sufficient for finding solutions to a class of elliptic partial differential equations with stochastic coefficients is that the aforementioned coefficients be in the space $(\mathcal{E})_\infty$. Our deliberations in this chapter will, for simplicity's sake, be restricted to second order divergence class elliptic operators whose lower order terms are zero, however, generalizations of the results follow using the same arguments.

The results of this dissertation show that we can quite simply, without the transformation methods of Øksendal, determine the existence and uniqueness of solutions to a quite large range of stochastic partial differential equations and in the case of elliptic partial differential equations satisfying a certain coercivity conditions solve the stochastic Dirichlet problem with zero boundary conditions, even in the event that the coefficients of the operator are stochastic.

2 Background

2.1 Topological Preliminaries

In this section we introduce some specific topologies and types of topological vector spaces which will be discussed later on. Of particular importance will be the projective limit and inductive limit topologies. The central results of this paper will make strong use of the fact that the spaces of stochastic test functions and distributions as defined by Hida and Kondratiev are nuclear. The nuclear spaces have some very nice properties which we will make use of in proving the existence and uniqueness of certain classes of stochastic partial differential equations.

We first recall some well known definitions of typical topologies on the dual spaces of topological vector spaces. Recall that the dual space of a topological vector space E is the linear space of continuous linear functionals on E and will be denoted by E^*.

Definition 2.1 *Let $A \subseteq E$ a topological vector space. The subset of E^*, $\{x' \in E^* : \sup_{x\in A} |< x', x >| \leq 1\}$ is called the polar of A.*

The polars of certain families of subsets of E will be our candidates for the basis of neighbourhoods of zero in E^*. These families must satisfy the following two properties in order to ensure that the topology generated is a locally convex topology on E^*. Consider the familiy F of bounded subsets of E with the following properties

1. $A, B \in \mathcal{F} \Longrightarrow$ there exists $C \in \mathcal{F}$ such that $A \cup B \subset C$
2. $A \in \mathcal{F}$, $\lambda \in \mathbb{R}$, there exists $B \in \mathcal{F}$ such that $\lambda A \subset B$

Definition 2.2 *The $\mathcal{F}$ topology on E^* is the locally convex topology defined by taking as a basis of neighbourhoods of zero the family $\mathcal{F}'$.*

Example 2.3 *The weak dual topology is obtained by taking $\mathcal{F} =$ the family of finite subsets of E. We denote this topology by $\sigma(E^*, E)$.*

Example 2.4 *The strong dual topology is obtained by taking $\mathcal{F} =$ the family of bounded subsets of E. We denote this topology by $b(E^*, E)$.*

Example 2.5 *The topology of convex compact convergence is obtained by taking $\mathcal{F} =$ the family of convex compact subsets of E. We denote this topology by $\gamma(E^*, E)$.*

It is clear that the following inclusions hold: $\sigma(E^*, E) \leq \gamma(E^*, E) \leq b(E^*, E)$.

Recall the following property of the strong dual topology.

Proposition 2.6 *If E is a normed space then E_b^* is a normed space with the norm being given by $\|x'\| = \sup_{\|x\|\leq 1} |< x', x >|$. Moreover, E_b^* is complete, i.e. E_b^* is a Banach space.*

Let X_α be a family of locally convex topological vector spaces and let X be a vector space.

Definition 2.7 *Suppose for every index α, we have a linear mapping $\phi_\alpha : X \longrightarrow X_\alpha$. The projective limit topology on X is the least fine topology so that each of the ϕ_α are continuous and X equipped with this topology is called the projective limit of the spaces X_α.*

In what follows, X will be a locally convex topological vector space whose topology is generated by an ordered family of seminorms p_α. The spaces X_α will be defined as the completion of the normed space $X \backslash Ker(p_\alpha)$ with respect to p_α (which is clearly a norm on this linear subspace of X). Indeed, X will usually be equipped with a countable family of increasing norms p_α and the linear maps ϕ_α will be the canonical injection of X into X_α.

A basis of this topology can be obtained as follows. Suppose $U_{\alpha,\beta}$ is a basis of neighbourhoods of zero in E_α and β is an element of an index set B. Let $V_{\alpha,\beta}$ be the pre-image of $U_{\alpha,\beta}$ under the mapping ϕ_α. A basis of neighbourhoods of zero in this topology is obtained by taking all finite intersection of sets of the form $V_{\alpha,\beta}$ letting α and β vary in all possible ways. As can easily be seen, a sequence converges in the projective limit topology iff it converges with respect to each semi-norm on X. It is clear that this topology is locally convex if all of the spaces X_α are locally convex and is Hausdorff if the X_α are Hausdorff and if for every α there is at least one $x \in X$ so that $\phi_\alpha(x) \neq 0$.

A closely related topology is the inductive limit topology. It is defined as follows

Definition 2.8 *Suppose we have a family of linear maps $\{\psi_\alpha\}_{\alpha\in\mathbb{N}}$ where $\psi_\alpha : X_\alpha \longrightarrow X$ and $X = \cup\psi_\alpha(X_\alpha)$. Suppose further that the X_α are all locally convex. Then we define on X the finest locally convex topology on X so that the mappings ψ_α are continuous. X equipped with this topology is called the inductive limit of the spaces X_α.*

A convex subset U is a neighbourhood of zero in this topology if for every α there exists a neighbourhood V_α of zero in X_α such that $\phi_\alpha(V_\alpha) = U \cap \phi(X_\alpha)$.

The projective limit and the inductive limit are related by the following consideration. Let X be the projective limit of X_α with respect to the mappings ϕ_α where the X_α are the completion of $X \backslash Ker(p_\alpha)$ with respect to the continuous seminorm p_α and ϕ_α is the canonical map from X into X_α. The transpose ϕ_α^t of ϕ_α is the mapping from $(X_\alpha)^*$ into $(X)^*$ defined by

$$y^* \mapsto (x \mapsto < y^*, \phi_\alpha(x) >) \forall x \in X$$

where $< \cdot, \cdot >$ is the dual form on $X^* \times X$. As a consequence of the Hahn-Banach theorem, we know that if E and F are locally convex spaces and $u : E \longrightarrow F$ is a continuous linear map then the transpose u^t of u is one-to-one if and only if u has a dense image (Chapter 18, Corollary 5 [Tr]). Furthermore, if E and F are normed spaces and we give the dual spaces the strong dual toplogies, they become Banach spaces with respect to the norm $\|x^*\| := \sup_{\|x\|=1} | < x^*, x > |$. Therefore we know that $\psi_\alpha = \phi_\alpha^t$ the transpose of ϕ_α is an injection and thus we can regard the Banach space X_α^* as a linear space in X^*. In this case we know that X_α^* are the linear functionals which are continuous with respect to the seminorm p_α. Therefore, $(X)^*$ which is by definition equal to the continuous linear functionals on X must be equal to $\cup_\alpha\psi((X)^*) = \cup_\alpha(X)_\alpha^*$. Thus we know in this case that the projective limit X has $\cup_\alpha(X)_\alpha^*$ as its dual space. It remains to be seen how the inductive limit topology as described above compares with the usual dual topologies on $(X)^*$. It can be shown that if X is a reflexive Frechet space then the inductive limit topology on $\cup_\alpha(X)_\alpha^*$ is equivalent to the strong dual topology on $(X)^*$ (see [Sch] Chapter 4 section 4 for a detailed discussion).

2.2 Countable Hilbert Spaces

Let E be a topological vector space. Suppose that, on E, we are given a countable system of scalar products $(\cdot,\cdot)_n$ which are compatible in the following sense: If a Cauchy sequence ϕ_k in E converges to zero in the norm $\|\phi_k\|_m^2 = (\phi_k, \phi_k)_m$ and is a Cauchy sequence with respect to the norm $\|\cdot\|_n$, then it converges to zero in the norm $\|\cdot\|_n$. We give E the locally convex topology induced by these norms, i.e. a basis of neighbourhoods of zero is the collection of sets $U_{n,\epsilon} := \phi \in E : \|\phi\|_n \leq \epsilon$.

Definition 2.9 *The space E is said to be countably Hilbert if it is complete relative to the topology described above.*

Proposition 2.10 *A countably Hilbert is a Frechet Space.*

Proof: Every topological vector space topologized by a countable family of seminorms (in this case norms) is metrizable. In this case we can define the metric as follows

$$d(x,y) = \sum_{n=0}^{\infty} \frac{\|x-y\|_n}{2^n(1+\|x-y\|_n)} \forall x,y \in E.$$

It is quite simple to see that the function defined above is a true metric. $\square$

We will also need the following proposition in our consideration of stochastic test functionals. As already described above, let E_n be the completion of E with respect to the norm $\|\cdot\|_n$. Then we have the following proposition.

Proposition 2.11 *A space E with a countable collection of scalar products which is equipped with the topology generated by their respective norms is countably Hilbert if and only if $E = \cap_{n=1}^{\infty} E_n$.*

Proof: See [Ge] volume 2. $\square$

Proposition 2.12 *Let E be a countably Hilbert space. Then E is reflexive.*

Proof: To each element $\phi \in E$ there corresponds a linear functional $\hat{\phi} \in E^*$ defined by the equation

$$<\hat{\phi}, \Phi> := <\Phi, \phi>$$

for every $\Phi \in E^*$. The mapping $\phi \mapsto \hat{\phi}$ is a one-to-one imbedding of the space E into E^{**}. We need to show that it is onto. A linear functional on the space $E^* = \cup_{n=1}^{\infty} E_n^*$ with the strong dual topology is a linear functional on each of the Hilbert spaces E^*. But by the Riesz representation theorem, each Hilbert space is reflexive and therefore $\hat{\phi}$ can be considered as an element of each of the Hilbert spaces which one obtains by completing E with respect to $\|\cdot\|_n$. Since $E = \cap_{n=1}^{\infty} E_n$ we find that $\hat{\phi}$ is an element of E. Thus the two spaces coincide as a space of elements, i.e. the space E is semi-reflexive. We now make use of the fact that a semi-reflexive Frechet space is reflexive (see Proposition 36.5 in [Tr]). $\square$

3 Hilbert Space Solutions to Elliptic Partial Differential Equations

In this section, we will describe a Hilbert space method used in showing the existence and uniqueness of a weak (Hilbert space) solution to the Dirichlet problem for certain classes of elliptic partial differential equations.

Let Ω be an open, bounded subset of $\mathbb{R}^n$ with a locally Lipschitz boundary denoted $\delta\Omega$ and L a differential operator on $C_0^\infty(\Omega)$ defined by

$$Lu := -\frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}u + b_i(x)u) + c_i(x)\frac{\partial}{\partial x_i}u + d(x)u.$$

We assume for the moment that a_{ij}, b_i, c_i and d are smooth functions defined on the closure of Ω. We are interested in finding solutions to the problem

$$\begin{aligned} Lu &= f && (1)\\ u(x) &= 0 \quad x \in \delta\Omega && (2) \end{aligned}$$

where f will in general will be an element from the space of distributions with support in Ω and u is zero on the boundary in the sense that it is an element of the nullspace of the Trace operator which basically maps functions to their restriction on the boundary. See [Ad] for a detailed description.

We begin by proposing the following interpretation of L. We define the mapping $u \mapsto Lu$ to be a linear map from $C_0^\infty(\Omega)$ into $\mathcal{D}(\Omega)$. The action of the element Lu on $C_0^\infty(\Omega)$ is defined by

$$Lu(\phi) =< Lu, \phi >= \int_\Omega Lu(x)\phi(x)dx.$$

Here, $< \cdot,\cdot >$ is the canonical bilinear form on $\mathcal{D}(\Omega) \times C_0^\infty(\Omega)$. We easily obtain that

$$Lu(\phi) =< Lu, \phi >= \int_\Omega a_{ij}(x)\frac{\partial}{\partial x_i}u(x)\frac{\partial}{\partial x_j}\phi(x) - b_i(x)\frac{\partial}{\partial x_i}\phi(x) + c_i(x)\frac{\partial}{\partial x_i}u(x)\phi(x) + d(x)\phi(x)u(x)dx$$

after integrating by parts. Note that the boundary terms vanish as the functions ϕ have compact support in Ω.

Thus, for every u, Lu interpreted as above is an element of the space of distributions with support in Ω. By applying Hölder's inequality to the above equation we easily find that

$$| < Lu, \phi > | \leq K(\|u\|_2 + \|\nabla u\|_2)(\|\phi\|_2 + \|\nabla\phi\|_2)$$

where the constant, K depends on th a_{ij}, b_i, c_i,d and Ω and $\|\cdot\|_2$ denotes the $L^2(\Omega)$ norm. If we equip $C_0^\infty(\Omega)$ with this norm and take its completion, we obtain the space $W_0^{1,2}(\Omega)$ where 1 denotes the number of (weak) derivatives, 2 the fact that the norm involves the L^2 norm of the function and its derivatives and 0 that the function is in a limiting sense equal to zero on the boundary. Since $C_0^\infty(\Omega)$ is dense in $W_0^{1,2}(\Omega)$, we can find a unique continuous extension of the functional $< Lu, \cdot >$ to a linear functional on $W_0^{1,2}(\Omega)$. We associate $< Lu, \cdot >$ with its extension and continue to call the extension $< Lu, \cdot >$. Thus for every $u \in C_0^\infty(\Omega)$, $< Lu, \cdot >$

is an element of the dual space $(W_0^{1,2}(\Omega))^*$ of $W_0^{1,2}(\Omega)$, a linear space in $\mathcal{D}'(\Omega)$ which we will denote by $W^{-1,2}(\Omega)$ and the map

$$u \mapsto < Lu, \cdot >$$

is a map from $C_0^\infty(\Omega)$ into $W^{-1,2}(\Omega)$. We equip $W^{-1,2}(\Omega)$ with the strong dual topology which makes it a Banach space. Since $C_0^\infty(\Omega)$ is dense in $W_0^{1,2}(\Omega)$ and $W^{-1,2}(\Omega)$ is complete, we can find a unique continuous extension of the map

$$u \mapsto < Lu, \cdot >$$

to all of $W_0^{1,2}(\Omega)$.

To summarize, let $L : W_0^{1,2}(\Omega) \longrightarrow W^{-1,2}(\Omega)$ be defined by $u \mapsto (v \mapsto < Lu, v >)$ where

$$< Lu, v > := \int_\Omega a_{ij} \frac{\partial}{\partial x_i} u \frac{\partial}{\partial x_j} v(x) - b_i(x) u(x) \frac{\partial}{\partial x_i} v(x) + c_i(x) \frac{\partial}{\partial x_i} u(x) v(x) + d(x) v(x) u(x) dx$$

for every $v \in W_0^{1,2}(\Omega)$. We call $u \in W_0^{1,2}(\Omega)$ a weak solution to the problem

$$-\frac{\partial}{\partial x_i}(a_{ij}(x) \frac{\partial}{\partial x_j} u + b_i(x) u) + c_i(x) \frac{\partial}{\partial x_i} u + d(x) u = f \quad (3)$$

$$u|_\Omega = 0 \quad (4)$$

if u satisfies

$$< Lu, v > = < f, v > \forall v \in W_0^{1,2}(\Omega)$$

where $< \cdot, \cdot >$ is the canonical dual form on $W^{-1,2}(\Omega) \times W_0^{1,2}(\Omega)$.

Showing existence and uniqueness of weak solutions is clearly equivalent to showing the surjectivity and injectivity of the above defined map L. We will show that provided that the coefficients satisfy certain conditions, there exists a unique solution to the above problem which is continuously dependent on the data, thus proving that the operator L is an isomorphism of topological vector spaces from $W_0^{1,2}(\Omega)$ onto $W^{-1,2}(\Omega)$. To do this we will make use of the following theorem of Lax-Milgram.

Theorem 3.1 *Let $\mathcal{H}$ be a Hilbert space and $B : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{R}$ a bounded coercive bilinear form on $\mathcal{H} \times \mathcal{H}$. Then for every $f \in \mathcal{H}^*$ there exists a unique $u \in \mathcal{H}$ such that*

$$B(u, v) = f(v) = < f, v > \forall v \in \mathcal{H}.$$

Proof: See, for example, [G-T].

The above theorem basically states that a bilinear form which is bounded and coercive is equivalent to the natural scalar product on the Hilbert space. We know by the Riesz representation theorem that a Hilbert space is isometrically isomorphic to its dual.

We proceed by associating to the above defined operator L, the bilinear form $\mathcal{L} : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\mathcal{L}(u, v) := < Lu, v >$$

and require that the bilinear form $\mathcal{L}$ be bounded and coercive. Applying the Lax-Milgram theorem provides the existence of a unique solution, i.e. that the operator L is a bijection. The boundedness of $\mathcal{L}$ guarantees the continuity of L and the coercivity assures us that the inverse L^- is continuous as well, i.e. that L is an isomorphism onto.

In the following proposition, we state necessary conditions for the coercivity of the bilinear form $\mathcal{L}$.

Proposition 3.2 *Let $\mathcal{L} : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \longrightarrow \mathbb{R}$ be given by*

$$\mathcal{L}(u,v) = \int_\Omega a_{ij}(x)\frac{\partial}{\partial x_i}u(x)\frac{\partial}{\partial x_j}v(x) + a(x)u(x)v(x)dx.$$

Let $a_{ij} \in L^\infty(\Omega)$ for all $i,j \in \mathbb{N}$, $1 \leq i,j \leq n$. Assume further that $a_{ij} = a_{ji}$ and that there exists $\lambda > 0$ such that $\sum a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$ for all $x \in \Omega$, $\xi \in \mathbb{R}^n$. Then the bilinear form $\mathcal{L}$ is coercive if one of the two following conditions holds:

- $\inf\{a(x) : x \in \Omega\} > 0$
- *Ω lies between two parallel hyperplanes at distance $\delta < \infty$, then $\mathcal{L}$ is coercive as long as* $\inf\{a(x) : x \in \Omega\} > -\mu/\delta^2$

.

Proof: See [R].

We will now extend the class of operators L which are isomorphisms from $W_0^{1,2}(\Omega)$ onto $W^{-1,2}(\Omega)$ by means of the Reisz-Schauder Theory. The idea is to decompose the operator L into the sum of a compact operator and an invertible operator, show that this operator is injective and subsequently applying the Fredholm alternative to show that it has a bounded inverse.

We first specify the conditions on the coefficients a_{ij},b_i, c_i, and d which we will need to guarentee that L is an isomorphism.

Assume a_{ij},b_i, c_i, and d are measurable, uniformly bounded functions on Ω. Let $(a_{ij})_{1\leq i,j\leq n}$ satisfy $\sum |a_{ij}(x)|^2 \leq \Lambda^2$ for all $x \in \Omega$. Furthermore, let

$$\sum_{i=1}^n \sum_{j=1}^n a_{ij}\xi_{\xi_j} \geq |\xi|^2 > 0$$

and

$$\lambda^{-2}\sum_{i=1}^d |b_i(x)|^2 + |c_i(x)|^2 + \lambda^{-1}|d(x)| \leq \nu^2.$$

In addition, let d and b_i be such that

$$\int_\Omega (dv - b_i\frac{\partial}{\partial x_i}v)dx \leq 0 \;\; \forall \;\; v \in W_0^{1,1}(\Omega) v \geq 0 \quad a.e..$$

We proceed to state the weak maximum principle for elliptic operators satisfying the above conditions.

Theorem 3.3 *Let $u \in W_0^{1,2}(\Omega)$. Suppose $Lu = 0$ where the coefficients of L satisfy the conditions stipulated above. Then*

$$\sup_{x\in\Omega} u(x) \leq \sup_{x\in\partial\Omega} u^+ \text{and} \inf_{x\in\Omega} u \geq \inf_{x\in\partial\Omega} u^-$$

Proof: see [G-T]

We will also need the following coercivity result:

Proposition 3.4 *Let L satisfy the conditions in the above theorem and let $\mathcal{L}$ be the associated bilinear form. Then*

$$\mathcal{L}(u,u) \geq \frac{\lambda}{2}\int_\Omega |\nabla u|^2 dx - \lambda\nu^2 \int_\Omega u^2 dx$$

Proof:

$$\begin{aligned}
\mathcal{L}(u,u) &\geq \int_\Omega \lambda|\nabla u|^2 + (c_i(x) - b_i(x))\frac{\partial}{\partial x_i}u \cdot u - d(x)u^2 dx \\
&\geq \int_\Omega \lambda|\nabla u|^2 + (c_i(x) - b_i(x))\frac{\partial}{\partial x_i}u \cdot u - |d(x)|u^2 dx \\
&\geq \int_\Omega \lambda|\nabla u|^2 - \left(\int_\Omega (c_i(x) - b_i(x))^2 (\frac{\partial}{\partial x_i}u)^2 dx\right)^{\frac{1}{2}} \\
&\left(\int_\Omega (\frac{\partial}{\partial x_i}u)^2 dx\right)^{\frac{1}{2}} - \int_\Omega |d(x)|u^2 dx \\
&\geq \int_\Omega \lambda|\nabla u|^2 - \left(\int_\Omega (c_i(x) - b_i(x))^2 (\frac{\partial}{\partial x_i}u)^2 dx\right)^{\frac{1}{2}} \lambda^{\frac{1}{2}}\lambda^{-\frac{1}{2}} \left(\int_\Omega (\frac{\partial}{\partial x_i}u)^2 dx\right)^{\frac{1}{2}} \\
&- \int_\Omega |d(x)|u^2 dx \\
&\geq \int_\Omega \lambda|\nabla u|^2 - \frac{1}{2\lambda}\int_\Omega (c_i(x) - b_i(x))^2 u^2(x) dx \\
&- \frac{\lambda}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega |d(x)|u^2(x) dx \\
&= \int_\Omega \lambda|\nabla u|^2 - \frac{1}{2}\int_\Omega ((c_i^2(x) + b_i^2(x))/\lambda + |d(x)|)u^2(x) dx + \int_\Omega c_i(x)b_(x)/\lambda u^2(x) dx \\
&- \frac{\lambda}{2}\int_\Omega |\nabla u|^2 dx \\
&\geq \int_\Omega \frac{\lambda}{2}|\nabla u|^2 - \frac{1}{2}\int_\Omega \nu^2\lambda u^2(x) dx - \frac{1}{2\lambda}\int_\Omega (c_i^2(x) + b_i^2(x))u^2(x) dx \\
&\geq \int_\Omega \frac{\lambda}{2}|\nabla u|^2 - \int_\Omega \nu^2\lambda u^2(x) dx
\end{aligned}$$

We now consider a modified operator $L_\sigma := L + \sigma Im$ where $\sigma \in \mathbb{R}$, $\sigma > 0$ and $Im : W_0^{1,2}(\Omega) \longrightarrow W^{-1,2}(\Omega)$ is the canonical imbedding operator. Denote by $\mathcal{L}_\sigma$ the associated bilinear form. From the above theorem, we know that is we choose σ large enough, $\mathcal{L}_\sigma$ is coercive. Choosing such a σ we get that L_σ is an isomorphism of $W_0^{1,2}(\Omega)$ onto $W^{-1,2}(\Omega)$.

We now use the result due to Rellich to show the compactness of the imbedding operator Im.

Theorem 3.5 *Let Ω be a bounded domain in $\mathbb{R}^n$. Then the imbedding*

$$W_0^{1,2}(\Omega) \longrightarrow L^2(\Omega)$$

is compact for $n \geq 2$.

Proof: See [Ad].

Writing Im as the composition of the imbedding I_1 of $W_0^{1,2}(\Omega)$ into $L^2(\Omega)$ with the imbedding I_2 of $L^2(\Omega)$ into $W^{-1,2}(\Omega)$ we have that Im is compact since, by the Rellich theorem, I_1 is compact and I_2 is obviously continuous. Since the compact operators are a closed two-sided ideal in the algebra of bounded operators, we have that Im is compact.

Thus, L can be written as $L = L_\sigma - \sigma Im$, in particular, L is the sum of a compact operator and an invertible operator. If we can show that the map $(L_\sigma^{-1}L) := \mathbb{I} - \sigma L_\sigma^{-1} Im : W_0^{1,2}(\Omega) \longrightarrow W_0^{1,2}(\Omega)$ (here $\mathbb{I}$ denotes the identity map) is invertible then we can find the inverse of the operator L by setting $L^{-1} = (L_\sigma^{-1}L)^{-1}L_\sigma^{-}1$. Showing that $(L_\sigma^{-1}L)$ is invertible is a simple consequence of the Fredholm Alternative:

Theorem 3.6 *Let V be a compact operator with $V \in L(X, X)$ where X is a normed vector space. Then if $\mu \neq 0$ is not an eigenvalue of V, then μ is in the resolvent set of V, i.e. $V - \lambda\mathbb{I} : X \longrightarrow X$ is invertible and the inverse is bounded.*

Proof: see [Yo]

As a result of the weak maximum principle, we know that $L_\sigma^{-1}Lu = \mathbb{I} + \sigma L_\sigma^{-1} Im(u) = 0$ implies that $u = 0$. Thus, by the Fredholm alternative, $L_\sigma^{-1}Lu$ is invertible and the inverse is bounded. We know then that L has a continuous inverse and is thus an isomorphism from $W_0^{1,2}(\Omega)$ onto $W^{-1,2}(\Omega)$.

We summarize the above statements in the following theorem:

Theorem 3.7 *Let$L : W_0^{1,2}(\Omega) \longrightarrow W^{-1,2}(\Omega)$ be defined by $u \mapsto (v \mapsto < Lu, v >)$ where*

$$< Lu, v >:= \int_\Omega a_{ij} \frac{\partial}{\partial x_i} u \frac{\partial}{\partial x_j} v(x) - b_i(x) u(x) \frac{\partial}{\partial x_i} v(x) + c_i(x) \frac{\partial}{\partial x_i} u(x) v(x) + d(x) v(x) u(x) dx$$

for every $v \in W_0^{1,2}(\Omega)$. Assume a_{ij},b_i, c_i, and d are measurable, uniformly bounded functions on Ω. Let $(a_{ij})_{1 \le i,j \le n}$ satisfy $\sum |a_{ij}(x)|^2 \le \Lambda^2$ for all $x \in \Omega$. Furthermore let

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \xi_{\xi_j} \ge \lambda |\xi|^2 > 0$$

and

$$\lambda^{-2} \sum_i |b_i(x)|^2 + |c_i(x)|^2 + \lambda^{-1} |d(x)| \le \nu^2.$$

In addition, let d and b_i be such that

$$\int_\Omega (dv - b_i \frac{\partial}{\partial x_i} v) dx \le 0 \quad \forall \quad v \in W_0^{1,1}(\Omega) \quad v \ge 0 \quad a.e..$$

Then the operator L is an isomorphism from $W_0^{1,2}(\Omega)$ onto $W^{-1,2}(\Omega)$.

4 Topological Tensor Products

Definition 4.1 *Let X, Y, M be topological vector spaces over the field of complex numbers. Let Φ be a bilinear map from $E \times F$ into M. We say that X and Y are Φ-linearly dijoint if the following holds:*

Let $x_1, \cdots, x_r$ be a finite subset of X, and $y_1, \ldots, y_r$ a finite subset of Y, consisting of the same number of elements and satisfying the relation

$$\sum_{j=1}^{r} \Phi(x_j, y_j) = 0.$$

Then, if $x_1, \ldots, x_r$ are linearly independent, $y_1 = \cdots = y_r = 0$, and if $y_1, \ldots, y_r$ are linearly independent, $x_1 = \cdots = x_r = 0$.

Definition 4.2 *A tensor product of X and Y is a pair (M, Φ) consisting of the vector space M and a bilinear map Φ of $X \times Y$ into M such that the image of $X \times Y$ spans the whole space M and X and Y are Φ-linearly disjoint.*

The tensor product of the spaces X and Y will be denoted by $X \otimes Y$. We will now state three well-known properties of the tensor product.

Theorem 4.3 *Let E,F be two vector spaces.*

1. *There exists a tensor product of E and F*
2. *Let (M, Φ) be a tensor product of X and Y. Let Z be any vector space, and b a bilinear map of $X \times Y$ into Z. There exists a unique map $\tilde{b}$ from M into Z such that the corresponding diagram commutes.*
3. *If (M_1, Φ_1) and (M_2, Φ_2) are two tensor products of X and Y, there is a one-to-one linear map U of M_1 onto M_2 such that the corresponding diagram commutes.*

The second property is the so-called universal property of the tensor product and the third states the uniqueness of the tensor product up to isomorphism.

Proof: See Theorem 39.1 in [Tr] □

Example 4.4 *The tensor product of functions.*

Let X, Y be two sets and f (resp. g) a complex valued function defined on X(resp. Y). The function defined on $X \times Y$ denoted by $f \otimes g$ is defined by the following map:

$$(x, y) \mapsto f(x)g(y).$$

The tensor product $X \otimes Y$ is the space of all complex functions defined on $X \times Y$ spanned by functions of the form $f \otimes g$ where f (resp. g) varies over X (resp. Y).

It can occur that the tensor product of two spaces (for example, spaces of functions) can be identified with a linear subspace of a larger space. If this larger space carries a topology, we can of course give the tensor product space the subspace topology. The tensor product, $C_0^\infty(\Omega_1) \otimes C_0^\infty(\Omega_2)$ where Ω_i, $i = 1, 2$ are open subsets of R^n and R^m respectively, is clearly a linear subspace of $C_0^\infty(\Omega_1 \times \Omega_2)$. In fact, it is sequentially dense in $C_0^\infty(\Omega_1 \times \Omega_2)$ as the following theorem states.

Theorem 4.5 *Let Ω_1,Ω_2 be open subsets of R^n and R^m respectively. Then the tensor product $C_0^\infty(\Omega_1) \otimes C_0^\infty(\Omega_2)$ is sequentially dense in $C_0^\infty(\Omega_1 \times \Omega_2)$*

Proof: Let $\phi \in C_0^\infty(\Omega_1 \times \Omega_2)$. It is well known (see Corollary 2 of Lemma 15.1 in [Tr]) that ϕ is the limit in $C^\infty(\Omega_1 \times \Omega_2)$ of a sequence of polynomials $\{P_k(x, y)\}$, $k \in \mathbb{N}$. Let K be the support of ϕ, K_1 and K_2 the projection of K into Ω_1 and Ω_2, respectively. Clearly, K_1 and K_2 are compact sets. Let $g \in C_0^\infty(\Omega_1)$ and $h \in C_0^\infty(\Omega_2)$ be equal to 1 on a neighbourhood of K_1 and K_2, respectively. Then $g \otimes h$ is identically equal to 1 on a neighbourhood of K and belongs to $C_0^\infty(\Omega_1) \otimes C_0^\infty(\Omega_2)$. This tensor product thus contains the sequence $(g \otimes h)P_k$ for $k \in \mathbb{N}$. This sequence converges in $C^\infty(\Omega_1 \times \Omega_2)$ and therefore also in $C_0^\infty(\Omega_1 \times \Omega_2)$ to the function $(g \otimes h)\phi = \phi$. $\square$

Corollary 4.6 *Ω_1,Ω_2 are as above. Then $L^p(\Omega_1) \otimes L^p(\Omega_2)$ is sequentially dense in $L^p(\Omega_1 \times \Omega_2)$ for $p \in [1, \infty)$*

Proof: Results from the fact that $C_0^\infty(\Omega_1 \times \Omega_2)$ is dense in $L^p(\Omega_1 \times \Omega_2)$ for $p \in [1, \infty)$. Thus, for any $\epsilon > 0$ and for all $\phi \in L^p(\Omega_1 \times \Omega_2)$ we can find an element $\phi_\epsilon \in C_0^\infty(\Omega_1 \times \Omega_2)$ such that $\|\phi - \phi_\epsilon\|_p < \epsilon/2$. By the above theorem we can find an element $\hat{\phi}_\epsilon \in C_0^\infty(\Omega_1) \otimes C_0^\infty(\Omega_2)$ which is arbitrarily close to ϕ_ϵ in $C_0^\infty(\Omega_1 \times \Omega_2)$ and as a consequence in $L^p(\Omega_1 \times \Omega_2)$ (since ϕ_ϵ has compact support in $\Omega_1 \times \Omega_2$). Choosing $\hat{\phi}_\epsilon$ so that $\|\hat{\phi}_\epsilon - \phi_\epsilon\|_p < \epsilon$ gives that $\|\hat{\phi}_\epsilon - \phi\|_p < \epsilon$, by the triangle inequality. Since ϵ is arbitrary the result follows. $\square$

For general tensor product spaces, there is no obvious natural topological vector space which provides it with the subspace topology as with the C_0^∞ or L^p spaces in the above examples. We will wish however, to give the tensor product of two topological vector spaces a topological structure. There are two possibilities at our disposal: we can try, as in the examples above, to associate the tensor product with a linear subspace of a larger space whose topology is known, or we can construct a seminorm topology on the tensor product $X \otimes Y$ from the seminorm topologies of X and Y.

In order to realise the first possibility, we observe that there exists a canonical bilinear map $\phi : X \times Y \to B(X_\sigma, Y_\sigma)$, the space of continuous bilinear forms on $X_\sigma^* \times Y_\sigma^*$ defined by

$$(x, y) \mapsto \phi(x, y) := < x', x >< y', y > .$$

Proposition 4.7 *$B(X'_\sigma, Y'_\sigma)$ is a tensor product of X and Y*

$B(X'_\sigma, Y'_\sigma)$ is a linear subspace of the space $\mathcal{B}_\epsilon(X'_\sigma, Y'_\sigma)$, the space of separately continuous bilinear forms on $X \times Y$ equipped with the topology of uniform convergence on the products of an equicontinous subset A of X' and B of Y'. Due to the above result, $B(X'_\sigma, Y'_\sigma)$ with the subspace topology provides the $X \otimes Y$ with the ϵ-topology. The space $X \otimes Y$ equipped with this topology is denoted by $X \otimes_\epsilon Y$.

The second important topology on the tensor product space is the projective or $\pi - topology$.

Definition 4.8 *The strongest locally convex topology on $X \otimes Y$ such that the canonical bilinear map $(x, y) \mapsto x \otimes y$ is continuous is called the π-topology . Equipped with this topology, the space $X \otimes Y$ is denoted by $X \otimes_\pi Y$.*

Another characterization of this topology is obtained by means of tensor products of seminorms. Let p and q be two seminorms on X and Y. Taking U_p and V_q as the closed unit semi-balls of in X and Y , respectively, and defining W as the convex hull of $U_p \otimes V_q$, we define the tensor product of p with q by

$$p \otimes q(\theta) = \inf_{\rho > 0} \{\rho \in \mathbb{R} : \theta \in \rho W\}.$$

We thus obtain a family of seminorms on $X \otimes Y$ which induce the $\pi - topology$. It can be easily shown that the tensor product of seminorms have the following properties:

Proposition 4.9 *For all $\theta \in X \otimes Y$*

$$(p \otimes q)(\theta) = \inf \sum_j p(x_j) q(y_j)$$

where the infimum is taken over all representations of θ of the form $\theta = \sum_j x_j \otimes y_j$. Furthermore, for all $x \in X$ and $y \in Y$, $(p \otimes q)(x \otimes y) = p(x)q(y)$.

Proposition 4.10 *Let E and F be two Frechet spaces. Every element $\theta \in E \otimes^c_\pi F$ is the sum of an absolutely convergent series*

$$\theta = \sum_{n=0}^{\infty} \lambda_n x_n \otimes y_n$$

where $\{\lambda_n\}$ is a sequence of complex numbers such that $\sum_n |\lambda_n| < 1$ and $\{x_n\}$ and $\{y_n\}$ are sequences converging to zero in E and F, respectively.

Remark 4.11 *In general, the π-topology is finer than the ϵ-topology.*

We now introduce the notion of the tensor product of two linear mappings defined on $X \otimes Y$.

Proposition 4.12 *Let X_1,Y_1,X_2 and Y_2 be locally convex Hausdorff vector spaces over the field of complex numbers. Let $u : X_1 \to X_2$ and $v : Y_1 \to Y_2$ be continuous linear maps. Then there exists a unique linear map of $X_1 \otimes Y_1$ into $X_2 \otimes Y_2$ called the tensor product $u \otimes v$of u and v, such that*

$$u \otimes v(x \otimes y) = u(x) \otimes v(y)$$

for every $x \in X$, $y \in Y$. Moreover, if u and v are continuous maps, then $u \otimes v$ is continuous when $X \otimes Y$ is equipped with either the π- or ϵ-topologies.

Remark 4.13 *The completion of topological vector spaces is accomplished exactly as in the case of metric spaces. Cauchy sequences in a topological vector space X are defined to be those sequences $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that for every open neighbourhood U of zero, we can find n, m large enough so that $x_n - x_m \in U$.*

We will denote the completion of the tensor product $X \otimes Y$ of X and Y with respect to the π(resp. ϵ)-topology by $X \otimes_\pi^c Y$(resp. $X \otimes_\epsilon^c Y$). If $u \otimes_\tau v$ is a continuous linear map from $X_1 \otimes_\tau Y_1$ into $X_2 \otimes_\tau Y_2$ where τ can be either π or ϵ, there exists a unique continuous extension which we will denote by $u \otimes_\tau^c v$ as a continuous linear map from $X_1 \otimes_\tau^c Y_1$ into $X_2 \otimes_\tau^c Y_2$.

Proposition 4.14 *Let X_1,Y_1,X_2 and Y_2 be locally convex Hausdorff vector spaces over the field of complex numbers. Let $u : X_1 \to X_2$ and $v : Y_1 \to Y_2$ be continuous linear maps. Furthermore, suppose that u and v are isomorphisms from X_1 into X_2 and Y_1 into Y_2 respectively. Then the $u \otimes_\epsilon^c v$ is an isomorphism from $X_1 \otimes_\epsilon^c Y_1$ into $X_2 \otimes_\epsilon^c Y_2$.*

Proposition 4.15 *Let X_1,Y_1,X_2 and Y_2 be locally convex Hausdorff vector spaces over the field of complex numbers. Let $u : X_1 \to X_2$ and $v : Y_1 \to Y_2$ be continuous linear maps. Further, suppose that u (resp. v) is a homomorphism of X_1 (resp. Y_1) onto a dense linear subspace of X_2(resp. Y_2). Then $u \otimes_\pi^c v$ is a homomorphism from $X_1 \otimes_\epsilon^c Y_1$ onto a dense linear subspace of $X_2 \otimes_\epsilon^c Y_2$.*

4.1 Nuclear Spaces

Consider two locally convex, Hausdorff topological vector spaces E,F and the tensor product $E^* \otimes F$ of the dual of E with F. We can regard this as a linear subspace of the space of continuous linear maps from E into F, i.e. the space of continuous linear maps with finite dimensional range. If we take an element of $E^* \otimes F$ we know that it is of the form $\sum x_i' \otimes y_i$ whose sum is finite. We can identify this element with the linear map

$$x \mapsto \sum < x_i', x > y_i.$$

Suppose for the moment that E and F are Banach spaces. It is known that the norm $\|\cdot\|_b$ induced by $L_b(E; F)$, the continuous linear maps with the norm topology, such that $\|\cdot\|_b \leq \|\cdot\|_\pi$. Thus, the injection of $E^* \otimes F$ into $L_b(E; F)$ can be extended to a continuous linear map of $E^* \otimes_\pi^c F$ into $L_b(E; F)$. This leads us to the following definition.

Definition 4.16 *Let E, F be two Banach spaces. the image of $E^* \otimes_\pi^c F$ into $L(E; F)$ is denoted by $L_1(E; F)$. Its elements are called the nuclear mappings of E into F.*

We can now define the nuclear spaces.

Definition 4.17 *A locally convex Hausdorff topological vector space E is said to be nuclear if to every continuous seminorm p on E there is another continuous seminorm q on E with $q \geq p$, such that the canonical mapping from $E_q \longrightarrow E_p$ is nuclear where E_q and E_p are the completion of the spaces E mod $Ker(q)$ and the completion $E \backslash Ker(p)$ with respect to q and p, respectively.*

Nuclear spaces have the following properties:

Theorem 4.18 *The following properties of a locally convex Hausdorff TVS are equivalent:*

1. *E is nuclear.*

2. *To every continuous seminorm p on E there is another continuous seminorm q on E, $q \geq p$ such that the canonical injection of E_p^* into E_q^* is nuclear.*

3. *Every continuous linear map of E into a Banach space is nuclear.*

4. *For every locally convex Hausdorff TVS F the canonical map of $E \otimes_\pi^c F$ into $E \otimes_\epsilon^c F$ is an isomorphism onto.*

Proof: See Theorem 50.1 in [Tr]. □

We will also make use of the following proposition.

Proposition 4.19 *A Frechet space E is nuclear if and only if its strong dual is nuclear.*

Proof: See Proposition 50.6 in [Tr]. □

4.2 Gaussian Spaces

Consider a stochastic process $\{X_t\}_{t \in T}$, for T an index set. This is a parameterized family of random variables on a probability measure space $\{\Omega, \mathcal{F}, \mathcal{P}\}$. This process is called Gaussian if, for every $t \in T$, the distribution function $F(x,t)$ is of the form

$$F(x,t) = (2\pi\sigma(t))^{-1/2} \int_{-\infty}^{x} \exp\left[-\frac{(y-\mu(t))}{\sigma^2(t)}\right] dy$$

or equivalently,

$$E(\exp[izX_t]) = \exp[i\mu(t)z - (1/2)z^2\sigma(t)^2], z \in \mathbb{R}.$$

If we take, for example, $T = \mathbb{R}_+$ and fix $\omega \in \Omega$ we can associate ω with the map $t \mapsto X_t(\omega) =: \omega(t)$. Intuitively, one can think of t as 'time', each ω with a 'particle' and the map $t \mapsto X_t(\omega)$ as the evolution of the particle in time. This mapping is called a 'sample path'. Thus, we can think of the process $\{X_t\}_{t \in [0,\infty)}$ as a Gaussian measure on the the space of functions mapping $\mathbb{R}_+$ into $\mathbb{R}$ by

$$\mu_X(\omega \in \Omega : \omega(t_1) \in A_1, \ldots \omega(t_n) \in A_n) = P(\omega \in \Omega : X_{t_1}(\omega) \in A_1, \ldots x_{t_n}(\omega) \in A_n)$$

for all Borel sets $A_1 \ldots A_n \subseteq \mathbb{R}$ and $t_1, \ldots t_n \in T$. However, the 'sample paths' are in general not given by functions but rather by generalized functions (consider the example of the white noise process refered to in Øksendal). To account for this, we construct a Gaussian measure on the much larger space of tempered distributions $S^*(\mathbb{R}^d)$. Recall that this is the dual space of the Schwartz space of rapidly decreasing smooth functions equipped with its usual topology (the topology of the Schwartz space will be discussed in more detail in what follows). In order to do this, we will need the following structure.

Let $\mathcal{D}$ be a real vector space equipped with a countable family of scalar products $(\cdot,\cdot)$ which induce an ordered sequence of norms $\|\cdot\|_p$, i.e. $p \leq q$ implies $\|\cdot\|_p \leq \|\cdot\|_q$. We also require that the norms be compatible. The compatibility condition requires that every Cauchy sequence converging to zero with respect to the norm $\|\cdot\|_p$, converges to zero with respect to $\|\cdot\|_q$ for every $q \geq p$. Now we define $\mathcal{N}_p$ as the completion of $\mathcal{D}$ with repect to the norm $\|\cdot\|_p$. It is not difficult to show that the compatability of the norms is equivalent to the fact that the imbeddings of $\mathcal{N}_q$ into $\mathcal{N}_p$ are continuous injections for all $q \geq p$. Now we set $\mathcal{N} = \cap_{k=1}^{\infty}\mathcal{N}_k$ and give $\mathcal{N}$ the projective limit topology. From the results in chapter 2, we know that $\mathcal{N}$ is a countably Hilbert, Frechet space. We will also need to assume that $\mathcal{N}$ is nuclear. Recall also that the dual space $\mathcal{N}^*$ of $\mathcal{N}$ is given by $\mathcal{N}^* = \cup_{k\in\mathbb{N}}\mathcal{N}_p^*$ and, in this case is given the weak dual topology. For $\mathcal{N}$ satisfying these conditions, we can state the following theorem due to Minlos:

Theorem 4.20 *Let $C : \mathcal{N} \longrightarrow \mathbb{R}$ be a function on $\mathcal{N}$ such that the following holds:*

- *C is continuous*
- *C is positive definite*
- *$C(0) = 1$*

Then there exists a unique probability measure μ_C on $(\mathcal{N}^*, \mathcal{B})$ (here $\mathcal{B}$ denotes the Borel σ-algebra generated by the weakly open sets in $\mathcal{N}^*$) such that the characteristic function of the random variable $X_\xi :=< \cdot, \xi >$ is given by the following:

$$E(\exp(iX_\xi)) = \int_{\mathcal{N}^*} \exp(i < x, \xi >) d\mu(x) = C(\xi)$$

.

Proof: See [Hi]. □

In what follows, we will take as $\mathcal{N}$ the linear space of Schwartz class functions $S(\mathbb{R}^d)$.

Proposition 4.21 *$S(\mathbb{R}^d)$ is a countably Hilbert, nuclear, Frechet space.*

Proof: See [Hi]. □

Thus, $S(\mathbb{R}^d)$ fulfills the conditions of the Minlos theorem. Define the map $C : S(\mathbb{R}^d) \longrightarrow \mathbb{R}$ by

$$C(\xi) = \exp(-\|\xi\|_{L^2(\mathbb{R}^d)}/2)).$$

It is easy to see that C is continuous, positive definite and that $C(0) = 1$. Thus there exists a measure μ_C for which the random variables X_ξ defined by $x \mapsto < x, \xi >$ for $x \in S^*(\mathbb{R}^d)$ is a centered (expected value equals zero) Gaussian random variable with variance $\|\xi\|^2_{L^2(\mathbb{R}^d)}$.

Note that we can calculate the covariance of the random variables X_{ξ_1} and X_{ξ_2} by polarization. Since the random variables are centered, the covariance is given by

$$\int_{S^*(\mathbb{R}^n)} < x, \xi_1 >< x, \xi_2 > d\mu(x) = \frac{1}{4}\int_{S^*(\mathbb{R}^n)} < x, \xi_1 + \xi_2 >^2 - < x, \xi_1 - \xi_2 >^2 d\mu(x)$$

.

The integral on the left is by definition 1/4 the difference of the variance of the $X_{\xi_1+\xi_2}$ and $X_{\xi_1-\xi_2}$. This is given by

$$\begin{aligned}
&\frac{1}{4}\|\xi_1+\xi_2\|^2_{L^2(\mathbb{R}^d)} - \|\xi_1+\xi_2\|^2_{L^2(\mathbb{R}^d)} \\
&\qquad = \frac{1}{4}[\|\xi_1\|^2_{L^2(\mathbb{R}^d)} + 2(\|\xi_1,\xi_2\|)_2 \\
&\qquad\quad + \|\xi_2\|^2_{L^2(\mathbb{R}^d)} - \|\xi_1\|^2_{L^2(\mathbb{R}^d)} + 2(\|\xi_1,\xi_2\|)_2 - \|\xi_2\|^2_{L^2(\mathbb{R}^d)} \\
&\qquad = (\xi_1,\xi_2)_2.
\end{aligned}$$

Since uncorrelated Gaussian random variables are independent, it is obvious that X_{ξ_1} and X_{ξ_2} are independent if and only if the functions ξ_1 and ξ_2 are orthogonal in $L^2(\mathbb{R}^d)$.

Having constructed the measure space $(S^*(\mathbb{R}^d),\mathcal{B},\mu_C)$ we proceed by defining the $L^2(S^*(\mathbb{R}^d),\mu_C)$ in the usual way, i.e. as the linear space of square-integrable, measurable functions on $(S^*(\mathbb{R}^d),\mathcal{B},\mu_C)$ (From now on we will denote this space by (L^2)). Subsequently, a complete orthogonal basis will be constructed using a system of orthogonal polynomials of random variables of the form X_ξ.

Definition 4.22 *Let $\xi_1,\cdots,\xi_n \in S(\mathbb{R}^d)$ and $P:\mathbb{R}^n \longrightarrow C$ be a polynomial. The polynomials on $S^*(\mathbb{R}^d)$ are given by mappings of the form*

$$P_{\xi_1,\cdot,\xi_n}(x) = P(X_{\xi_1}(x),\cdots,X_{\xi_n}(x)).$$

Proposition 4.23 *The polynomials on $S^*(\mathbb{R}^d)$ are elements of (L^2).*

Proof: Let $P(u) = \sum_{|\alpha|\le n} c_\alpha u^\alpha$ be a polynomial of order n, where $\alpha = (\alpha_1,\cdots,\alpha_n)$ is a multi-index. Then

$$\begin{aligned}
\int_{S^*(\mathbb{R}^d)} |P|^2_{(\xi_1,\cdots,\xi_n)}(x)d\mu &= \int_{S^*(\mathbb{R}^d)} \left(\sum_{|\alpha|\le n} c_\alpha X_\xi^\alpha\right)|^2 d\mu \\
&\le K(d)\sum_{|\alpha|\le n}|c_\alpha|^2 \int_{S^*(\mathbb{R}^d)} X_{\xi_1}^{2\alpha_1}(x)\cdots X_{\xi_n}^{2\alpha_d}(x)d\mu
\end{aligned}$$

If we successively apply the Hölder inequality to the last integral above, it is clear that the (L^2) norm of the polynomial $P_{(\xi_1,\cdots,\xi_n)}$ is finite provided that $\int_{S^*(\mathbb{R}^d)} X_{\xi_i}^{2m}d\mu$ is finite for every m. These integrals, however are merely the 2m-th moments of Gaussian random variables with expected value 0 and variance $\|\xi\|_{L^2(\mathbb{R}^d)}$. The 2m-th moment of such a random variable is given by

$$E(X_{\xi_i}^{2m}) = \frac{2m!}{m!2^m}\|\xi_i\|^2 m_{L^2(\mathbb{R}^d)} < \infty$$

for all $m \in \mathbb{N}$. □

That the polynomials are dense in (L^2) is stated in the following theorem.

Theorem 4.24 *Let $B = \{\xi_i, i \in \mathbb{N}\}$ be an orthonormal basis of $L^2(\mathbb{R}^d)$. Let $\mathcal{P}_{\mathcal{B}}$ denote the algebra of polynomials generated by X_{ξ_i} for $\xi_i \in B$. Then $\mathcal{P}_B$ is a dense subset of (L^2).*

Proof: See [Ba]. □

We will next use the fact that the poynomials are dense to construct a basis of orthogonal polynomials for (L^2).

Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis for $L^2(\mathbb{R}^d)$. Let $\alpha \in I^n$ for $n \in \mathbb{N}$. We define

$$H_\alpha(x) = \prod_{k=1}^{\infty} h_{n_k(\alpha)}(< x, e_k >)$$

where h_m is the m-th hermite polynomial, α a multi-index and $n_k(\alpha)$ is equal to the number of entries α_i which are equal to k.

Proposition 4.25 *The collection $\{H_\alpha, \alpha \in I^n, n \in \mathbb{N}_0\}$ forms an orthogonal basis of (L^2).*

Proof:

$$\begin{aligned}
& (H_\alpha, H_\beta)_2 \\
= \; & \int_{S^*(\mathbb{R}^d)} \prod_{k=1}^{\infty} h_{n_k}(\alpha)(< x, e_k >) \prod_{j=1}^{\infty} h_{n_j}(\beta)(< x, e_j >) d\mu(x) \\
= \; & \int_{S^*(\mathbb{R}^d)} \prod_{k=1}^{\infty} h_{n_k}(\alpha)(< x, e_k >) h_{n_k}(\beta)(< x, e_k >) d\mu(x) \\
= \; & \prod_{k=1}^{\infty} \int_{S^*(\mathbb{R}^d)} h_{n_k}(\alpha)(< x, e_k >) h_{n_k}(\beta)(< x, e_k >) d\mu(x) \\
= \; & \prod_{k=1}^{\infty} \int_{\mathbb{R}} h_{n_k}(\alpha)(u) h_{n_k}(\beta)(u) \exp(-1/2u^2)/\sqrt{2\pi} du \\
= \; & \prod_{k=1}^{\infty} \delta_{n_k(\alpha), n_k(\beta)} n_k(\alpha) \\
= \; & \delta_{\alpha,\beta} n(\alpha)!
\end{aligned}$$

where $\prod_{k=1}^{\infty} n_k(\alpha) =: n(\alpha)!$ and the empty product equals 1. Thus, this basis, known as the Hermite basis, is orthogonal. We know by the previous theorem that the algebra of polynomials $calP_B$ generated by the basis $\{e_k\}_{k\in\mathbb{N}}$ is dense in (L^2). Since every polynomial of degree n can be written as a linear combination of Hermite monomials of degree less than or equal to n, it follow that the basis is complete. □

We can see from the previous theorem that each element ϕ in (L^2) has a unique representation

$$\phi(x) = \sum_{n=0}^{\infty} \phi^{(n)}(x)$$

where

$$\phi^{(n)}(x) = \sum_{\alpha \in I^n} a_\alpha H_\alpha(x).$$

We also know that span$\{H_\alpha : \alpha \in I^n\}$ is a closed linear subspace of (L^2) which is orthogonal to the space span$\{H_\alpha : \alpha \in I^n\}$ for $m \neq n$. Therefore,

$$\|\phi\|_2^2 = \|\sum_{n=0}^{\infty} \phi^{(n)}\|_2^2 = \sum_{n=0}^{\infty} \|\phi^{(n)}\|_2^2 = \sum_{n=0}^{\infty} \sum_{\alpha \in I^n} |a_\alpha|^2 n(\alpha)!$$

If we denote with $\mathcal{H}^{(n)}$, the span $\{H_\alpha : \alpha \in I^n\}$, we can represent (L^2) as the topological direct sum of the $\mathcal{H}^{(n)}$, i.e.

$$(L^2) = \oplus_{n=0}^{\infty} \mathcal{H}^{(n)}$$

.

4.3 Fock Space and Symmetric Fock Space

In this section it will be shown that the space (L^2) is isomorphic as a topological vector space to a subspace of $\oplus_{n=0}^{\infty} L_c^2(\mathbb{R}^d)^{\otimes n}$, where here $L_c^2(\mathbb{R}^d)$ denotes the standard complexification of $L^2(\mathbb{R}^d)$, i.e. $L_c^2(\mathbb{R}^d) = L^2(\mathbb{R}^d) \oplus \mathrm{i}\mathrm{L}^2(\mathbb{R}^\mathrm{d})$ and $L_c^2(\mathbb{R}^d)^{\otimes n} = L_c^2(\mathbb{R}^d) \otimes \cdots \otimes L_c^2(\mathbb{R}^d)$ where the tensor product is the completed tensor product with respect to the Hilbert-Schmidt norm, taken n times (we will supress the notation denoting the complexification when it is obvious from the context what is meant). Note, this yields the same topology on the tensor product when completed with respect to the subspace topology induced by identifying $L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d)$ with a linear subspace of $L^2(\mathbb{R}^{2d})$. Recall, from our example in chapter 4, that the algebraic tensor product $L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d)$ is sequentially dense in $L^2(\mathbb{R}^{2d})$, thus the completion is isomorphic to $L^2(\mathbb{R}^{2d})$. We will refer to the space $\oplus_{n=0}^{\infty} L_c^2(\mathbb{R}^{nd})$ as the Fock space.

The subspace of the Fock space refered to above, will consist of sequences $(f^{(n)}, n \in \mathbb{N}_0)$ with $f^{(n)} \in \mathrm{L}^2(\mathbb{R}^{nd})$ which are symmetric in the following sense :$f^{(n)}(u_1, \ldots, u_n) = f^{(n)}(u_{\pi(1)}, \ldots, u_{\pi(n)})$ for all π elements of the group of permutations of n letters, $u_i \in \mathbb{R}^d$. This linear subspace will be refered to as the symmetric Fock space.

We will make use of the isomorphism between (L^2) and the symmetric Fock space to construct spaces of stochastic test functions by identifying, via the isomorphism, subspaces of (L^2) with subspaces of the symmetric Fock space consisting of sequences of smooth functions, more precisely, Schwartz class functions. As usual, the space of stochastic distributions will be taken as the dual space of the space of test functions.

We will first more precisely describe the Fock and symmetric Fock spaces and construct a complete orthonormal system for each.

Suppose H_1 and H_2 are complex Hilbert spaces with respective scalar products $(\cdot, \cdot)_1, (\cdot, \cdot)_2$. Taking the tensor product $H_1 \otimes H_2$, we obtain a pre-Hilbert space with scalar product

$$(\cdot, \cdot)_{H_1 \otimes H_2} := (\cdot, \cdot)_1 (\cdot, \cdot)_2.$$

Completing with respect to the norm induced by this scalar prduct, (this completion is equivalent to completing with respect to the π topology) we obtain the space $H_1 \otimes_\pi^c H_2$. In what follows we will suppress the notation designating the completion and refer to this space as $H_1 \otimes H_2$.

Proceding inductively, we can define the space $\mathcal{F}^n(H) = H\otimes\cdots\otimes H =: H^{\otimes n}$ and the topological direct product $\mathcal{F}(H) := \sum_{n=0}^{\infty}\mathcal{F}^n(H)$ taking $\mathcal{F}^0(H)$ to be the complex numbers. Then every element ϕ of $\mathcal{F}(H)$ has a unique decomposition $\phi = \sum_{n=0}^{\infty}\phi^{(n)}$ with $\phi^{(k)} \in \mathcal{F}^k(H)$ for all $k \in \mathbb{N}_0$ and $\mathcal{F}(H)$ is a Hilbert space equipped with the scalar product $(\phi,\psi)_{\mathcal{F}} := \sum_{n=0}^{\infty}(\phi^{(n)},\psi^{(n)})_{\mathcal{F}^n}$.

Assuming that the Hilbert space H is separable, we can construct a complete orthonormal basis for $\mathcal{F}(H)$. Let $\{e_k, k\in\mathbb{N}\}$ be a complete orthonormal basis of H. Let $\alpha := (\alpha_1,\ldots,\alpha_n), \beta := (\beta_1,\ldots,\beta_n)$ be multi-indices,and $e_\alpha := \otimes_{i=1}^n e_{\alpha_i}$, $e_\beta := \otimes_{i=1}^n e_{\beta_i}$. From the above definitions, it is clear to see that these elements form a complete orthonormal basis for $\mathcal{F}^n(H)$. To obtain a basis for $(\mathcal{F})^(H)$, we simply take sequences of the form $(0,\ldots,0,e_\alpha,0,\ldots,)$ where $\alpha := (\alpha_1,\ldots,\alpha_n)$ and e_α is the $n+1st$ element in the sequence along with the sequence $(1,0,0,\ldots)$.

The symmetric tensor product will now be defined.

Definition 4.26 *Let $f_1,\ldots,f_n \in H$. Then the symmetric tensor product $\Gamma^{(n)}(H) := H\hat{\otimes}\cdots\hat{\otimes}H$ is the completion with respect to the norm induced by $\mathcal{F}^{(n)}(H)$ of the linear space of finite linear combinations of elements of the form*

$$f_1\hat{\otimes}\ldots\hat{\otimes}f_n := (n!)^{-1}\sum_{\pi\in\Sigma(n)} f_{\pi(1)}\otimes\cdots\otimes f_{\pi(n)}$$

where $\Sigma(n)$ is the permutation group of n letters.

Using the scalar product on $\mathcal{F}^n$ and extending by anti-linearity, yields the space $\Gamma^{(n)}(H)$. Again, taking the direct sum $\oplus_{n=0}^{\infty}\Gamma^{(n)}(H) =: \Gamma(H)$, we obtain the symmetric Fock space over H.

An orthonormal basis of symmetric Fock space is obtained as follows: Let $n\in\mathbb{N}$, I^n the set of naturally ordered elements $\alpha = (\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^n$. If $\alpha \mathrm{I}^{(n)}$ and $i\in\mathbb{N}$, let $n_i(\alpha)$ be the number of entries of α equal to i. Set

$$n(\alpha)! := \prod_{i=1}^{\infty} n_i(\alpha)!$$

and $e_\alpha := \hat{\otimes}_{i=1}^n e_{\alpha_i}$, then

$$\begin{aligned}
(e_\alpha, e_\beta)_{\Gamma^{(n)}(H)} &= (\hat{\otimes}_{i=1}^n e_{\alpha_i}, \hat{\otimes}_{i=1}^n e_{\beta_i}) \\
&= \sum_{\pi\in\Sigma(n)} (e_{\alpha_i}, e_{\beta_{\pi(i)}}) \\
&= n(\alpha)!\delta_{\alpha,\beta}.
\end{aligned}$$

Therefore, an orthonormal basis for $\Gamma(H)$ is the set of sequences $(n(\alpha)!)^{-1/2}(0,0,\ldots,e_\alpha,0,\ldots)$ where e_α is the $(n+1)$st entry, together with the sequence $(1,0,0,\ldots)$. Thus, as remarked upon at the beginning of the section,

$$\Gamma^{(n)}(L^2(\mathbb{R}^d)) \cong L^2(\mathbb{R}^d)^{\wedge n} := \{f\in L^2(\mathbb{R}^{nd}) : f(u_1,\ldots,u_n) = f(u_{\pi(1)},\ldots,u_{\pi(n)})\forall\pi\in\Sigma(n)\}.$$

Giving the space $L^2(\mathbb{R}^d)^{\wedge n}$ the subspace topology, the isomorphism is also an isometry up to a factor of $\sqrt{n!}$.

Remark 4.27 *If H is separable, $\mathcal{F}^{(n)}(H)$ can equivalently be defined as the space of all multi-linear n-forms on $H^n := H \times H \cdots \times H$ where the product is taken n times, equipped with the Hilbert-Schmidt norm given by*

$$\|F\|_{H.S.} := \sum_{k=(k_1,\dots k_n)\in I^n} |F(e_{k_1},\dots,e_{k_n})|^2,$$

where the $\{e_k, k \in \mathbb{N}\}$ form an orthonormal basis of H. Note that with this interpretation, the symmetric space $\Gamma^{(n)}(H)$ is the space of symmetric multi-linear forms on H^n.

4.4 The S Transformation

The S-transformation will serve as our isomorphism from (L^2) onto the symmetric Fock space $\Gamma(H)$.

Definition 4.28 *Let $\phi \in (L^2)$, $\xi \in S(\mathbb{R}^d)$. Then the S transformation is given by*

$$S\phi(\xi) := \int_{S^*(\mathbb{R}^d)} \phi(x) : e^{<x,\xi>} : d\mu(x)$$

where

$$: e^{<x\xi>} : \quad := \exp(< x, \xi > -1/2\|\xi\|^2_{L^2(\mathbb{R}^d)}) x \in S^*(\mathbb{R}^d)$$

.

Proposition 4.29 *Let $\phi \in (L^2)$, $\xi \in S(\mathbb{R}^d)$. Then*

$$S\phi(\xi) = \int_{S^*(\mathbb{R}^d)} \phi(x+\xi) d\mu$$

and

$$S(: e^{<;\eta>} :)(\xi) = \exp(\eta, \xi)_{L^2(\mathbb{R}^d)}$$

Proof: See Hida. □

In particular, we see from the above result that $\int : e^{<x,\xi>:} d\mu = 1$.

We will now show that this mapping implements the isomorphism between (L^2) and the symmetric Fock space $\Gamma(L^2(\mathbb{R}^d))$. Consider the S-transformation of the stochastic Hermite polynomial

$$SH_\alpha(\xi) = \int_{S^*(\mathbb{R}^d)} \exp(< x, \xi > -1/2\|\xi\|_2^2) \prod_k :< x, e_k >^{n_k} (\alpha) : d\mu(x).$$

We proceed by decomposing ξ into

$$\xi = \sum_{k:n_k(\alpha)\neq 0} (e_k, \xi) e_k + \xi'$$

where $(e_k, \xi') = 0$ for all $k \in \{j : n_j(\alpha) \neq 0\}$, i.e. ξ' is the projection of ξ on the orthogonal complement of the space spanned by the $\{e_k : n_k(\alpha) \neq 0\}$. Then

$$
\begin{aligned}
SH_\alpha &= e^{-1/2|\xi|^2}\int e^{<x,\xi'>}\exp(\sum_k (e_k,\xi)_2 < x, e_k >)\prod_k h_{n_k(\alpha)}(< x, e_k >)d\mu(x) \\
&= e^{-1/2|\xi|^2}\int e^{<x,\xi'>}\prod_k \exp((e_k,\xi)_2 < x, e_k >)h_{n_k(\alpha)}(< x, e_k >)d\mu(x) \\
&= e^{-1/2|\xi|^2}\int e^{<x,\xi'>}d\mu(x)\prod_k \int \exp((e_k,\xi)_2 < x, e_k >)h_{n_k(\alpha)}(< x, e_k >)d\mu(x) \\
&= e^{-1/2|\xi|^2}\int : e^{<x,\xi'>} : e^{|\xi'|^2/2}d\mu(x)\prod_k \int \exp((e_k,\xi)_2 < x, e_k >)h_{n_k(\alpha)}(< x, e_k >)d\mu(x) \\
&= e^{-1/2(|\xi|^2-|\xi'|^2)}\prod_k \int \exp((e_k,\xi))_2 < x, e_k > h_{n_k(\alpha)}(< x, e_k >)d\mu(x).
\end{aligned}
$$

but the integral can be calculated to be

$$
\begin{aligned}
&\int \exp((e_k,\xi)_2 < x, e_k >)h_{n_k(\alpha)}(< x, e_k >)d\mu(x) \\
&= \int \exp((e_k,\xi)_2 u)h_{n_k(\alpha)}(u)(2\pi)^{-1/2}e^{-u^2/2}d\mu(x) \\
&= \exp(\frac{1}{2}(e_k,\xi)_2^2)(e_k,\xi)_2^{n_k(\alpha)})
\end{aligned}
$$

and thus,

$$
\begin{aligned}
SH_\alpha(\xi) &= e^{-1/2(|\xi|^2-|\xi'|^2)}\prod_k \exp(\frac{1}{2}(e_k,\xi)_2^2)(e_k,\xi)_2^{n_k(\alpha)}) \\
&= e^{-1/2(|\xi|^2-|\xi'|^2)}e^{1/2(|\xi|^2-|\xi'|^2)}\prod_k (e_k,\xi)_2^{n_k(\alpha)}) \\
&= \prod_k (e_k,\xi)_2^{n_k(\alpha)}) \\
&= \hat{\otimes}_{k=1}^{\infty} e_k^{\otimes n_k(\alpha)}(\xi,\ldots,\xi) \\
&= \hat{\otimes}_{i=1}^{n} e_{\alpha_i}(\xi,\ldots,\xi) \\
&= e_\alpha(\xi,\ldots,\xi)
\end{aligned}
$$

We can therefore interpret the S-transformation as a map from the basis elements H_α of $\mathcal{H}^{(n)}$ onto the basis elements e_α of $\Gamma^{(n)}(L^2(\mathbb{R}^d))$ since the symmetric multilinear forms are uniquely determined by their values on the diagonal. Making use of the linearity of the S-transform and that $\|SH_\alpha\|_{\Gamma^{(n)}(L^2(\mathbb{R}^d))} = \|e_\alpha\|_\Gamma^{(n)}(L^2(\mathbb{R}^d)) = \|H_\alpha\|_{(L^2)}$ taking limits, we obtain, for all $\phi \in (L^2)$,

$$\begin{aligned} S\phi(\xi) &= S\sum_{n\in\mathbb{N}} \phi^{(n)}(\xi) \\ &= \sum_{n\in\mathbb{N}} S\phi^{(n)}(\xi) \\ &= \sum_{n\in\mathbb{N}} S \sum_{\alpha\in I^n} a_\alpha H_\alpha(\xi) \\ &= \sum_{n\in\mathbb{N}} \sum_{\alpha\in I^n} a_\alpha SH_\alpha(\xi) \\ &= \sum_{n\in\mathbb{N}} \sum_{\alpha\in I^n} a_\alpha e_\alpha(\xi,\ldots,\xi). \end{aligned}$$

Setting $\sum_{\alpha\in I^n} a_\alpha e_\alpha$ we find that this is an element of $L_c^2\hat{(}\mathbb{R}^d)$, (where we have used the notation $L_c^2\hat{(}\mathbb{R}^d)$ to denote the space of symmetric L^2 functions on $\mathbb{R}^d$) and that

$$\begin{aligned} \|f^{(n)}\|^2_{\Gamma^{(n)}(L^2(\mathbb{R}^d))} &= \sum_{\alpha\in I^n} \|a_\alpha e_\alpha\|^2_{\Gamma^{(n)}(L^2(\mathbb{R}^d))} \\ &= \sum_{\alpha\in I^n} |a_\alpha|^2 \|e_\alpha\|^2_{\Gamma^{(n)}(L^2(\mathbb{R}^d))} \\ &= \sum_{\alpha\in I^n} |a_\alpha|^2 n(\alpha)! \\ &= \|\phi^{(n)}\|_2^2. \end{aligned}$$

It is clear that both $f^{(n)}$ and $\phi^{(n)}$ are uniquely determined by the sequence $\{a_\alpha\}_{\alpha\in I^n}$. Thus the mapping

$$\phi^{(n)} \mapsto f^{(n)}$$

is bijective. By the above calculations, we know that the mapping is continuous and has a continuous inverse, and thus is an isomorphism from $\mathcal{H}^{(n)}$ onto $\Gamma^{(n)}(L^2(\mathbb{R}^d))$.

It follows from the chaos expansion that for every $\phi \in (L^2)$, there exists a uniquely determined sequence $\{\phi^{(n)}\}_{n\in\mathbb{N}_0}$ which is in one-to-one correspondence with a sequence $\{f^{(n)}\}_{n\in\mathbb{N}_0}$ in $\Gamma(L_c^2(\mathbb{R}^d))$. We summarize these considerations in the following proposition.

Proposition 4.30 *Let $\phi \in (L^2)$. Then there exists a unique sequence $\{f^{(n)}\}_{n\in\mathbb{N}_0} \in \Gamma(L_c^2(\mathbb{R}^d))$ such that*

$$\|\phi\|_2 = \sum_{n=0}^{\infty} n! \|f^{(n)}\|_{\Gamma^{(n)}(L_c^2(\mathbb{R}^d))}.$$

Furthermore,

$$S\phi(\xi) = \sum_{n=0}^{\infty} f^{(n)}(\xi,\ldots,\xi).$$

It is useful to represent the function $\phi \in (L^2)$ directly in terms of the sequence $f^{(n)}$.

Proposition 4.31 *Let $\phi : S^*(\mathbb{R}^d) \longrightarrow \mathbf{C}$ be an (L^2) with integral kernal $f^{(n)}$, $n \in \mathrm{N}_0$. Then*

$$\phi(x) := \sum_{n=0}^{i} nfty <: x^{\otimes n}, f^{(n)} > \quad \mu - almost\ all \quad x$$

where the $: x^{\otimes n} :\in S^*(\mathbb{R}^n d)$ are given by the recursion formula

$$\begin{aligned} : x^{\otimes 0} : &= 1 \\ : x^{\otimes 1} : &= x \\ : x^{\otimes n} : &= : x^{\otimes n-1} : \hat{\otimes} x - : x^{\otimes n-2} : \hat{\otimes} Tr \end{aligned}$$

where $Tr \in S^*(\mathbb{R}^2 d)$ is given by the map $f \mapsto \int f(x,x)dx$

Proof: See Lemma 2.11 in [Hi et al.]

4.5 Stochastic Test Functions and Distibutions

We are now prepared to construct spaces of test functions on the measure space $(S^*(\mathrm{R}^d), \mathcal{B}, \mu)$. We shall begin by considering the case $d = 1$. Here, we use the fact that $S(\mathbb{R})$ is the projective limit of a decreasing sequence of Hilbert spaces $S_p(\mathbb{R})$, $p \in \mathbb{N}_0$ with $S_0(\mathbb{R}) = L^2(\mathbb{R})$, each equipped with a norm $\|\cdot\|_p$ induced by a scalar product $(\cdot,\cdot)_p$ defined by $(f,g)_p := (H^p f, H^p g)_2$ where H is the densely defined, self-adjoint operator on $L^2(\mathbb{R})$ defined at the beginning of this chapter. Recall that these scalar products are compatible, the norms satisfy $\|\cdot\|_p \leq \|\cdot\|_q$ for $q > p$ and that $S_p(\mathbb{R})$ is the completion of $S(\mathbb{R})$ with respect to the p-norm. Recall also that the canonical injection of $S_q(\mathbb{R})$ into $S_p(\mathbb{R})$ is a continuous injection for $q > p$ and is a nuclear map for q sufficiently larger than p, i.e. $S(\mathbb{R})$ is a nuclear space. We proceed by considering the inverse image under the S-transformation (S_p) of the linear subspaces $\Gamma(S_p(\mathbb{R}))$ of $\Gamma(L^2(\mathbb{R}))$. The space of stochastic test functions will be defined as the projective limit of these spaces. Taking the union of the dual spaces $(S_p)^*$ of the spaces (S_p) and providing it with the inductive limit topology yield the stochastic distributions on $(S^*(\mathrm{R}^d), \mathcal{B}, \mu)$.

Theorem 4.32 *Let*

$$\cdots \supset (S_p)^* \supset \cdots (S_1)^* \supset L^2(\mathbb{R}) \supset \cdots S_p(\mathbb{R}) \supset \cdots$$

be the chain of continuous dense imbeddings described above. Then

$$\cdots \supset \Gamma(S_p(\mathbb{R})^*) \supset \cdots \supset \Gamma(S_1(\mathbb{R})^*) \supset \Gamma(L^2(\mathbb{R})) \supset \Gamma(S_1(\mathbb{R})) \supset \cdots \supset \Gamma(S_p(\mathbb{R})^*) \supset \cdots$$

is a chain of continuous, dense imbeddings.

Proof: See Hida. $\square$

Definition 4.33 *We define the space (S_p) to be the inverse image under the S-transformation of the closed linear subspace $S_p(\mathbb{R})$ of $L^2(\mathbb{R})$, i.e. $(S_p) := \{\phi \in (L^2) : S\phi \in \Gamma(S_p(\mathbb{R}))\}$. This space becomes a Hilbert space by defining the scalar product $(\phi,\psi)_{(S_p)} := (S\phi, S\psi)_p$. We denote the dual space $(S_p)^*$ by (S_{-p}) and equip it with the strong dual topology.*

Proposition 4.34 *(S_{-p}) is isomorphic to $\Gamma(S_{-p}(\mathbb{R}))$*

Proof: It follows from the fact that (S_p) and $\Gamma(S_p(\mathbb{R}))$ are isomorphic. If $u : (S_p) \longrightarrow \Gamma(S_p(\mathbb{R}))$ is an isomorphism and u^{-1} its inverse, then the transpose $u^t : \Gamma(S_p(\mathbb{R}))^* \longrightarrow (S_p)^*$ and u^{-1^t} are the inverses of one another. $\square$

From the above proposition we see that we can extend the definition of the inverse of the S-transformation by setting $S^{-1} := S^t : \Gamma(S_p(\mathbb{R}))^* \longrightarrow (S_p)^*$.

The action of elements $\Phi \in (S_{-p})$ on $\phi \in (S_p)$ resulting from this definition can also be expressed via the S-transformation as

$$< \Phi, \phi >=< S\Phi, S\phi >,$$

where $< \cdot, \cdot >$ denotes the canonical dual forms for each of the respective spaces.

In this way, we obtain a chain of continuous dense imbeddings

$$\cdots \supset (S_{-p}) \supset \cdots \supset (L^2) \supset (S_1) \cdots \supset (S_p) \supset \cdots$$

.

Proposition 4.35 *The norm on the space (S_p) is given by*

$$\|\phi\|_{2,p}^2 := \|\phi\|_{(S_p)}^2 = \sum_{n=0}^{\infty} \sum_{\alpha \in I^n} c_\alpha^2 n(\alpha)! \prod_{i=0}^{\infty} (2i+2)^{n_i(\alpha)2p}$$

for every integer p.

Proof: We take as our basis $\{e_k, k \in \mathbb{N}_0\}$ of $L^2(\mathbb{R})$ the hermite functions, i.e., we take as e_j the j-th Hermite function (see for example, [Ho et.al] for a more detailed discussion of the Hermite functions and their properties) . It is well known that these form a complete orthonormal system for $L^2(\mathbb{R})$ and are the eigenfunctions of the self-adjoint, on $L^2(\mathbb{R})$ densely defined differential operator $A := -\frac{d^2}{du^2} + 1 + u^2$. Further, the k-th hermite function e_k has eigenvalue $(2k+2)$.

Thus,

$$
\begin{aligned}
(\phi,\phi)_{2,p} &= (S\phi, S\phi)_{\Gamma(S_p(\mathbb{R}))} \\
&= \sum_{n=0}^{\infty}(S\phi^{(n)}, S\phi^{(n)})_{S_p(\mathbb{R}^n)} \\
&= \sum_{n=0}^{\infty}((A^{\otimes n})^p S\phi^{(n)}, (A^{\otimes n})^p S\phi^{(n)})_{L^2(\mathbb{R}^n)} \\
&= \sum_{n=0}^{\infty}((A^{\otimes n})^p \sum_{\alpha\in I^n} c_\alpha e_\alpha, (A^{\otimes n})^p \sum_{\alpha\in I^n} c_\alpha e_\alpha)_{L^2(\mathbb{R}^n)} \\
&= \sum_{n=0}^{\infty}\sum_{\alpha\in I^n}(c_\alpha (A^p)\otimes n e_\alpha, c_\alpha (A^p)\otimes n e_\alpha)_{L^2(\mathbb{R}^n)} \\
&= \sum_{n=0}^{\infty}\sum_{\alpha\in I^n}(c_\alpha (A^p)\otimes n\hat{\otimes}_{i=1}^{n} e_{\alpha_i}, c_\alpha (A^p)\otimes n\hat{\otimes}_{i=1}^{n} e_{\alpha_i})_{L^2(\mathbb{R}^n)} \\
&= \sum_{n=0}^{\infty}\sum_{\alpha\in I^n}(c_\alpha \hat{\otimes}_{i=1}^{n} A^p(e_{\alpha_i}), c_\alpha \hat{\otimes}_{i=1}^{n} A^p(e_{\alpha_i}))_{L^2(\mathbb{R}^n)} \\
&= \sum_{n=0}^{\infty}\sum_{\alpha\in I^n}(c_\alpha \hat{\otimes}_{i=1}^{n}(2\alpha_i+2)^p e_{\alpha_i}, c_\alpha \hat{\otimes}_{i=1}^{n}(2\alpha_i+2)^p e_{\alpha_i})_{L^2(\mathbb{R}^n)} \\
&= \sum_{n=0}^{\infty}\sum_{\alpha\in I^n}(c_\alpha \prod_{i=1}^{n}(2\alpha_i+2)^p e_\alpha, c_\alpha \prod_{i=1}^{n}(2\alpha_i+2)^p e_\alpha)_{L^2(\mathbb{R}^n)} \\
&= \sum_{n=0}^{\infty}\sum_{\alpha\in I^n}(c_\alpha \prod_{i=1}^{\infty}(2i+2)^{pn_i(\alpha)} e_\alpha, c_\alpha \prod_{i=1}^{\infty}(2i+2)^{pn_i(\alpha)} e_\alpha)_{L^2(\mathbb{R}^n)} \\
&= \sum_{n=0}^{\infty}\sum_{\alpha\in I^n} c_\alpha^2 n(\alpha)! \prod_{i=1}^{\infty}(2i+2)^{2pn_i(\alpha)}
\end{aligned}
$$

□

Definition 4.36 *Let $(S) := \cap_{i=0}^{\infty}$. Give (S) the projective limit topology with respect to the imbeddings $i_p : (S_p) \longrightarrow (S)$. This space is called the* **Hida space of white noise test functionals**. *The dual space $(S)^*$ is called the* **Hida space of white noise generalized functionals**.

Proposition 4.37 *(S) is a countable Hilbert, reflexive Frechet space.*

Proof: It follows from the construction and considerations from chapter 2. □

Proposition 4.38 *The dual space $(S)^*$ of (S) with the strong dual topology is isomorphic to the space $\cup_{p:p\in\mathbb{N}}(S_p)^*$ with the inductive limit topology induced by the mappings $i_p^t : (S)^* \longrightarrow (S_p)^*$ where the i^t is the transpose of the imbedding in the above definition.*

Proof: This follows from the considerations from chapter 2. $\square$

Recall that the $\{e_k, k \in \mathbb{N}\}$ form a basis for $L^2(\mathbb{R})$. For d-dimensional white noise space, we must consider a countable basis for $L^2(\mathbb{R}^d)$. We construct this basis by taking tensor products of Hermite functions. Let $\{e_k, k \in \mathbb{N}\}$ be the Hermite function basis for

$L^2(\mathbb{R})$. Then, the set $\{e_\delta := e_{\delta_1} \otimes \cdots \otimes e_{\delta_d} : \delta := (\delta_1, \ldots, \delta_d) \in \mathbb{N}^n\}$ forms an orthonormal basis for $L^2(\mathbb{R}^d)$. We construct a countable basis $\{\eta_j, j \in \mathbb{N}\}$ with respect to an ordering of the n-tuples δ which satisfies $i < j \Longrightarrow$ that $\sum_{k=1}^d \delta_k^{(j)} \leq \sum_{k=1}^d \delta_k^{(i)}$. We then can define the Hida spaces of test functionals and generalized functionals as above, by defining $H_\alpha(x) := \prod_i h_{\alpha_i}(< x, \eta_i >)$ and defining the spaces $(S_p) := \{\phi \in (L^2) : \|\phi\|_{2,p}^2 := \sum_{n=0}^\infty \sum_\alpha \in I^n c_\alpha^2 n(\alpha)! \prod_{j=1}^\infty (2j+2)^{2n_j(\alpha)p}\}$. The space (S) is again the intersection over p of (S_p) and is equipped with the projective limit topology. The dual spaces are constructed as above.

Proposition 4.39 *The spaces (S) and $(S)^*$ are nuclear.*

Proof: For the proof that (S) is nuclear see [Ho et.al]. The nuclearity of $(S)^*$ follows from the fact that the strong dual of a nuclear Frechet space is nuclear. $\square$

Remark: One should note that in the one dimensional case, we obtain the weighted norms

$$\|\phi\|_{2,p} = \sum_{n=0} \sum_{\alpha \in I^n} c_\alpha^2 n(\alpha)! \prod_{j=1}^\infty (2j+2)^{2n_j(\alpha)p}.$$

The weights $\prod_{j=1}^\infty (2j+2)^{2n_j(\alpha)p}$ come from the fact that the $L^2(\mathbb{R})$ functions can be represented as a sum of the eigenvectors $\{e_k, k \in \mathbb{N}\}$ of the harmonic oscilator $A := -\frac{d^2}{dx^2} + (1+x^2)$ (the hermite functions), and that the e_j have eigenvalues $2j+2$. If we consider d-dimensional white noise space, a basis of $L^2(\mathbb{R}^d)$ consists of the d-fold tensor products of the hermite functions. A complete orthonormal system consists of elements of the form$\{\xi_{\delta_1} \otimes \cdots \otimes \xi_{\delta_d}\}$ where $\delta = (\delta_1, \ldots, \delta_d) \in \mathbb{N}^d$. If we order these elements in such a way that

$$\begin{aligned} \eta_j &:= \{\xi_{\delta_1^{(j)}} \otimes \cdots \otimes \xi_{\delta_d}^{(j)}\} \\ i < j &\Longrightarrow \sum_{k=1}^d \delta_k^{(i)} \leq \sum_{k=1}^d \delta_k^{(j)} \end{aligned}$$

then $\eta_j, j \in \mathbb{N}_0$ forms a complete orthonormal system for $L^2(\mathbb{R}^d)$. If we wish to proceed analogously to that which is described above, we would need to define $\Gamma(S_p\hat{(}(\mathbb{R}^d))$ as the direct sum of $S_p\hat{(}(\mathbb{R}^d)$ the completion of the symmetric Schwartz class functions with respect to the norm generated by the scalar product

$$(f^{(n)}, g^{(n)})_{S_p(\mathbb{R}^{nd})} = (((A_d^{\otimes n})^p) f^{(n)}, (A_d^{\otimes n})^p g^{(n)})_{L^2(\mathbb{R}^n d)}$$

where $A_d := -\Delta + (1+|x|^2)$, Δ is the d-dimensional Laplacian and $|x|^2 = \sum_{i=1}^d x_i^2$. The functions η_j remain eigenfunctions of this operator but the eigenvalue corresponding to the j-th eigenfunction is no longer $2j+2$ but rather $\sum_{k=1}^{d}(2\delta_k^j+2)$. However, in their consideration of the d-dimensional white noise space, Holden, Øksendal etc. have defined the (S_p) spaces and subsequently the space of test functionals and distributions with respect to the (L^2) norm with the weights $\prod_{j=1}^{\infty}(2j+2)^{2n_j(\alpha)p}$ rather than $\prod_{j=1}^{\infty}(2(\sum_{k=1}^{d}2\delta_k^j+2)^{2n_{(\sum_{k=1}^{d}2\delta_k^j)}(\alpha)p}$. For simplicity, we will retain this convention in the next chapter.

5 Stochastic Partial Differential Operators with Deterministic Coefficients

In this chapter, we will use the tools acquired in the previously discussed material to solve stochastic partial differential equations. This will be done by looking for solutions of the PDE's in spaces which are tensor products of function spaces on $\mathbb{R}^d$ with the space of generalized white noise functionals (or spaces of generalized functionals lying therein).

Let D be an open bounded subset of $\mathbb{R}^d$ with a smooth boundary ∂D and $(\Omega, \mathcal{B}, \mu)$ the d-dimensional white noise space. Consider a stochastic partial differential equation of second order in the x variable of the form

$$\begin{aligned} L_x u(x,\omega) &:= \sum_{1\le i,j\le d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}u(x,\omega) + \sum_{i=1}^{d} b_i(x)u(x,\omega) + c(x)u(x,\omega) \\ &= f(x,\omega) \quad x \in D, \omega \in \Omega \\ u(x,\omega) &= 0 \quad x \in \partial D \end{aligned}$$

where $f(x,\cdot)$ is a random variable for each fixed $x \in D$. This is the stochastic version of the classical Dirichlet problem with zero boundary conditions. Since the data f is a random variable with d-dimensional parameter x, we expect that the solution be a random variable as well. Øksendal solves this type of problem in the case of the Poisson equation with white noise forcing, i.e. $L_x = \Delta_x$ and f equal to the negative white noise functional $f(x,\omega) = -W(x,\omega)$. The functional W is an element of $(S)^*$ for every $x \in D$ and he proceeds to find a solution u such that, for every $x \in D$, $u(x,\cdot) \in (S)^*$. In what follows, we will take a different approach. We will consider that data f as an element of a space defined as the tensor product $W^{-1,2}(D) \otimes^c (S)^*$ and seek a solution $u \in W_0^{1,2}(D)$. Note that in the equation above, the differential operator L_x acts only on the 'spatial' variable x. Thus, to facilitate the tensor product approach, we interpret the stochastic differential operator as the tensor product $L_x \otimes Id$ of the operators L_x and Id where Id is the identity operator. If $u(x,\omega) = u_1(x)u_2(\omega) =: u_1 \otimes u_2(x,\omega)$ then $L_x \otimes Id(u_1 \otimes u_2)(x,\omega) = L_x(u_1) \otimes Id(u_2)(x,\omega) = L_x(u_1)(x) \cdot u_2(\omega)$ whenever $L_x(u_1)$ and u_2 are pointwise almost everywhere defined functions. With this interpretation of the differential operator we can define the operator $L_x \otimes Id : W_0^{1,2}(D) \otimes (S)^* \longrightarrow W^{-1,2}(D) \otimes (S)^*$. If we complete both spaces and extend the operator continuously on the completion, the existence and uniqueness of a solution to the stochastic partial differential equation above is equivalent to showing that the unique continuous extension $L_x \otimes_c Id$ is an isomorphism of the topological vector spaces.

To motivate our treatment of the Dirichlet problem and to perhaps clarify why it is reasonable to look for solutions in tensor product spaces, we will start by considering a special case where the mapping

$$x \mapsto f(x,\cdot)$$

is a smooth mapping taking values in some Frechet space of functionals, call it F, in the generalized Hida functionals. We will begin by showing the existence of a function u, also a smooth function taking values in F which satisfies the stochastic PDE in the set D without requiring that it satisfy the boundary condition.

Definition 5.1 *Let D be an open subset of $\mathbb{R}^d$, F a topological vector space, and $f : D \subseteq \mathbb{R}^d \longrightarrow F$ a continuous map. Then f is said to be differentiable at $y_0 \in D$ if is there are vectors $\{e_1, \ldots, e_d\}$ in F such that*

$$|y - y_0|^{-1}\{f(y) - f(y_0) - \sum_{j=1}^{n}(y_j - (y_0)_j)e_j\}$$

converges to zero in F as the number $|y - y_0| > 0$ goes to zero. The vectors e_j are called the first partial derivatives of f at the point y_0 and are denoted by

$$E_j = \frac{\partial f}{\partial y_i}(y_0)$$

for all $j = 1, \ldots, d$.

We say that $f : D \longrightarrow F$ is a smooth function with values in F if it is infinitely continuously differentiable. The set of all such functions will be denoted by $C^\infty(D; F)$.

We topologize this space in an analogous way as we do with smooth functions.

Definition 5.2 *The C^∞ topology on $C^\infty(D; F)$ is the topology of uniform convergence on compact subsets of D for f and its partial derivatives of all orders.*

Proposition 5.3 *If F is complete, so is $C^\infty(D; F)$.*

Proof: Since F is complete, a Cauchy sequence in $C^\infty(D; F)$ converges pointwise to a function $f : D \longrightarrow F$. Since D is locally compact, the convergence is uniform on a neighbourhood of each point (since each point has a compact neighbourhood). Thus the limit is continuous. Now proceed inductively. □

Proposition 5.4 *Let D be an open subset of $\mathbb{R}^d$. Then $C_0^\infty(D) \otimes F$ is dense in $C^\infty(D; F)$*

Proof: See proposition 44.2 in [Tr] □

Theorem 5.5 *If F is complete then $C^\infty(D; F) \cong C^\infty(D) \otimes_\epsilon^c F)$.*

For the proof, we shall need the following lemma.

Lemma 5.6 *$C^\infty(D; F)$ induces the ϵ-topology on $C^\infty(D) \otimes F$.*

Proof: See p.449 in [Tr] □

Proof: $C^\infty(D;F)$ induced the ϵ-topology on $C^\infty(D)\otimes F$. Thus the natural injection can be extended to an isomorphism into the completion of $C^\infty(D;F)$, since this space is, by the previous proposition, complete. But since $C^\infty(D)\otimes F$ is dense in $C^\infty(D;F)$ we know that the isomorphism must be onto. □

We will now wish to make use of the properties of nuclear spaces discussed in chapter 4 in particular the fact that the ϵ- and π- topologies on the tensor product of a nuclear space and a locally convex, Hausdorff topological vector space are equivalent. To use this in our case we will use the following well known fact:

Proposition 5.7 *Let D be an open subset of $\mathbb{R}^d$. Then the space $C^\infty(D)$ with the topology of uniform convergence on compact subsets of D of the function and its partial derivatives of all orders is a nuclear space.*

Proof: See the corollary to theorem 51.1 in [Tr]. □

Thus, by theorem 4.18, we have that

$$C^\infty(D;F)\cong C^\infty(D)\otimes^c_\epsilon F\cong C^\infty(D)\otimes^c_\pi F.$$

The point of these considerations is that we want to equip the tensor product space with the π-topology. The tensor product with this topology has the nice property that the tensor product of two linear homomorphisms is a homomorphism onto.

Theorem 5.8 *Let D be an open subset of $\mathbb{R}^d$ and L be a differential operator with the property that $L(C^\infty(D))=C^\infty(D)$. Let F be a Frechet space. Then the operator $L\otimes^c_\pi F=L\otimes^c_\epsilon F$ and $L\otimes^c F(C^\infty(D)\otimes^c F)=C^\infty(D)\otimes^c F$.*

Proof: The result follows from the nuclearity of $C^\infty(D)$ from Proposition 5.7 above and the fact that the maps L and Id are homomorphisms onto. Since F is Frechet, we have by Proposition 4.14, that $L\otimes^c F$ is a homomorphism onto. □

The operators for which the condition $L(C^\infty(D))=C^\infty(D)$ are satisfied include the elliptic differential operators with smooth coefficients. (See, for example Theorem 9.19 in [G-T])

Example 5.9 *Let $W(x,\omega)$ be the d-parameter white noise functional described in [Ho et al.] Then $W(x,\omega)$ has the formal chaos expansion given by*

$$W(x,\omega)=\sum_{n=0}^{\infty}\sum_{i=1}^{n}\eta_i(x)H_{\epsilon_i^{(n)}}(\omega)$$

where $\epsilon_i^{(n)}=(0,\dots,1)\in I^n$ and $\eta_i, i\in\mathbb{N}_0$ is the basis of $L^2(\mathbb{R}^d)$ constructed from taking d-fold tensor product of the Hermite functions on $L^2(\mathbb{R})$ as described above. We can approximate

$W(x,\omega)$ by $W_N(x,\omega) = \sum_{n=0}^{N}\sum_{i=1}^{n}\eta_i(x)H_{\epsilon_i^{(n)}}(\omega)$. Let D be an open, bounded subset of $\mathbb{R}^d$ and consider the stochastic partial differential equation

$$L_x \otimes^c Idu(x,\omega) := (\sum_{1\le i,j\le d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) + c(x)\otimes^c Id)u(x,\omega) = W_N(x,\omega) x \in D, \omega \in \Omega.$$

Assume that the coefficients a_{ij},b_i and c are all smooth functions on the closure $\overline{D}$ of D. In addition, let the a_{ij} satisfy the ellipticity condition, i.e. $\sum_{1\le i,j\le d} a_{ij}(x)\xi_i\xi_j > \lambda|\xi|^2$ for all vectors $(\xi_1,\ldots,\xi_d) \in \mathbb{R}^d$ and $x \in D$, $\lambda > 0$.

It is well known that the functions $\{\eta_i, i \in \mathbb{N}_0\}$ are Schwartz class functions (see p. 18 [Ho et al.]). Thus, their restriction to a bounded subdomain of $\mathbb{R}^d$ are elements of $C^\infty(D)$. Since the chaos expansion of $W_n(x,\omega)$ is finite, it is clear that the map

$$x \mapsto W_N(x,\cdot)$$

is an element of $C^\infty(D;(L^2))$ and thus of $C^\infty(D)\otimes^c(L^2)$. Thus, by the above theorem, we know that there is an element $u \in C^\infty(D)\otimes^c(L^2)$ such that $L_x \otimes^c Idu(x,\omega) = W_N(x,\omega)$

In what we have done so far, the nuclearity of $C^\infty(D)$ is essential. We have taken advantage of it when we used the property that the completion of a tensor product with respect to the ϵ-topology and the π-topology are the same when one of the two spaces is nuclear. This not only allows us to take advantage of the nice properties which the completion with respect to the π-topology enjoys with regard to homomorphisms onto but also gives us the result that the completed tensor product spaces of smooth functions on an open subset or $\mathbb{R}^d$ with Frechet spaces in the space of Hida generalized functionals. If we consider the chaos expansion of a stochastic process with a d-dimensional parameter $x \in \mathbb{R}^d$ on the white noise space, the coefficients will depend on x. In the case that the chaos expansion, when interpreted as a mapping $x \mapsto \sum_{n=0}^{\infty}\sum_{\alpha\in I^n} c_\alpha(x)H_\alpha(\cdot)$, is a smooth map (into any arbitrary Frechet distribution space which is a subset of $(S)^*$), for example if the chaos expansion is a finite sum, and if the coefficients are smooth functions of x we know that there exists a solution to the stochastic PDE. This is illustrated in the above example.

The disadvantage of the above result is that we are working in a relatively small space, i.e. the space of smooth function. Therefore, we can as yet only solve a stochastic PDE if the data f is in a very restricted class of mappings. Not only this, but when considering elliptic operators, we have been confined to considering those differential operators whose coefficients are smooth functions. We have also to account for the satifaction of the boundary conditions.

To address these problems, we will seek stochastic solutions which are weak solutions in the 'spatial variable' x. For simplicity, we will only consider the class of elliptic operators discussed in chapter 3. This method can, however, be extended to apply to a broader class of operators.

Let L be a differential operator of the form

$$Lu = \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}u + b_i(x)u) + c_i(x)\frac{\partial}{\partial x_i}u + d(x)u$$

whose coefficients are assumed to be bounded, measurable functions on the closure of a open, bounded subset of D of $\mathbb{R}^d$. Furthermore, let the coefficients satisfy the conditions: there exists a positive number λ, and non-negative numbers Λ and ν such that

$$\begin{aligned}
a_{ij}(x)\xi_i\xi_j &\geq \lambda|\xi|^2 \forall x \in D, \xi \in \mathbb{R}^d \\
\sum |a_{ij}(x)|^2 &\leq \Lambda^2 \forall x \in D \\
\lambda^{-2}\sum(|b_(x)|^2 + |c_i(x)|^2 + \lambda^{-1}|d(x)| &\leq \nu^2 \\
\int_D (dv - b_i \frac{\partial}{\partial x_i} v)dx &\leq 0 \forall v \geq 0, a.e., v \in W_0^{1,1}(D).
\end{aligned}$$

Let $f \in W^{-1,2}(D) \otimes^c (S)^*$. We define a weak stochastic solution to the Dirichlet problem with zero boundary conditions as $u \in W_0^{1,2}(D) \otimes^c (S^*)$ such that

$$L \otimes Id(u) = f.$$

In chapter 3, we have shown that an elliptic operator whose coefficients satisfy the conditions mentioned above are isomorphisms from $W_0^{1,2}(D)$ onto $W^{-1,2}(D)$. Thus we have the following theorem.

Theorem 5.10 *Let D be an open, bounded subset of $\mathbb{R}^d$ with a boundary ∂D which is locally Lipschitz. Let L be a second order elliptic partial differential equation which satisfies the above mentioned conditions. Then the mapping $L \otimes^c Id$ is an isomorphism of $W_0^{1,2}(D) \otimes^c (S)^*$ onto $W^{-1,2}(D) \otimes^c (S)^*$.*

Proof: Since, by theorem 3.37 we have that the operator $L : W_0^{1,2}(D) \longrightarrow W^{-1,2}(D)$ is an isomorphism onto, as is $Id : (S)^* \longrightarrow (S)^*$, we have by Proposition 4.15 that $L \otimes_\epsilon^c Id : W_0^{1,2}(D) \otimes_\epsilon^c (S)^* \longrightarrow W^{-1,2}(D) \otimes_\epsilon^c (S)^*$ is an isomorphism into. We know also that, by proposition 4.14 $L \otimes_\pi^c Id : W_0^{1,2}(D) \otimes_\pi^c (S)^* \longrightarrow W^{-1,2}(D) \otimes_\pi^c (S)^*$ is a homomorphism onto a dense subset of $W^{-1,2}(D) \otimes_\pi^c (S)^*$. By Proposition 4.38 which shows that $(S)^*$ is nuclear and Theorem (4.18) stating that the spaces $W^{-1,2}(D) \otimes_\pi^c (S)^*$ and $W^{-1,2}(D) \otimes_\epsilon^c (S)^*$ are isomorphic, we know that $L \otimes_\pi^c Id = L \otimes_\pi^c Id = L \otimes_I^c d$ is an isomorphism onto a dense subset of $W^{-1,2}(D) \otimes_(^{c} S)^*$. However, the space $W^{-1,2}(D) \otimes^c (S)^*$ is complete and thus its image under the mapping $L \otimes^c Id$ is closed. Thus, $L \otimes^c Id$ is an isomorphism of $W_0^{1,2}(D) \otimes^c (S)^*$ onto $W^{-1,2}(D) \otimes^c (S)^*$. □

Corollary 5.11 *For every $f \in W^{-1,2}(D) \otimes^c (S)^*$, there exists a unique weak solution $u \in W_0^{1,2}(D) \otimes^c (S)^*$ to the stochastic Dirichlet problem with zero boundary conditions.*

Example 5.12 *Let D be an open, bounded subset of $\mathbb{R}^d$ and consider the stochastic partial differential equation*

$$L_x \otimes^c Idu := (\sum_{1 \leq i,j \leq d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) + c(x) \otimes^c Id)u = Wx \in D, \omega \in \Omega.$$

Assume that the coefficients a_{ij},b_i and c are all smooth functions on the closure $\overline{D}$ of D which satisfy the conditions given above. This time, the white noise functional W given by its chaos expansion

$$W(\cdot,\cdot) = \sum_{n=0}^{\infty}\sum_{i=1}^{n} \eta_i(\cdot)H_{\epsilon_i}(\cdot)$$

is an element of $f \in W^{-1,2}(D) \otimes^c (S)^*$ if we interpret the sum as

$$W = \sum_{n=0}^{\infty}\sum_{i=1}^{n} < \eta_i, \cdot >< H_{\epsilon_i}, \cdot >,$$

where the $< \cdot, \cdot >$ are the canonical dual forms on $W^{-1,2}(D) \times W_0^{1,2}(D)$ and $(S)^* \times (S)$, respectively. From proposition 4.10 on the representation of an element in the completions of a tensor product space with respect to the π- topology, we know that every element in $W^{-1,2}(D) \otimes^c (S)^*$ can be represented as $\sum_{i=1}^{\infty} x_i \otimes y_i$ where the sequences $\{x_i\}_{i\in\mathbb{N}}$, $\{y_i\}_{i\in\mathbb{N}}$ converge to zero in $W^{-1,2}(D)$ and $(S)^*$, respectively, and the sum converges absolutely. This is clearly the case here. A simple calculation shows that the (S_{-p}) norm of H_{ϵ_k} is $(2k+2)^{-p}$, which is clearly convergent for $p > 0$. This implies convergence in $(S)^*$. Furthermore, it is also easily seen that the Hermite functions η_k on a bounded domain of $\mathbb{R}^d$ are of order $O(k^{-1/2})$ in $W^{-1,2}(D)$. The absolute summability of the series in the π-topology is clear since we know that

$$\sum_{n=0}^{\infty}\sum_{i=1}^{n} \|\eta_i\|_{W^{-1,2}(D)}\|H_{\epsilon_i}\|_{(S_{-p})} \leq \sum_{i=1}^{\infty}(2i+2)^{-p} < \infty$$

for $p > 1$.

6 Partial Differential Operators with Stochastic Coefficients

In this chapter, we will consider stochastic partial differential operators which are themselves random, i.e. the coefficients of the second order differential operator are stochastic processes, and finding solutions to the corresponding Dirichlet problem with 0 boundary conditions. The idea is again to seek solutions in the tensor product of topological vector spaces and extending the methods in the classical theory of partial differential equations appropriately to correspond to the stochastic case. The methods of the previous sections are, in many cases, not directly applicable. Recall that we described a stochastic partial differential operator as the tensor product of two operators, one operating on spaces of functions defined on open subsets of $\mathbb{R}^d$ the other on the random variables on the probability space Ω. The solvability of the stochastic PDE depended, among other things, on the invertibility of each of the respective operators. In what follows, we will study partial differential operators of the form

$$\frac{\partial}{\partial x_i}\left(a_{ij}(x,\omega)\frac{\partial}{\partial x_j}\right) + b_i(x,\omega)\frac{\partial}{\partial x_i} + c(x,\omega).$$

This operator makes apparent the difficulties resulting from an attempt at trying to extend the results from the previous sections as it is not in general clear whether such an operator can be expressed as the tensor product of two operators. However, our previous deliberations give us a hint as to the spaces in which one should search for solutions. Again, it will be shown that the tensor product of spaces completed with respect to the appropriate topology is the right choice.

Difficulties arise in situations where the product of random variables appear. First of all, it is not clear what it means to find a solution $u = u(x, \omega)$ of a PDE for which $u(x, \cdot) \in (S)^*$ for almost all x if the coefficients $a_{ij}(x, \omega)$ are not elements of (S), i.e. we must ensure in our requirements on the coefficients are sufficiently strong to ensure that the product in within the operator are well-defined when one is looking for solutions in the spaces of generalized functions. This would suggest that we assume that the coefficients are at least elements of (S) when searching for distributional solutions in the ω-variable. This, however, will prove to be insufficient for our needs. This is because even for elements $\phi \in (S)$ the space of stochastic test functions, the mapping defined by pointwise multiplication with functions $f \in (S)_p$, $p \geq 0$

$$f \mapsto \phi \cdot f$$

is not, in general, a continuous linear mapping from $(S)_p$ into $(S)_p$. Consider, for example the operator

$$L := \frac{d}{dx}\left(p(x, \omega)\frac{d}{dx}\right).$$

We define L's domain of definition as $C_0^\infty(D) \otimes_\tau^c (S)_p$ where $\otimes_\tau^c$ denotes the closure of the tensor product with respect to the topology τ and the differential operator $\frac{d}{dx}$ is, as in the previous section, interpreted as the continuous extention of the operator $\frac{d}{dx} \otimes Id$ to $C_0^\infty(D) \otimes_\tau^c (S)_p$. The range of this map will not necessarily be a subset of the domain in general, even if the function $\omega \mapsto p(x, \omega) \in (S)$ for every x since the product of the multiplication of functions in $(S)_p$ with stochastic test functions are in $(S)_{p-q}$ for with $q > \frac{\ln(2+\sqrt{2})}{2\ln 2}$ (see Potthoff and Yan, 1992). Since, however, is a property that we will need for what follows, we begin by discussing suitable requirements on the coefficients of stochastic partial differential operators.

6.1 The Spaces $(\mathcal{E})$, $(\mathcal{E})_{p,\infty}$ and $(\mathcal{E})_\infty$

Definition 6.1 *Let $\xi \in S(\mathbb{R}^d)$. We will denote by X_ξ the random variable defined by the map $\xi \mapsto < x, \xi >$ where $x \in S^*(\mathbb{R}^d)$ and $< \cdot, \cdot >$ canonical dual form. The mapping $\xi \mapsto X_\xi$ is called the canonical coordinate process over $S(\mathbb{R}^d)$.*

The next definition will be useful in the computations that follow.

Definition 6.2 *Let $\xi \in S(\mathbb{R}^d)$. Then we define the function $: \exp < \cdot, \xi >:$ as follows:*

$$: \exp < x, \xi >: \ := \exp(< x, \xi > -\frac{1}{2}\|\xi\|^2), x \in S^*(\mathbb{R}^d)$$

where $\| \cdot \|$ denotes the norm of $L^2(\mathbb{R}^d)$.

Proposition 6.3 *Let $\alpha \in \mathbf{C}, \xi \in S(\mathbb{R}^d), p \in [1, \infty)$. Furthermore Then $\exp(\alpha X_\xi) \in (L^p)$. If $Re(\alpha) = 0$, then $\exp(\alpha X_\xi) \in (L^\infty)$.*

Proof: From the Bochner-Minlos Theorem, we know that

$$\int_{S^*(\mathbf{R}^d)} |\exp(\alpha X_\xi(x)|^p d\mu(x) = \int_{S^*(\mathbf{R}^d)} |\exp(\alpha < \xi, x > |^p d\mu(x)$$
$$= \int_{S^*(\mathbf{R}^d)} \exp(pRe(\alpha) < \xi, x >)^p d\mu(x)$$
$$= \exp(1/2(pRe(\alpha)|\xi|^2)) < \infty.$$

If the real part of α equals zero, it is clear that $\exp(\alpha X_\xi)$ is bounded by 1.

Definition 6.4 *Let $\{\xi_i\}_{i\in \mathrm{N}} \subseteq S(\mathbb{R}^d)$ be the orthonormal basis of $L^2(\mathbb{R}^d)$ obtained from the eigenfunctions of the self-adjoint, semi-bounded operator A defined in Section 1. $(\mathcal{E})$ denotes the algebra of trigonometric polynomials generated by $: \exp(iX_{\xi_k}) :$.*

Lemma 6.5 *Suppose that $\xi_i, \xi_j \in S(\mathbb{R}^d)$ are orthogonal with respect to the $L^2(R^d)$ inner product. Then*

$$: \exp i < \cdot, \xi_i >: \cdot : \exp i < \cdot, \xi_j >:=: \exp i < \cdot, \xi_i + \xi_j >: .$$

Proof: By definition

$$: \exp i < \cdot, \xi_i >: \cdot \exp i < \cdot, \xi_j >:= \exp(i < \cdot, \xi_i > + \frac{1}{2}\|\xi_i\|^2) \cdot \exp(i < \cdot, \xi_j > + \frac{1}{2}\|\xi_j\|^2)$$
$$= \exp(i < \cdot, \xi_i + \xi_j > + \frac{1}{2}(\|\xi_i\|^2 + \|\xi_j\|^2))$$
$$= \exp(i < \cdot, \xi_i + \xi_j > + \frac{1}{2}(\|\xi_i + \xi_j\|^2))$$
$$=: \exp(i < \cdot, \xi_i + \xi_j >:$$

where the third equality follows from the orthogonality of ξ_i and ξ_j and the Parsevals identity.

Theorem 6.6 *$(\mathcal{E})$ is dense in (L^p) for $1 \le p < \infty$.*

Proof: See Theorem 1.9 in Hida, Kuo, Potthoff and Streit.

Definition 6.7 *Let $\phi \in (\mathcal{E})$. Then ϕ is of the form*

$$\phi = \sum_{m=1}^{M} a_m : \exp(i \sum_{n=1}^{N} \alpha^{(m)} X_{\xi_{k_n}}) : .$$

We define the norm $\|\cdot\|_{\infty,p}$ by

$$\begin{aligned} \|\phi\|_{\infty,p} &= \| \sum_{m=1}^{M} a_m : \exp(i \sum_{n=1}^{N} \alpha_n^{(m)} X_{\xi_n}) : \|_{\infty,p} \\ &:= \sum_{m=1}^{M} |a_m| \exp(\sum_{n=1}^{N} (\alpha_n^{(m)})^2 \|A^p \xi_n)\|^2 \end{aligned}$$

We denote be $(\mathcal{E})_{p,\infty}$ the completion of $(\mathcal{E})$ with respect to the norm $\|\cdot\|_{\infty,p}$.

Obviously, $(\mathcal{E})_{p,\infty}$ is a Banach space.

Definition 6.8 *We denote by $(\mathcal{E})_\infty$ the vector space defined by $(\mathcal{E})_\infty := \bigcap_{p\in \mathrm{N}} (\mathcal{E})_{p,\infty}$.*

Proposition 6.9 *The space $(\mathcal{E})_\infty$ equipped with the projective limit topology is a Fréchet Space.*

Proof:

We need only show that the family $\|\cdot\|_{p,\infty}$, $p \in \mathrm{N}$ is an increasing family of norms since, as we have seen, the projective limit topology is then equivalent to the metric topology on $(\mathcal{E})_\infty$ with the metric

$$d(x,y) = \sum_{p=0}^{\infty} \frac{\|x-y\|_{p,\infty}}{2^p(1+\|x-y\|_{p,\infty})}.$$

It suffices to show that $\|\phi\|_{p,\infty} \leq \|\phi\|_{p+1,\infty}$ for all $\phi \in \mathcal{E}$ and $p \in \mathrm{N}_0$. Let $\phi \in (\mathcal{E})$. Then ϕ is of the form

$$\phi = \sum_{m=1}^{M} a_m : \exp(i \sum_{n=1}^{N} \alpha_n^{(m)} X_{\xi_n}) : \; = \sum_{m=1}^{M} a_m : \exp(i < \cdot, \alpha^{(\mathbf{m})} \cdot \xi >) :$$

where $\alpha^{(\mathbf{m})} \cdot \xi = \sum_{n=1}^{N} < \cdot, \alpha_n^{(m)} \xi_n >, \alpha_n^m \in \mathrm{N}_0$. Then

$$\begin{aligned}
&\|\phi\|_{\infty,p} = \| \sum_{m=1}^{M} a_m : \exp(i < \cdot, \alpha^{(\mathbf{m})} \cdot \xi >) : \|_{\infty,p} \\
&= \sum_{m=1}^{M} |a_m| \exp\left(\frac{1}{2}\sum_{n=1}^{N} (\alpha_n^{(m)})^2 \|A^p \xi_n\|^2\right) \\
&\leq \sum_{m=1}^{M} |a_m| \exp\left(\frac{1}{2}\sum_{n=1}^{N} (\alpha_n^(m))^2 \|A^{p+1} \xi_n\|^2\right) \\
&= \| \sum_{m=1}^{M} a_m : \exp(i < \cdot, \alpha^{(\mathbf{m})} \cdot \xi >) : \|_{\infty,p+1} \\
&= \|\phi\|_{\infty,p+1}
\end{aligned}$$

where the inequality follows from the fact that all elements in the sum are positive and that the eigenvalues of the operator A are greater than 2.

Note that the norms $\|\cdot\|_{\infty,0}$ and $\|\cdot\|_\infty$ are different with $\|\phi\|_\infty \leq \|\phi\|_{\infty,0}$. Thus $(\mathcal{E})_{0,\infty} \subseteq (L^\infty)$.

Definition 6.10 *Suppose $\phi : S^*(\mathbb{R}^d) \longrightarrow \mathbb{R}$ is of the form*

$$\phi(x) := \sum_{m=1}^{M} a_m \prod_{n=1}^{N} : \exp(i < x, \eta_n >) :, \eta_n \in S(\mathbb{R}^n).$$

Then define $\phi^{(p)}(x) := \sum_{m=1}^{M} a_m \prod_{n=1}^{N} : exp < x, A^p \eta_n >:$. We know by Theorem 1.9 in Hida, Kuo, Potthoff and Streit that the set of functions having the form of ϕ in the above definition form a dense subset of (L^∞). We will denote by $(L^\infty)_p$ the completion of this subset with respect to the norm given by $|||\phi|||_{p,\infty} := \|\phi^{(p)}\|_\infty$.

Proposition 6.11 *Suppose $\phi \in (\mathcal{E})_\infty$. Then $\phi \in (S)$ and as such has a strongly continuous version given pointwise by*

$$\phi(x) = \sum_{k=0}^{\infty} <: x^{\otimes n} :, f^{(n)} >$$

where $f^{(n)} \in S(\mathbb{R}^d)\hat{\otimes}\cdots\hat{\otimes}S(\mathbb{R}^d)$, the symmetric tensor product being taken n-times are the elements of the symmetric Fock space corresponding to ϕ.

Proof: We need to show that if $\phi \in (\mathcal{E})_\infty$ then $\phi \in (S)_p$ for every $p \in \mathrm{N}_0$. If $\phi \in (\mathcal{E})_\infty$ then $\phi \in (\mathcal{E})_{p,\infty}$ for all $p \in \mathrm{N}_0$ and thus that $\phi^{(p)} \in (L^\infty)$ for all $p \in \mathrm{N}_0$. But since $\phi \in (L^\infty)$ implies that $\phi \in (L^2)$ we have that $\phi^{(p)} \in (L^2)$ for all $p \in \mathrm{N}_0$. We will now need the following lemmas.

Lemma 6.12 *The S-transformation of* $: \exp(i < x, \xi >) :$ *is given by*

$$S(: \exp(i < x, \xi >) :) = \exp i(\xi, \cdot).$$

Proof: Follows from a simple computation. For details see Proposition 2.4 in Hida, Kuo, Potthoff, and Streit.

Lemma 6.13 *Suppose $\phi \in (S)_p$ with $\phi(x) : \sum_{n=0}^{\infty} <: x^{\otimes n}, f^{(n)} >$. There exists a sequence $\{\phi\}_k \in (\mathcal{E})$ such that $\phi_k \longrightarrow \phi$ in $(S)_p$. Then $\phi_k^{(p)}$ converges in (L^2) for every $p \in \mathrm{N}_0$ to a function $\phi^{(p)}$ defined by*

$$\phi^{(p)}(x) := \sum_{n=0}^{\infty} <: x^{\otimes n}, (A^p)^{\otimes n} f^{(n)} >$$

Proof: Since the polynomials are dense in $(S)_p$ for all p it follows that the trigonometric polynomials are dense these spaces as well thus, let

$$\phi_k := \sum_{j=1}^{N(K)} a_j^{(k)} : \exp(i < \cdot, \sum_{l=1}^{M} a_l^{(i)} \xi_l >)$$

and suppose $\phi_k \longrightarrow \phi$ in $(S)_p$. Then $S(\phi_k)(\cdot) = \sum_{j=1}^{N(K)} a_j^{(k)} \exp i(\sum_{l=1}^{M} a_l^{(j)} \xi_l, \cdot))$ and by definition of convergence in (S) we have that

$$\begin{aligned}(A^p)^{\times n} &\left(\sum_{j=1}^{N(K)} \frac{a_j^k i^n (\sum \alpha_l^{(j)} \xi_l)^{\otimes n}}{n!} \right) \\ &= \sum_{j=1}^{N(K)} \frac{a_j^k i^n (\sum \alpha_l^{(j)} A^p(\xi_l))^{\otimes n}}{n!} \longrightarrow (A^p)^{\otimes n}(f^{(n)})\end{aligned}$$

by the linearity of $(A^p)^{\otimes n}$ and the definition of the completed tensor product of operators where the convergence here is in (L^2). This, however, is equivalent saying $\phi_k^{(p)} \longrightarrow \phi^{(p)}$ in (L^2).

Now $\phi_k \longrightarrow \phi$ in $(\mathcal{E})_\infty$ implies that $\phi_k \longrightarrow \phi$ in $(L^\infty)_p$ for every $p \in \mathrm{N}_0$, i.e. $\phi_k^{(p)} \longrightarrow \phi^{(p)}$ in (L^∞) which in turn implies $\phi_k^{(p)} \longrightarrow \phi^{(p)}$ in (L^2) for every $p \in \mathrm{N}_0$. But by the above Lemma, this means that $\phi \in (S)_p$ for every $p \in \mathrm{N}_0$ and thus is in (S).

Theorem 6.14 *Every $\phi \in (S)$ has a strongly continuous version given pointwise by*

$$\phi(x) = \sum_{k=0}^{\infty} <: x^{\otimes n} :, f^{(n)} >$$

where $f^{(n)} \in S(\mathbb{R}^d)\hat{\otimes} \cdots \hat{\otimes} S(\mathbb{R}^d)$, the symmetric tensor product being taken n-times are the elements in the Fock space corresponding to ϕ.

Proof:

See Theorem 4.7 in Hida, Kuo, Potthoff and Streit.

Corollary 6.15 *Every $\phi \in (\mathcal{E})_\infty$ is continuous and is given pointwise by*

$$\phi(x) = \sum_{k=0}^{\infty} <: x^{\otimes n} :, f^{(n)} >$$

6.2 The Multiplication of Random Variables

In this section we will discuss the multiplication of random variables with the purpose of investigating the properties of the linear map

$$f \mapsto \phi \cdot f$$

where $(\phi \cdot f)$ denotes the pointwise product. As was mentioned at the beginning of this section, $f : (S)_p \longrightarrow (S)_p$ as defined in the above equation need not be a continuous linear operator even if $f \in (S)$. This subsection will be devoted to showing the continuity of this map for $f \in (\mathcal{E})_\infty$. We begin with a representation of the pointwise product of two (L^2) random variables ϕ and ψ.

Proposition 6.16 *Let $\phi, \psi \in (L^2)$ have the integral kernels $(f^{(n)})_{n \in \mathrm{N}}$ and $(g^{(n)})_n \in \mathrm{N}$, respectively. Then*

$$(\phi \cdot \psi)(x) = \\ = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{m \wedge n} k! \binom{m}{k} \binom{n}{k} <: x^{\otimes(m+n-2k)} :, f^{(m)} \hat{\otimes}_k g^{(n)} >, \mu - a.e. x \in S^*(\mathbb{R}^d)$$

where $f^{(m)} \hat{\otimes}_k g^{(n)}$ is the symmetrization of

$$\int_{\mathbb{R}^k} f^{(m)}(s_1, \ldots, s_k, \cdot) g^{(n)}(s_1, \ldots, s_k, \cdot) ds_1 \ldots ds_k.$$

Proof:

See, for example Osaka.

Using this identity, we will prove the following theorem.

Theorem 6.17 *Let $\phi \in (\mathcal{E})_\infty$. Then the mapping defined by*

$$\psi \mapsto \phi \cdot \psi, \quad \psi \in (S)_p$$

is a continuous linear map from $(S)_p$ into $(S)_p$

Proof: Since $\phi \in (\mathcal{E})_\infty$ by Corollary 9.15 we know that ϕ has the a strongly continuous representation given by $\phi(x) = \sum_{m=0}^{\infty} <: x^{\otimes m} :, f^{(m)} >$. Likewise, $\psi(x) = \sum_{n=0}^{\infty} <: x^{\otimes n} :, g^{(n)} >$, where here the functions $g^{(n)} \in S_{\hat{p}}(\mathbb{R}^n)$. Consider the $(S)_p$ norm of the product $\phi \cdot \psi$. Using the representation formula given above for the product of two random variables, we find that

$$\phi \cdot \psi(x) = \sum_{j=0}^{\infty} <: x^{\otimes j} :, h^{(j)} >$$

with

$$h(j) = \sum_{I(m,n,k)} k! \binom{n}{k} \binom{m}{k} f^{(n)} \hat{\otimes}_k g^{(m)} > .$$

Where, for simplicity, denote the set $\{(m,n,k) \in N_0^2 : m+n-2k = j, k \leq m \wedge n\}$ by $I(m,n,k)$. We will first show that $\|\phi \cdot \psi\|_{p,2} = \|(\phi \cdot \psi)^{(p)}\|_2 \leq \|(\phi^{(p)} \cdot \psi^{(p)})\|_2$. We will have proved the result if we can show that

$$\|A^{\otimes j} h^{(j)}\|_2^2 \leq \| \sum_{I(m,n,k)} k! \binom{n}{k} \binom{m}{k} A^{\otimes n} f^{(n)} \hat{\otimes}_k g^{(m)} \|_2^2.$$

for every $j = 0, 1, 2, \cdots$.

$$\|A^{\otimes j} h^{(j)}\|_2^2 = \| \sum_{I(m,n,k)} k! \binom{m}{k} \binom{n}{k} A^{\otimes j} (f^{(n)} \hat{\otimes}_k g^{(m)}) \|_2^2$$

From the definition of $(f^{(n)} \hat{\otimes}_k g^{(m)})$ we have that

$$A^{\otimes j}(f^{(n)} \hat{\otimes}_k g^{(m)}) = A^{\otimes j} \mathbf{sym} \left(\int_{\mathbb{R}^k} f^{(n)}(s_1, \ldots, s_k, \cdot) \otimes g^{(m)}(s_1, \ldots, s_k, \cdot) ds_1 \ldots ds_k \right)$$

Since $f^{(n)}, g^{(m)} \in L^2(\mathbb{R}^n), L^2(\mathbb{R}^m)$, respectively, they can be represented as elements of the tensor product $L^2(\mathbb{R}^k) \hat{\otimes}^c_{H.S.} L^2(\mathbb{R}^{m-k})$ and $L^2(\mathbb{R}^k) \hat{\otimes}^c_{H.S.} L^2(\mathbb{R}^{n-k})$, where the completion of the tensor product is taken with respect to the Hilbert-Schmidt norm. Thus, we have for $x \in \mathbb{R}^k$, $y \in \mathbb{R}^{m-k}$, $f^{(m)}(x,y) = \sum_{o,p=0}^{\infty} \lambda_{op}^{(n)} v_o(x) w_p(y)$ where v_o, w_p are elements of the orthonormal bases of $L^2(\mathbb{R}^k)$

and $L^2(\mathbb{R}^{m-k})$, resp., each generated by the k-times and m-k times tensor product of the Hermite functions on $\mathbb{R}$. Likewise for $g^{(m)}$, we have that $g^{(m)}(x,y) = \sum_{q,r=0}^{\infty} \mu_{qr}^{(m)} v_q(x)\hat{w}_r(y)$. Thus

$$
\begin{aligned}
&A^{\otimes j}(f^{(n)} \hat{\otimes}_k g^{(m)}) = A^{\otimes j}\mathbf{sym}\left(\int_{\mathbb{R}^k} f^{(n)}(s_1,\dots,s_k,\cdot)\otimes g^{(m)}(s_1,\dots,s_k,\cdot)ds_1\dots ds_k\right)\\
&= A^{\otimes j}\mathbf{sym}\left(\int_{\mathbb{R}^k}\sum_{o,p=0}^{\infty}\lambda_{op}^{(n)}v_o(s_1,\dots,s_k)w_p(\cdot)\sum_{q,r=0}^{\infty}\mu_{qr}^{(m)}v_q(s_1,\dots,s_k)\hat{w}_r(\cdot)ds_1\dots ds_k\right)\\
&= A^{\otimes j}\mathbf{sym}\sum_{o,p=0}^{\infty}\sum_{q,r=0}^{\infty}\lambda_{op}^{(n)}\mu_{qr}^{(m)}w_p\otimes\hat{w}_r\left(\int_{\mathbb{R}^k}v_o(s_1,\dots,s_k)v_q(s_1,\dots,s_k)ds_1\dots ds_k\right)\\
&= A^{\otimes j}\sum_{p,q,r=0}^{\infty}\lambda_{qp}^{(n)}\mu_{qr}^{(m)}w_p\hat{\otimes}\hat{w}_r\\
&= A^{\otimes j}\sum_{p,q,r=0}^{\infty}\lambda_{qp}^{(n)}\mu_{qr}^{(m)}A^{\otimes(n-k)}w_p\hat{\otimes}A^{\otimes(m-k)}\hat{w}_r
\end{aligned}
$$

where we have made use of the uniform convergence of the eigenfunction expansion to interchange the summation and integration in the third step and the fourth equality is due to the fact that v_i's form an orthonormal basis. Thus, it follows that

$$
\begin{aligned}
&\|A^{\otimes j}h^{(j)}\|_2^2 = \|\sum_{I(m,n,k)} k!\binom{m}{k}\binom{n}{k}\sum_{p,q,r=0}^{\infty}\lambda_{qp}^{(n)}\mu_{qr}^{(m)}A^{\otimes(n-k)}w_p\hat{\otimes}A^{\otimes(m-k)}\hat{w}_r\|_2^2\\
&= \|\sum_{I(m,n,k)} k!\binom{m}{k}\binom{n}{k}\sum_{p,q,r=0}^{\infty}\lambda_{qp}^{(n)}\mu_{qr}^{(m)}\Lambda_{pr}w_p\hat{\otimes}\hat{w}_r\|_2^2\\
&= \|\sum_{p,r=0}^{\infty}\left(\sum_{q=0}^{\infty}\sum_{I(m,n,k)} k!\binom{m}{k}\binom{n}{k}\lambda_{qp}^{(n)}\mu_{qr}^{(m)}\right)\Lambda_{pr}w_p\hat{\otimes}\hat{w}_r\|_2^2\\
&= \sum_{p,q=0}^{\infty}\left(\sum_{r=0}^{\infty}\sum_{I(m,n,k)} k!\binom{m}{k}\binom{n}{k}\lambda_{qp}^{(n)}\mu_{qr}^{(m)}\right)^2\Lambda_{pr}^2
\end{aligned}
$$

the last equality follows from the fact that the functions $\{w_p\hat{\otimes}\hat{w}_r\}_{p,r\in\mathbb{N}_0}$ form an orthonormal basis of $L^2(\mathbb{R}^j)$ which are generated by the eigenfunctions of $A^{\otimes j}$ and Parseval's identity.

Now consider

$$
\|\sum_{I(m,n,k)} k!\binom{n}{k}\binom{m}{k}A^{\otimes n}f^{(n)}\hat{\otimes}_k g^{(m)}\|_2^2.
$$

Analogous to the considerations above we have that

$$
\begin{aligned}
& A^{\otimes n} f^{(n)} \underset{k}{\hat{\otimes}} g^{(m)} \\
& = \mathbf{sym}\left(\int_{\mathbb{R}^k} A^{\otimes n} f^{(n)}(s_1, \dots, s_k, \cdot) \otimes A^{\otimes m} g^{(m)}(s_1, \dots, s_k, \cdot) ds_1 \dots ds_k\right) \\
& = \mathbf{sym}\left(\int_{\mathbb{R}^k} A^{\otimes n} \sum_{o,p=0}^{\infty} \lambda_{op}^{(n)} v_o(s_1, \dots, s_k) w_p(\cdot) \otimes A^{\otimes m} \sum_{q,r=0}^{\infty} \mu_{qr}^{(m)} v_q(s_1, \dots, s_k) \hat{w}_r(\cdot) ds_1 \dots ds_k\right) \\
& = \mathbf{sym}\left(\int_{\mathbb{R}^k} \sum_{o,p=0}^{\infty} (\lambda_{op}^{(n)} A^{\otimes k} v_o(s_1, \dots, s_k) A^{\otimes n-k} w_p(\cdot)) \otimes \sum_{q,r=0}^{\infty} \mu_{qr}^{(m)} A^{\otimes k} v_q(s_1, \dots, s_k) A^{\otimes m-k} \hat{w}_r(\cdot) ds_1 \dots ds_k\right) \\
& = \sum_{o,p=0}^{\infty} \sum_{q,r=0}^{\infty} (\lambda_{op}^{(n)} \mu_{qr}^{(m)} A^{\otimes n-k} w_p \hat{\otimes} A^{\otimes m-k} \hat{w}_r \left(\int_{\mathbb{R}^k} A^{\otimes k} v_o(s_1, \dots, s_k) A^{\otimes k} v_q(s_1, \dots, s_k) ds_1 \dots ds_k\right) \\
& = \sum_{p,q,r=0}^{\infty} (\lambda_{qp}^{(n)} \mu_{qr}^{(m)} A^{\otimes n-k} w_p \hat{\otimes} A^{\otimes m-k} \hat{w}_r {\Omega_q^k}^2
\end{aligned}
$$

where we have used the fact that v_o and v_q are orthogonal and are eigenfunctions of the operator $A^{\otimes k}$ and we have denoted by Ω_q^k the eigenvalue of v_q. Thus, it follows that

$$
\begin{aligned}
& \| \sum_{I(m,n,k)} k! \binom{n}{k} \binom{m}{k} A^{\otimes n} f^{(n)} \underset{k}{\hat{\otimes}} g^{(m)} \|_2^2 \\
& = \| \sum_{I(m,n,k)} k! \binom{n}{k} \binom{m}{k} \sum_{p,q,r=0}^{\infty} (\lambda_{qp}^{(n)} \mu_{qr}^{(m)} A^{\otimes n-k} w_p \hat{\otimes} A^{\otimes m-k} \hat{w}_r {\Omega_q^k}^2 \|_2^2 \\
& = \| \sum_{I(m,n,k)} k! \binom{n}{k} \binom{m}{k} \sum_{p,q,r=0}^{\infty} (\lambda_{qp}^{(n)} \mu_{qr}^{(m)} {\Omega_q^k}^2 \Lambda_{pr} (w_p \hat{\otimes} \hat{w}_r \|_2^2 \\
& = \sum_{p,q,r=0}^{\infty} (\| \sum_{I(m,n,k)} k! \binom{n}{k} \binom{m}{k} {\Omega_q^{(k)}}^2) \lambda_{pr} (w_p \hat{\otimes} \hat{w}_r) \|_2^2 \\
& = \sum_{p,r=0}^{\infty} (\sum_{q=0}^{\infty} (\| \sum_{I(m,n,k)} k! \binom{n}{k} \binom{m}{k} {\Omega_q^{(k)}}^2)^2 \lambda_{pr}^2 \\
\geq \quad & \sum_{p,r=0}^{\infty} (\sum_{q=0}^{\infty} (\| \sum_{I(m,n,k)} k! \binom{n}{k} \binom{m}{k})^2 \lambda_{pr}^2 \\
& = \| A^{\otimes j} h^{(j)} \|_2^2
\end{aligned}
$$

where the inequality follows from the fact that $A \geq 2$ and thus $\Omega_q^{(k)} > 2^{2k}$. Carrying out the exact same argument inductively in p gives that $\|(\phi \cdot \psi)^{(p)}\|_2^2 \leq \|(\phi^{(p)} \cdot \psi^{(p)}\|_2^2$ and thus

$\|(\phi \cdot \psi)^{(p)}\|_2 \leq \|(\phi^{(p)} \cdot \psi^{(p)}\|_2$. Finally, since $\phi \in (\mathcal{E})_\infty$, we have that $\|(\phi^{(p)}\|_\infty < \infty$ and $\|(\phi \cdot \psi)^{(p)}\|_2 \leq \|(\phi^{(p)} \cdot \psi^{(p)}\|_2 \leq \|(\phi^{(p)}\|_\infty \|\psi^{(p)}\|_2 \leq C(p)\|\psi^{(p)}\|_2 = \|\psi\|_{2,p}$ and thus follows the result.

The purpose of the deliberations above was to introduce the linear mapping

$$a : L^2(D)\otimes^c_{H.S.}(S)_p \longrightarrow L^2(D)\otimes^c_{H.S.}(S)_p$$

defined by

$$\Phi \mapsto a \cdot \Phi$$

where $a \in C^\infty(\bar{D})\otimes^c(\mathcal{E})_\infty$, D an open subset of $\mathbb{R}^d$ with smooth boundary.

Theorem 6.18 *Let $a \in C^\infty(\bar{D})\otimes^c(\mathcal{E})_\infty$. Then the mapping given by*

$$\Phi \mapsto a \cdot \Phi$$

is a continuous linear map from $L^2(D)\otimes^c_{H.S.}(S)_p$ into $L^2(D)\otimes^c_{H.S.}(S)_p$ for every $p = 0, 1, \ldots$

Proof: Since $a \in C^\infty(\bar{D})\otimes^c(\mathcal{E})_\infty$ we know that there exists sequences $x_{k\,k\in\mathrm{N}} \in C^\infty(\bar{D}), y_{k\,k\in(\mathcal{E})_\infty}$ and $\lambda_k \in l^1$ such that $x_k \longrightarrow 0, y_k \longrightarrow 0$ and

$$a = \sum_{k=0}^{\infty} \lambda_k x_k \otimes y_k$$

where the series is absolutely summable. Recall that a sequence $\{z_n\}$ in a locally convex topological vector space X is absolutely summable if, for every continuous seminorm p on X the sequence of non-negative numbers $p(z_n)$ is summable. This representation is a consequence of Proposition (?) and the fact that $C^\infty(\bar{D})$ and $(\mathcal{E})_\infty$ are Fréchet spaces. Now let Φ be an arbitrary element of $L^2(D) \otimes_H .S.^c(S)p$ for $p \in \mathrm{N}_0$. Then there exists a unique representation of Φ such that $\Phi = \sum_{i,j=0}^\infty \mu_{ij} e_i \otimes f_j$ where $e_{i\,i\in\mathrm{N}}$ and $f_{j\,j\in\mathrm{N}}$ are elements of an orthonormal basis of $L^2(D)$ and $(S)_p$, respectively and $\|\Phi\|^2 = \sum_{i,j=0}^\infty \lambda_{ij}^2$. Thus, the pointwise product $a\cdot\Phi(x,\omega) = \sum_{k=0}^\infty \lambda_k x_k(x) y_k(\omega) \sum_{i,j=0}^\infty \mu_{ij} e_i(x) f_j(\omega)$ for a.e. $x \in D$ with respect to the Lebesgue measure and μ-a.e.$\omega \in S^*(\mathbb{R}^d)$. We claim that this function is an element of $L^2(D)\otimes^c_{H.S.}(S)_p$.

Lemma 6.19 *Let $\{a_{\alpha,\beta}\}_{\alpha\in A,\beta\in B}$ be a sequence of real numbers, A and B index sets, with the property that $sum_{\beta\in B}\left(\sum_{\alpha\in A} a^2_{\alpha,\beta}\right) < \infty$. Then the inequality*

$$\left(\sum_{\alpha\in A}\left(\sum_{\beta\in B} a_{\alpha,\beta}\right)^2\right)^{1/2} \leq \sum_{\beta\in B}\left(\sum_{\alpha\in A} a^2_{\alpha,\beta}\right)^{1/2}$$

holds.

Proof: Define $g_\alpha := \sum_\beta a_{\alpha,\beta}$. Then

$$\begin{aligned}
\sum_{\alpha\in A}\left(\sum_{\beta\in B} a_{\alpha,\beta}\right)^2 &= \\
&= \sum_{\alpha\in A} g_\alpha^2 \\
&= \sum_{\alpha\in A}\left(\sum_{\beta\in B} a_{\alpha,\beta}\right) g_\alpha \\
&= \sum_{\alpha\in A}\sum_{\beta\in B} a_{\alpha,\beta} g_\alpha \\
&= \sum_{\beta\in B}\sum_{\alpha\in A} a_{\alpha,\beta} g_\alpha \\
&\leq \sum_{\beta\in B}\left(\sum_{\alpha\in A} a_{\alpha,\beta}^2\right)^{1/2}\left(\sum_{\alpha\in A} g_\alpha^2\right)^{1/2} \\
&= \sum_{\beta\in B}\left(\sum_{\alpha\in A} a_{\alpha,\beta}^2\right)^{1/2}\left(\sum_{\alpha\in A}\left(\sum_{\beta\in B} a_{\alpha,\beta}\right)^2\right)^{1/2}
\end{aligned}$$

Where we have applied Fubini's Theorem and Hölder's inequality. Dividing both sides by $\left(\sum_{\alpha\in A}\left(\sum_{\beta\in B} a_{\alpha,\beta}\right)^2\right)^{1/2}$ gives the result.

We will now apply this Lemma to our situation. Recall

$$\|a\cdot\Phi\|_{L^2(D)\otimes^c_{H.S.}} = \left(\sum_{\beta=(\beta_1,\beta_2)} (a\cdot\Phi, e_{\beta_1}\otimes f_{\beta_2})^2_{L^2(D)\otimes^c_{H.S.}}\right)^{1/2}.$$

Using the Lemma, we get the following inequality:

$$\left(\sum_{\beta} (a \cdot \Phi, e_{\beta_1} \otimes f_{\beta_2})^2_{L^2(D) \otimes^c_{H.S.}} \right)^{1/2}$$

$$= \left(\sum_{\beta} \left(\sum_{k,\alpha} \lambda_k \mu_\alpha (x_k e_{\alpha_1}, e_{\beta_1})_{L(D)} (y_k f_{\alpha_2}, f_{\beta_2})_{(S)_p} \right)^2 \right)^{1/2}$$

$$\leq \left(\sum_{k,\alpha} (\lambda_k \mu_\alpha)^2 \, \|x_k e_{\alpha_1}\|_2^2 \|y_k f_{\alpha_2}\|^2_{(S)_p} \right)^{1/2}$$

$$\leq \left(\sum_{k,\alpha} (\lambda_k \mu_\alpha)^2 \, \|x_k\|^2_\infty \|y_k^{(p)}\|^2_\infty \right)^{1/2}$$

$$\leq C(p) \left(\sum_k \lambda_k^2 \right)^{1/2} \left(\sum_\alpha \mu_\alpha^2 \right)^{1/2}$$

where we applied the lemma to obtain the first inequality Theorem (?) on the continuity of the multiplication mapping shown in the previous section for the second while the third inequality follows from the boundedness of the sets $\{x_k\}_{k \in \mathrm{N}}$ and $\{y_k\}_{k \in \mathrm{N}}$ in $C^(\bar{D})$ and $(\mathcal{E})_\infty$, resp. But as previously mentioned, $\left(\sum_\alpha \mu_\alpha^2\right)^{1/2} = \|\Phi\|_{L(D) \otimes^c_{H.S.} (S)_p}$ and thus

$$\|a \cdot \Phi\|_{L(D) \otimes^c_{H.S.} (S)_p} \leq C(p, a) \|\Phi\|_{L(D) \otimes^c_{H.S.} (S)_p}.$$

Clearly $\left(\sum_k \lambda_k^2\right)^{1/2} < \infty$ since for the sequence spaces l^1 and l^2 it holds that $l^1 \subseteq l^2$.

6.3 The Dirichlet Problem for Elliptic Stochastic Partial Differential Operators

We are now ready to pose the Dirichlet problem for elliptic stochastic partial differential operators. Although the treatment holds for more general operators, for the sake of simplicity we will only consider operators of the form

$$\frac{\partial}{\partial x_i} \left(a_{ij}(x, \omega) \frac{\partial}{\partial x_j} \right)$$

where, as usual, the partial differential operator $\frac{\partial}{\partial x_j}$ is interpreted as $\frac{\partial}{\partial x_j} \otimes^c Id$ where Id is the identity operator.

We pose the Dirichlet problem as follows.

Consider $D \subseteq \mathbb{R}^d$ an open subset of $\mathbb{R}^d$ with a C^2 boundary. Let $f \in W^{-1,2}(D) \otimes^c_{H.S.} (S)_{-p}$. We call $u = u(x, \omega)$, $u \in W_0^{1,2}(D) \otimes^c_{H.S.} (S)_p$ a solution to the equation

$$\frac{\partial}{\partial x_i}\left(a_{ij}(x,\omega)\frac{\partial}{\partial x_j}u(x,\omega)\right) = f(x,\omega) \qquad (x,\omega) \in D\times\Omega$$
$$u(x,\omega) = \qquad (x,\omega)\in \partial D\times\Omega$$

if

$$\left(a_{ij}(x,\omega)\frac{\partial}{\partial x_j}\otimes^c Id(u)\frac{\partial}{\partial x_i}\otimes^c Id(v)\right)_{L^2(D)\otimes^2_{H.S.}(S)_p} = f[v]$$

for every $v \in W_0^{1,2}(D)\otimes^c_{H.S.}(S)_p$.

Theorem 6.20 *Let $D \subseteq \mathbb{R}^d$ be an open subset of $\mathbb{R}^d$ with a C^2 boundary,$(\Omega,\mathcal{B},\mu)$ the Gaussian probability space with Gauss measure given by the Bochner-Minlos Theorem. Let A a $d\times d$ matrix which has components $a_{i,j} = a_{i,j}(x,\omega)$, $a_{i,j}\in C^\infty(\bar{D})\otimes^c_\pi(\mathcal{E})_\infty$. Futhermore, suppose there exist a strictly positive function $\lambda \in (\mathcal{E})_\infty$ and a matrix $(\tilde{a}_{i,j})$ and $\mu_0 > 0$ such that $a_{i,j}(x,\omega)\xi_i\xi_j \geq \lambda(\omega)\tilde{a_{i,j}}\xi_i\xi_j \geq \mu_0\lambda(\omega)$ for all $x\in D$, $\omega\in\Omega$. Then, for every $f \in W^{-1,2}(D)\otimes^c_{H.S.} S_{-p}$, there exists a unique $u \in W_0^{1,2}(D)\otimes^c_{H.S.}(S)_p$ that solves*

$$\frac{\partial}{\partial x_i}\left(a_{ij}(x,\omega)\frac{\partial}{\partial x_j}u(x,\omega)\right) = f(x,\omega) \qquad (x,\omega) \in D\times\Omega$$
$$u(x,\omega) = 0 \qquad (x,\omega)\in \partial D\times\Omega$$

Proof: The idea of the proof is to use the theorem of Lax-Milgram. Thus, we need to show that the bilinear form on $W_0^{1,2}(D)\otimes^c_{H.S.}(S)_p \times W_0^{1,2}(D)\otimes^c_{H.S.}(S)_p$ given by

$$(u,v)\mapsto \left(a_{ij}(x,\omega)\frac{\partial}{\partial x_j}\otimes^c Id(u)\frac{\partial}{\partial x_i}\otimes^c Id(v)\right)_{L^2(D)\otimes^2_{H.S.}(S)_p}$$

is a well-defined, bounded, coercive, bilinear form. We know from our previous deliberations that the mapping $u\mapsto \frac{\partial}{\partial x_j}\otimes^c Id(u)$ is a continuous linear mapping from $W_0^{1,2}(D)\otimes^c_{H.S.}(S)_p$ into $L^2(D)\otimes^c_{H.S.}(S)_p$ and from Theorem 6.18 that since $a_{i,j}\in C^\infty(\bar{D})\otimes^c(\mathcal{E})_\infty$, $a_{i,j}\cdot\frac{\partial}{\partial x_j}\otimes^c Id(u) \in L^2(D)\otimes^c_{H.S.}(S)_p$. Thus, the bilinear form is well-defined. Furthermore,

$$\begin{aligned}
&\left(a_{i,j}\cdot\frac{\partial}{\partial x_j}\otimes^c Id(u), \frac{\partial}{\partial x_j}\otimes^c Id(v)\right)_{L^2(D)\otimes^c_{H.S.}(S)_p} \\
&\leq \|a_{i,j}\cdot\frac{\partial}{\partial x_j}\otimes^c Id(u)\|_{L^2(D)\otimes^c_{H.S.}(S)_p}\|\frac{\partial}{\partial x_j}\otimes^c Id(v)\|_{L^2(D)\otimes^c_{H.S.}(S)_p} \\
&\leq C(A,p)\|\frac{\partial}{\partial x_j}\otimes^c Id(u)\|_{L^2(D)\otimes^c_{H.S.}(S)_p}\|\frac{\partial}{\partial x_j}\otimes^c Id(v)\|_{L^2(D)\otimes^c_{H.S.}(S)_p}
\end{aligned}$$

where the last inequality again follows from Theorem 6.18. It follows that the bilinear form is bounded. To show the coercivity of the operator, we make use of the following lemmas.

Lemma 6.21 *Suppose $F, G \in L^2(D) \otimes^c_{H.S.} (L^2)$. Then $(F,G)_{L^2(D)\otimes^c_{H.S.}(L^2)}$ is given by*

$$(F,G)_{L^2(D)\otimes^c_{H.S.}(L^2)} = \int_\Omega \int_D F(x,\omega)G(x,\omega)dxd\mu$$

Proof: Suppose that $F = \sum_{k=1}^n w_k \otimes v_k, G = \sum_{l=1}^n f_l \otimes g_l$. Then

$$\begin{aligned}
&(F,G)_{L^2(D)\otimes^c_{H.S.}(L^2)} = \sum_{l,k} (w_k, f_l)(v_k, g_l) \\
&= \sum_{l,k} \int_\Omega \int_{\mathbb{R}^d} w_k(x) f_l(x) v_k(\omega) g_l(\omega) dxd\mu \\
&= \int_\Omega \int_d F(x,\omega)G(x,\omega)dxd\mu.
\end{aligned}$$

The result follows from the density of elements of this form in $L^2(D) \otimes^c_{H.S.} (L^2)$.

Lemma 6.22 *Let $u \in L^2(D) \otimes^c_{H.S.} (S)_p$. Then u has a unique representation given by $u = \sum_{\alpha\in I^2} \lambda_\alpha e_{\alpha_1} \otimes f_{\alpha_2}$ where the e_{α_1}, f_{α_2} are elements of an orthonormal basis of $L^2(D)$ and $(S)_p$, respectively. Define $u^{(p)} \in L^2(D) \otimes^c_{H.S.} (L^2)$ by $u^{(p)} := \sum_{\alpha\in I^2} \lambda_\alpha e_{\alpha_1} \otimes f^{(p)}_{\alpha_2}$, where $f^{(p)}_{\alpha_2}$ is as previously defined. Then $\|u\|_{L^2(D)\otimes^c_{H.S.}(S)_p} = \|u^{(p)}\|_{L^2(D)\otimes^c_{H.S.}(L^2)}$*

Proof: Follows easily from the definitions.

Applying the lemmas to $(a_{ij}\frac{\partial}{\partial x_i} \otimes^c Id(u), \frac{\partial}{\partial x_j} \otimes^c Id(u))_{L^2(D)\otimes^c_{H.S.}(S)_p}$ yields

$$\begin{aligned}
&(a_{ij}\frac{\partial}{\partial x_i} \otimes^c Id(u), \frac{\partial}{\partial x_j} \otimes^c Id(u))_{L^2(D)\otimes^c_{H.S.}(S)_p} = \\
&= \int_\Omega \int_D a^{(p)}_{ij} \frac{\partial}{\partial x_i} \otimes^c Id(u)^{(p)} \frac{\partial}{\partial x_j} \otimes^c Id(u)^{(p)} dxd\mu \\
&\geq \int_\Omega \int_D \tilde{a}^{(p)}_{ij} \lambda^{(p)}(\omega) \frac{\partial}{\partial x_i} \otimes^c Id(u)^{(p)} \frac{\partial}{\partial x_j} \otimes^c Id(u)^{(p)} dxd\mu \\
&\geq \int_\Omega \int_D c\lambda^{(p)}(\omega) |\nabla \otimes^c Id(u)^{(p)}|^2 (x,\omega) dxd\mu \\
&\geq c \inf_{\omega\in\Omega\lambda(\omega)} \int_\Omega \int_D |\nabla \otimes^c Id(u)^{(p)}|^2 (x,\omega) dxd\mu \\
&\geq c \inf_{\omega\in\Omega\lambda(\omega)} (\nabla \otimes^c Id(u), \nabla \otimes^c Id(u))_{W^{1,2}_0(D)\otimes^c_{H.S.}(S)_p}.
\end{aligned}$$

Thus we have shown that the bilinear form is coercive. The result now follows from an application of the Lax- Milgram Theorem.

References:

[Ad] Adams, R.A.: Sobolev Spaces. Academic Press, London 1970

[G-V] Gelfand, I.M.; Vilekin, N.Y.: Generalized Funcitions, Vol. 4. Academic Press, London 1964

[G-T] Gilberg, D.; Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order, Second Edition. Grundlehren der mathematischen Wissenschaften 224, Springer 1983

[Hi] Hida, T.: Brownian Motion. Applications of Mathematics 11, Springer 1980

[Hi et al.] Hida, T.; Kuo, H.-H.; Potthoff, J.; Streit, L.: White Noise. Kluwe Academic Publishers, Dordrecht/Boston/London 1993

[Ho et al.] Holden, H.; Øksendal, B.; Ubøe, J; Zhang, T.: Stochastic Partial Differential Equations. Birkhauser, Boston/Basel/Berlin 1996

[Hö] Hörmander, L.: The Analysis of Linear Partial Differential Operators I. Grundlehren der mathematischen Wissenschaften 256, Springer 1983

[Is] Istratescu, V.I.: Inner Product Structures. D. Reidel Publishing Company, Dordrecht/Boston/Lancaster/Tokyo 1987

[Kö] Köthe, G.: Topologische Lineare Räume. Grundlehren der mathematischen Wissenschaften 102, Springer 1966

[Ob] Obata, N.: White Noise Calculus and Fock Space. Springer 1994

[Ø] Øksendal, B.: Stochastic Differential Equations, Fifth Edition. Springer 1998

[Sch] Schaefer, H.H.: Toplogical Vector Spaces. Graduate Texts in Mathematics 3, Springer 1971

[Tr] Treves, I.: Topological Vector Spaces, Distributions and Kernels. Acadamic Press, London 1967

Curriculum Vitae

11.1.1966	Born in Paducah, Kentucky, U.S.A.
1971-1979	Attended Featherbed Lane Elementary School in Colonia, New Jersey, U.S.A., St. John Vianney Elementary School in Colonia, New Jersey, U.S.A., St. Thomas More Elementary School in Paducah, Kentucky, U.S.A., Lone Oak Middle School in Paducah, Kentucky, U.S.A.
1979-1983	Attended Lone Oak High School, Pauduch, Kentucky, U.S.A.
1983-1987	Began studies for the Bachelors of Arts Degree at Northwestern University, Evanston, Illinois, U.S.A.
December 1987	Awarded Bachelor of Arts degree in Mathematics
1987-1990	Served on board the U.S. Navy ship U.S.S. Moosbrugger DD980
1991-1992	Employed as statistician by K and D Facilities Resource Corp., Chicago, Illinois
1993-1994	Conducted graduate studies in mathematics at Indiana University, Bloomington, Indiana
December 1994	Awarded Master of Science degree in Mathematics
1995-1999	Attended the Georg-August-Universität Goettingen
Oct. 1999	Completed Diplom in Mathematics Advisor: Prof. H. Hering Field of Study: Stochastic Analysis
since Oct.1999	Doctoral studies with Prof. H. Hering at the Institute for Mathematical Stochastics