Heat Content Inequalities for Diffusions on Manifolds

Dissertation

zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultäten
der Georg-August-Universität zu Göttingen

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1 Introduction

1.1 About this Thesis

The main purpose of this thesis is to prove a conjecture of Bhattacharya and Götze which arose from their research on central limit theorems for diffusions with periodic drift vector fields (see [BhaGöt1] and [BhaGöt2]). They stated a result concerning the long-time behaviour of a class of diffusions with periodic, divergence-free drift vector fields. Their conjecture will be proved here in a more general context.
Let b be a vector field on a compact, connected, orientable Riemannian manifold $\mathbf{M}$ without boundary. The vector field b can be considered as a differential operator. Let $\triangle$ be the Laplace-Beltrami operator on $\mathbf{M}$. The generalised heat equation with initial condition f is given by

$$\begin{aligned}&\triangle u(x,t) + b(x)u(x,t) = \partial_t u(x,t),\\ &u(x,0) = f(x).\end{aligned}$$

It is well known that there exist fundamental solutions $q_{\mathbf{M},b} : \mathbf{M}\times\mathbf{M}\times I\!R^+ \longrightarrow I\!R^+$ for this problem. Every solution of the above equation can be written as

$$u(x,t) = K_{\mathbf{M},b,t}f(x) = \int_{\mathbf{M}} q_{\mathbf{M},b}(t,x,y)f(y)d\mu_{\mathbf{M}}(y),$$

where $\mu_{\mathbf{M}}$ is the volume measure on $\mathbf{M}$. Clearly, the fundamental solutions depend on the vector field b. The solution $u(x,t)$ describes the behaviour of a diffusion in a moving medium, whose local velocity is given by the vector field b. The fundamental solutions play the role of the transition probabilities for this diffusion. More details about the probabilistic interpretation of the fundamental solutions as transition probabilities for solutions of stochastic differential equations on manifolds can be found in the book of Ikeda and Watanabe [IkeWat]. For applications, one is interested in estimates for the fundamental solutions where not all information regarding the vector field b is available, for example, if only local properties of b are known. An important example is encountered in the problem of diffusions in the turbulent flow of an incompressible medium. In this case the only information available in general is the divergence-free property of b. Therefore, it is interesting to obtain estimates for the fundamental solutions which are uniform over the class of divergence-free vector fields. The main idea is to compare the equation on $\mathbf{M}$ with a simpler equation on a comparison manifold $\mathbf{M}^*$. The comparison manifold is a sphere $\mathbf{M}^* = \{x \in I\!R^{d+1}; \|x\| = a\}$ provided with a Riemannian metric g^*. This Riemannian metric has some special features. It is rotation invariant under rotations fixing the north pole $\mathbf{n}$ and the

south pole $\mathbf{s}$, it is invariant under reflection at the equator of $\mathbf{M}^*$ and it has the same Riemannian volume as $\mathbf{M}$ has. The most important property is that for all open subsets A of $\mathbf{M}$ with smooth boundary there exists a polar cap $\hat{A}$ on $\mathbf{M}^*$ having the same Riemannian volume as A and having smaller boundary. This condition is closely related to the isoperimetric function of $\mathbf{M}$. The isoperimetric problem for $\mathbf{M}$ can be stated as follows: for all $c > 0$ find the open sub-manifold of $\mathbf{M}$ having volume c and minimal boundary. In the euclidian space the solution is given by a sphere of volume c. For Riemannian manifolds, this problem can not be solved in general. Nevertheless, the isoperimetric function assigning to each $c > 0$ the minimal size of all boundaries enclosing the volume c holds a considerable amount of information about the heat flow on $\mathbf{M}$. The condition comparing the boundaries of polar caps in $\mathbf{M}^*$ with boundaries of open sets in $\mathbf{M}$ guarantees that the amount of heat escaping from an arbitrary set of a given size on $\mathbf{M}$ will be bigger than the amount of heat escaping from the polar cap with same volume on $\mathbf{M}^*$ (see Theorem 4). This property will be the main ingredient for the comparison results in this thesis. The existence of comparison manifolds is proved here under some small assumptions (see Theorem 1).
The following inequalities for the fundamental solutions will be proved in this thesis (see Theorem 7):

$$\begin{aligned}&\sup_{\mathrm{div}(b)=0}\ \sup_{\mu_{\mathbf{M}}(B)=o}\ \sup_{\mu_{\mathbf{M}}(A)=c} \int_B \int_A q_{\mathbf{M},b}(t,x,y)\,d\mu_{\mathbf{M}}(y)\,d\mu_{\mathbf{M}}(x)\\ &\leq \int_{\hat{B}_o} \int_{\hat{A}_c} q_{\mathbf{M}^*,0}(t,x,y)\,d\mu_{\mathbf{M}^*}(y)\,d\mu_{\mathbf{M}^*}(x)\end{aligned}$$

and

$$\begin{aligned}&\inf_{\mathrm{div}(b)=0}\ \inf_{\mu_{\mathbf{M}}(B)=o}\ \inf_{\mu_{\mathbf{M}}(A)=c} \int_B \int_A q_{\mathbf{M},b}(t,x,y)\,d\mu_{\mathbf{M}}(y)\,d\mu_{\mathbf{M}}(x)\\ &\geq \int_{\check{B}_o} \int_{\hat{A}_c} q_{\mathbf{M}^*,0}(t,x,y)\,d\mu_{\mathbf{M}^*}(y)\,d\mu_{\mathbf{M}^*}(x).\end{aligned}$$

Here $\hat{A}_c$ resp. $\hat{B}_o$ are balls around the north pole $\mathbf{n}$ with volume c resp. o and $\check{B}_o$ is a ball around the south pole $\mathbf{s}$ with volume o. From these inequalities we can directly derive the following inequalities for the suprema and infima of the fundamental solutions (see Theorem 9):

$$\sup_{\mathrm{div}(b)=0}\ \sup_{x,y\in\mathbf{M}} q_{\mathbf{M},b}(t,x,y) \leq q_{\mathbf{M}^*,0}(t,\mathbf{n},\mathbf{n})$$

and

$$\inf_{\mathrm{div}(b)=0}\ \inf_{x,y\in\mathbf{M}} q_{\mathbf{M},b}(t,x,y) \geq q_{\mathbf{M}^*,0}(t,\mathbf{s},\mathbf{n}).$$

It is worth to mention that the right hand side of the above inequalities is given by a very symmetric object in contrast to the situation on the left hand side. The equations on the right side can be reduced to a one dimensional diffusion problem. The only information we need from the left hand side is a local property of the vector field and the geometric data given by the isoperimetric function of $\mathbf{M}$. To each initial condition f on $\mathbf{M}$, one can construct a polar rearrangement $\hat{f}$ on $\mathbf{M}^*$ having the same value distribution. Furthermore, $\hat{f}$ is decreasing with the distance to $\mathbf{n}$ and invariant under rotations fixing the north pole $\mathbf{n}$ and the south pole $\mathbf{s}$ of $\mathbf{M}^*$. With this construction one can give estimates for the solutions of the generalised heat equation on $\mathbf{M}$ with initial condition f. We get estimates of the following type (see 6):

$$\sup_{\operatorname{div}(b)=0} \sup_{\mu_{\mathbf{M}}(A)=c} \int_A K_{\mathbf{M},b,t} f(x) d\mu_{\mathbf{M}}(x) \leq \int_{\hat{A}_c} K_{\mathbf{M}^*,0,t} \hat{f}(x) d\mu_{\mathbf{M}^*}(x)$$

and

$$\inf_{\operatorname{div}(b)=0} \inf_{\mu_{\mathbf{M}}(A)=c} \int_A K_{\mathbf{M},b,t} f(x) d\mu_{\mathbf{M}}(x) \geq \int_{\hat{A}_c} K_{\mathbf{M}^*,0,t} \hat{f}(x) d\mu_{\mathbf{M}^*}(x).$$

These estimates could be called non local heat content estimates, because they estimate the heat content regardless of its location. As a corollary, one obtains the following uniform estimates of the exponential decay of diffusions on compact manifolds (see Corollary 6):
There exist $\lambda > 0$ and $C > 0$ such that for all $t \geq 0$ and all f with $\int_{\mathbf{M}} f d\mu_{\mathbf{M}} = 0$

$$\sup_{\operatorname{div}(b)=0} \|K_{\mathbf{M},b,t} f\|_{\sup} \leq C \|f\|_{\sup} \exp(-t\lambda).$$

Uniform estimates of this type are useful in probability theory. As a result of the uniform estimates obtained in this thesis, one can prove a conjecture of Götze and Bhattacharya. In their work on central limit theorems for diffusions with periodic vector fields, they had to handle the following class of divergence-free vector fields on $\mathbb{R}^d$

$$b_n(x) := b(x) + \beta(xn^{-1}),$$

where b and β are fixed divergence-free vector fields of period 1 on $\mathbb{R}^d$ and n ranges over the natural numbers. Their research concerned the generalised heat equation with periodic initial condition f

$$\begin{aligned} &\triangle u(x,t) + b_n(x) u(x,t) = \partial_t u(x,t), \\ &u(x,0) = f(x). \end{aligned}$$

Since the solution to the equation is spatially periodic, the problem is equivalent to a generalised heat equation on the flat torus $\mathbf{T}^d$. They made the following statement (see [BhaGöt1]):
There exist constants $\lambda > 0$ and $C > 0$ such that for all $n \in I\!N$ and all $f \in \mathcal{B}_b(\mathbf{T}^d)$ with $\int_{\mathbf{T}^d} f d\mu_{\mathbf{T}^d} = 0$

$$\|K_{\mathbf{T}^d,b_n,t} f\|_{\sup} \leq C\|f\|_{\sup} \exp(-\lambda t).$$

Their proof contained an error, which was pointed out in [BhaGöt2]. Their statement is a special case of Corollary 6

This thesis is divided in two parts, each which is totally self contained. The first part deals with the general situation of compact Riemannian manifolds and ends with Section 5. The Second part begins in section 6 and deals with the special case of d-dimensional spheres $\mathbf{S}_a^d$. From the point of view of diffusions, there are no new results in these sections, but the results on the sphere can be derived from purely geometric constructions. For this reason, it is included in this thesis. The geometric constructions mentioned above are rearrangements of sets. The following concentration estimate is the key to understanding the heat flow on the sphere (see Theorem 10):

$$\begin{aligned}&\sup_{\mu_a^d(B)=c}\ \sup_{\mu_a^d(A)=b} \mu_a^d \times \mu_a^d(A \times B \cap \{(x,y); d(x,y) \leq r\}) \\ &= \mu_a^d \times \mu_a^d(\hat{A} \times \hat{B} \cap \{(x,y); d(x,y) \leq r\}),\end{aligned}$$

where $\hat{A}$ and $\hat{B}$ are balls around the north pole $\mathbf{n}$ of $\mathbf{S}_a^d$ with $\mu_a^d(\hat{B}) = c$ and $\mu_a^d(\hat{A}) = b$. The proof of this result is based on set rearrangement techniques. With this estimate, we can derive estimates for the fundamental solutions of the heat kernel on the sphere. Since the sphere is its own comparison manifold, we can compare the generalised heat equation to the classical heat equation without changing the underlying manifold. We get as an analogue to Theorem 7 the Theorem 16. This theorem compares the heat dissipation in an incompressible moving medium to that in a medium without motion. The moving medium has faster heat dissipation. This statement is not true in general for Riemannian manifolds. There could exist a Riemannian manifold and a motion on it, which slows down the heat dissipation. It would be interesting to construct such an example.

The idea of linking analytical estimates with isoperimetric properties is not new. There exist numerous results concerning the relationship between isoperimetric

properties of domains and the first eigenvalue of differential operators defined on those domains. In 1877 Lord Raleigh conjectured that among all membranes with same area the circular one has minimal bass mode. His conjecture was proved by E. Krahn (1924) and G. Faber (1923) independently (see [Fab], [Kra]). Their proof used the isoperimetric property of balls in euclidian spaces and rearrangement techniques. In the literature, rearrangements are sometimes called symmetrisation or transplantation. See the excellent book of C. Bandle [Ban] for more detailed information on the relatationship between rearrangements and geometric inequalities.
Berard and Gallot generalised this approach to Riemannian manifolds with finite volume in the eighties (see [Gal] and [Ber]). In doing so they constructed comparison manifolds. However, they did not focus on general diffusions, rather restricted their analysis to the heat equation. They were able to estimate the first eigenvalue of the Laplace-Beltrami operator, by comparing it to the spectrum of the Laplace-Beltrami operator on the comparison manifold. Their comparison manifolds are cylinders $\mathbf{S}^1 \times [0,1]$. In his book, Berard constructs a comparison manifold from the data given by the Cheegers isoperimetric constant (see [Ber]). From this he obtains estimates for the spectral gap of the Laplace-Beltrami operator. Regarding the fundamental solutions of the heat equation, he gives estimations of $\sup_{x,y\in\mathbf{M}} q_{\mathbf{M}}(t,x,y)$ by means of the fundamental solutions on the comparison manifold $\mathbf{M}^*$.

It may be possible to generalise the results presented here.
The requirements on the vector fields to be divergence-free is very important if one wishes to compare the generalised heat equation to the heat equation on the comparison manifold. But it might be dropped in a comparison to a generalised heat equation with suitable rotationally invariant vector field on $\mathbf{M}^*$, where the vectors point in direction of the south pole. The divergence of this constructed vector field must have the same value distribution as the divergence of the original vector field on $\mathbf{M}$ and decrease with increasing distance to the north pole of the comparison manifold $\mathbf{M}^*$.
A zero order term could be added to the equation on $\mathbf{M}$ without changing the philosophy. In this case it is necessary to make the comparison with a generalised diffusion on $\mathbf{M}^*$ with rotationally invariant, southwardly decreasing potential. Again we require that this potential has the same value distribution as the original potential on $\mathbf{M}$.
The results of this thesis could be generalised to evolution equations with time dependent vector fields and perhaps to nonlinear equations, where the vector field depends on the state of the system.

At this point I would like to take the opportunity to express my thanks to those people who, through their constant support, both mathematical and moral, contributed to the writing of this thesis. In particular, I would like to thank Professor Denker for many fruitful discussions and for the patience he showed throughout my time as his doctoral student. Special thanks also go to Professor Hering for his Seminar which provided a forum for the presentation and refinement of the ideas and results to be found here. My gratitude also goes to Professor Bhattacharya, for his collaboration during the time spent at Indiana University which gave me numerous mathematical impulses. Thanks also go to the DAAD for providing the financial means making my year at Indiana University possible. I also want to thank Paul Besold, Olaf Wittich, Wolfgang Schwarzwäller, Russell Lyons and Christopher Judge for their comments during and after talks I gave at Göttingen University and at Indiana University. Innumerable discussions with my friends Detlef Kossakowski and Carsten Kutzner gave me the opportunity to optimise the presentation of the material. I owe a unpayble debt of gratitude to my fiance, Huai-Yao. She was a constant source of support and inspiration throughout my doctoral studies.

2 Notation and Definitions

In this section, we define the comparison manifold $\mathbf{M}^*$ to a compact, connected, orientable d-dimensional Riemannian manifold $\mathbf{M}$ and we define the polar rearrangement $\hat{f}$ on $\mathbf{M}^*$ of a given real-valued function f on $\mathbf{M}$.
A comparison manifold is a d-dimensional sphere provided with a new Riemannian metric g^*. The metric is invariant with respect to rotations of the sphere fixing the north and the south pole. Further, it is invariant with respect to reflection at the equator. In addition, the volume of $\mathbf{M}^*$ is equal to the volume of $\mathbf{M}$. The most important feature of $\mathbf{M}^*$ is that a ball around the north pole of $\mathbf{M}^*$ with volume c has a smaller boundary than any open sub-manifold of volume c in $\mathbf{M}$. This property assures that the heat flux for any open sub-manifold of volume c in $\mathbf{M}$ is faster then that of the polar cap with volume c on $\mathbf{M}^*$. In order to study the heat propagation of a given heat distribution f on $\mathbf{M}$, we have to define the polar rearrangement. This is a function $\hat{f}$ on $\mathbf{M}^*$ with the same value distribution as f which is invariant with respect to rotations fixing the north and the south pole of $\mathbf{M}^*$ and decreasing with the distance to the north pole. This heat distribution on $\mathbf{M}^*$ has the property that its heat dissipation is slower than that of the heat distribution f. The following sections concern this comparison of the heat dissipation.
In the first subsection, we enumerate some important facts and indicate the notations which will be used constantly in what follows. In the second subsection, the comparison manifold is defined and its existence will be proved under a small assumption on $\mathbf{M}$. In the third subsection, the polar rearrangement is defined and some of its useful properties are proved.

2.1 Notation

Let $\mathbf{M}$ be a compact, connected, orientable d-dimensional Riemannian manifold without boundary. Denote by $\mathcal{C}(\mathbf{M})$ the space of continuous functions from $\mathbf{M}$ to $I\!R$. $\mathcal{C}(\mathbf{M})$ is a Banach space under supremum norm $||.||_{\text{sup}}$. Denote by $\mathcal{C}^n(\mathbf{M})$ the space of n-times continuously differentiable functions from $\mathbf{M}$ into $I\!R$. Furthermore, denote by $\mathcal{C}^\infty(\mathbf{M})$ the space of infinitely differentiable functions from $\mathbf{M}$ to $I\!R$. $\mathcal{C}^\infty(\mathbf{M})$ is dense in $\mathcal{C}(\mathbf{M})$ with respect to the supremum norm. Finally, denote by $\mathcal{B}_b(\mathbf{M})$ the space of all bounded Borel measurable functions from $\mathbf{M}$ to $I\!R$. The tangent bundle is denoted by $T\mathbf{M}$. The space of smooth sections in $T\mathbf{M}$ is denoted by $\Gamma T\mathbf{M}$. Let $g : T\mathbf{M} \times T\mathbf{M} \longrightarrow I\!R$ denote the Riemannian metric on $\mathbf{M}$ and let $d_{\mathbf{M}} : \mathbf{M} \times \mathbf{M} \longrightarrow [0, \infty[$ denote the resulting distance on $\mathbf{M}$. On $\mathbf{M}$ there exists a unique Levi-Civita connection $\bigtriangledown : \Gamma T\mathbf{M} \times \Gamma T\mathbf{M} \longrightarrow \Gamma T\mathbf{M}$ consistent with the Riemannian metric g (see [GalHulLaf] p.70). Let $\omega_{\mathbf{M}}$ be the volume form corresponding to the metric and $\mu_{\mathbf{M}}$ be the resulting volume measure on $\mathbf{M}$. In local coordinates, $\mu_{\mathbf{M}}$ has a density with respect to the Lebesques measure given by $\sqrt{\det(g_{ij})}$ and $\omega_{\mathbf{M}}$ is given by $\sqrt{\det(g_{ij})}dx_1 \wedge ... \wedge dx_d$ (see [GalHulLaf] p.140). Denote by $\mathcal{L}^2(\mathbf{M})$ the space of square integrable functions from $\mathbf{M}$ to $I\!R$ with respect to $\mu_{\mathbf{M}}$. The space $\mathcal{L}^2(\mathbf{M})$ is a real Hilbert space with scalar product given by $\langle f, g\rangle := \int_{\mathbf{M}} f(x)g(x)d\mu_{\mathbf{M}}(x)$. The gradient of a function $f \in \mathcal{C}^\infty(\mathbf{M})$ is defined to be the unique vector field $\mathrm{grad}(f)$ on $\mathbf{M}$ such that for all vector fields Y on $\mathbf{M}$ holds: $g(\mathrm{grad}(f), Y) = Yf$. The gradient at a point $x \in \mathbf{M}$ points in the direction of the greatest increase of f at x. The divergence of a vector field X on $\mathbf{M}$ is defined by $\mathrm{div}(X) := \mathrm{tr}(\bigtriangledown_{.}X)$, where $Y \mapsto \bigtriangledown_Y X(x)$ is interpreted as an endomorphism of $T_x\mathbf{M}$. The Laplace-Beltrami operator is defined on $\mathcal{C}^\infty(\mathbf{M})$ by $\triangle f := \mathrm{div}(\mathrm{grad}(f))$. With this definition, $\triangle$ is a negative operator (see [GalHulLaf] p.183). For an open sub-manifold D with smooth boundary one can consider ∂D as an orientable Riemannian sub-manifold of $\mathbf{M}$ (see: [GalHulLaf] p.181). For a point $x \in \partial D$ let $n_x \in T_x\mathbf{M}$ be the tangential vector of length one which is normal to $T_x\partial D$ pointing into the exterior of D. For vector fields $X_2, ..., X_d$ on ∂D and $x \in \partial D$ define $\omega_{\partial D}(X_2, ..., X_d)(x) := \omega_{\mathbf{M}}(n_x, X_2, .., X_d)$. The $d-1$-form $\omega_{\partial D}$ is simply the volume form corresponding to the restriction of the metric g to ∂D (see [GalHulLaf] p.183). Stokes theorem holds for all $d-1$-forms ω on D (see [Spi]), i.e.,

$$\int_D d\omega = \int_{\partial D} \omega.$$

From this one obtains the following version of Stokes theorem: (see [GalHulLaf] p.184)

$$\int_D \triangle f \omega_{\mathbf{M}} = \int_{\partial D} g(\mathrm{grad}(f), n)\omega_{\partial D}. \tag{1}$$

The heat equation on $\mathbf{M}$ with initial condition $f \in \mathcal{C}(\mathbf{M})$ is given by

$$\begin{aligned} \triangle u(x,t) &= \partial_t u(x,t), \\ u(x,0) &= f(x). \end{aligned}$$

There exist fundamental solutions

$$p_{\mathbf{M}} :]0, \infty[\times \mathbf{M} \times \mathbf{M} \longrightarrow]0, \infty[$$

with the following properties:

$$\begin{aligned} &\triangle_x p_{\mathbf{M}}(t,x,y) = \partial_t p_{\mathbf{M}}(t,x,y), \\ &\lim_{t\to 0} p_{\mathbf{M}}(t,x,y) = \delta_x(y), \end{aligned}$$

where the limit is taken in the space of distributions on $\mathbf{M}$. (see [GetBerVer] p.75).
Furthermore, since $\triangle$ is a symmetric operator in $\mathcal{L}^2(\mathbf{M})$ one gets the following representation of the fundamental solutions in $\mathcal{L}^2(\mathbf{M})$:

$$p(t,x,y) = \sum_{i=1}^{\infty} \exp(t\lambda_i)\Xi_i(x)\Xi_i(y),$$

where Ξ_i is the eigenfunction corresponding to the eigenvalue λ_i of $\triangle$ (see [GetBerVer] p.91). From this formula, one sees that the fundamental solutions of the Laplace-Beltrami operator are symmetric in x and y. Since the fundamental solutions are smooth (see [GetBerVer] p.86), they can be considered as Feller transition probabilities (see [Tai] p.329), and as such they generate a strongly continuous semigroup $T_{\mathbf{M}}$ on $\mathcal{C}(\mathbf{M})$ by the formula

$$T_t f(x) = \int_{\mathbf{M}} p_{\mathbf{M}}(t,x,y) f(y) d\mu_{\mathbf{M}}(y)$$

(see [Tai] p.333).

Let b be a smooth vector field on $\mathbf{M}$. In local coordinates b, can be described as $b(x) = b_i(x)\frac{\partial}{\partial x_i}$. Then $A_b f := \triangle f + bf$ is a generalised elliptic differential operator of second order on $\mathcal{C}^\infty(\mathbf{M})$. The generalised heat equation on $\mathbf{M}$ with initial condition $f \in \mathcal{C}(\mathbf{M})$ is given by

$$\begin{aligned} &\triangle u(x,t) + b(x)u(x,t) = \partial_t u(x,t), \\ &u(x,0) = f(x). \end{aligned}$$

There exist fundamental solutions

$$q_{\mathbf{M},b} :]0,\infty[\times \mathbf{M} \times \mathbf{M} \longrightarrow]0,\infty[$$

with the property

$$\begin{aligned} &A_{b,x} q_{\mathbf{M},b}(t,x,y) = \partial_t q_{\mathbf{M},b}(t,x,y), \\ &\lim_{t\to 0} q_{\mathbf{M},b}(t,x,y) = \delta_x(y), \end{aligned}$$

where the limit is taken in the space of distributions on $\mathbf{M}$ (see [GetBerVer] p.75). Since the fundamental solutions of the generalised heat equation are smooth (see [GetBerVer] p.86), they can be considered as Feller transition probabilities (see

[Tai] p.329). Therefore, they generate a strongly continuous semigroup $K_{\mathbf{M},b}$ on $\mathcal{C}(\mathbf{M})$ by the formula

$$K_{\mathbf{M},b,t}f(x) = \int_{\mathbf{M}} q_{\mathbf{M},b}(t,x,y)f(y)d\mu_{\mathbf{M}}(y).$$

(see [Tai] p.333). Notice that $K_{\mathbf{M},0} = T_{\mathbf{M}}$.
Let $\mathcal{V}_n := \mathrm{Span}(\Xi_i, i \leq n)$. The $\mathcal{V}_n$ are finite dimensional linear subspaces of $\mathcal{C}^\infty(M)$. The space $\mathcal{V}(\mathbf{M}) := \bigcup_{n\in I\!N} \mathcal{V}_n$ is a dense subspace of $\mathcal{L}^2(M)$. For $f \in \mathcal{V}(\mathbf{M})$ one has the following representation $f(x) = \sum_{i=1}^n a_i \Xi_i(x)$ with suitable coefficients a_i in $I\!R$. Now the solution of the heat equation with initial condition f is given by

$$T_{\mathbf{M},t}f(x) = \sum_{i=1}^n e^{-\lambda_i t} a_i \Xi_i(x).$$

This function is analytic in t. Furthermore, it is entire. Assume that there exists an open set $U \in \mathbf{M}$ and a $t \geq 0$ such that $T_{\mathbf{M},t}f(x)$ is constant on U. Then would hold $\triangle^k T_{\mathbf{M},t}f(x) = 0$ for all $x \in U$ and all $k \in I\!N$. From the heat equation, one would get $\partial_t^k T_{\mathbf{M},t}f(x) = 0$ for all $x \in U$ and all $k \in I\!N$. Since $T_{\mathbf{M}}f(x)$ is entire, this would imply that $T_{\mathbf{M}}f$ is constant on $U\times]-\infty,\infty[$. On a connected manifold this is only possible if f is constant everywhere. Therefore, we obtain the following Lemma about level sets.

Lemma 1 *For all non constant $f \in \mathcal{V}(\mathbf{M})$ and for all $t \geq 0$ one has*

$$\mu_{\mathbf{M}}(\{x \in \mathbf{M}; T_{\mathbf{M},t}f(x) = c\}) = 0.$$

2.2 The Comparison Manifold

For an $a > 0$ lets consider $\mathbf{N} := \{x \in I\!R^{d+1}, \|x\| = a\}$ provided with a Riemannian metric g^*. Denote by $\mu_{\mathbf{N}}$ the volume measure corresponding to g^*. The function

$$F_{\mathbf{N}} : [-a,a] \longrightarrow I\!R_0^+; r \mapsto \mu_{\mathbf{N}}(\mathbf{N} \cap \{x_{d+1} \geq r\})$$

is called the polar cap area function of $\mathbf{N}$. The polar cap area function is strictly decreasing on $[\inf_{x\in\mathbf{N}}(x_{d+1}), \sup_{x\in\mathbf{N}}(x_{d+1})]$. For $c \in [0, \mu_{\mathbf{N}}(\mathbf{N})]$ the set

$$\hat{A}_c := \mathbf{N} \cap \{x_{d+1} \geq F_{\mathbf{N}}^{-1}(c)\}$$

is called the polar cap of volume c. The function

$$L_{\mathbf{N}} : [0, \mu_{\mathbf{N}}(\mathbf{N})] \longrightarrow I\!R_0^+; c \mapsto \mu_{\partial\hat{A}_c}(\partial\hat{A}_c)$$

is called the latitudal function of $\mathbf{N}$.

Definition 1 *For a compact, connected, orientable Riemannian manifold* $\mathbf{M}$ *without boundary let* $S(\mathbf{M})$ *be the set of open subsets of* $\mathbf{M}$ *with smooth boundary. The function*

$$S_{\mathbf{M}} : [0, \mu_{\mathbf{M}}(\mathbf{M})] \longrightarrow [0, \infty[; c \mapsto \inf_{A \in S(\mathbf{M}), \mu_{\mathbf{M}}(A) = c} \mu_{\partial A}(\partial A)$$

is called the isoperimetric function of $\mathbf{M}$.

Clearly $S_{\mathbf{M}}$ is continuous and satisfies for all $x \in [0, \frac{1}{2}\mu_{\mathbf{M}}(\mathbf{M})]$

$$S_{\mathbf{M}}(x) = S_{\mathbf{M}}(\mu_{\mathbf{M}}(\mathbf{M}) - x).$$

Definition 2 *Let* $\mathbf{M}$ *be a compact, connected, orientable d-dimensional Riemannian manifold without boundary.*
$\mathbf{M}^* = \{x \in I\!R^{d+1}, \|x\| = a\}$ *with a Riemannian metric* g^* *will be called a comparison manifold of* $\mathbf{M}$ *if it satisfies the following properties:*
i) $\mu_{\mathbf{M}^*}(\mathbf{M}^*) = \mu_{\mathbf{M}}(\mathbf{M})$,
ii) the elements of $SO(d+1)$ *fixing the* x_{d+1}*-axis act as isometries on* $\mathbf{M}^*$,
iii) the reflection at the hyperplane $\{x \in I\!R^{d+1}; x_{d+1} = 0\}$ *is an isometry of* $\mathbf{M}^*$,
iv) $L_{\mathbf{M}^*}(c) \leq S_{\mathbf{M}}(c)$ *for all* $c \in [0, \mu_{\mathbf{M}}(\mathbf{M})]$.

With this definition comparison manifolds are not unique. We are not concerned about getting optimal estimates. The theorem below is an existence theorem for comparison manifolds. Here we are not interested in the most general case.

Theorem 1 *Let* $\mathbf{M}$ *be a compact, connected, orientable d-dimensional Riemannian manifold without boundary which satisfies* $S_{\mathbf{M}}(x) > 0$ *for all* $x \in]0, \mu_{\mathbf{M}}(\mathbf{M})[$. *If there exists an* $\epsilon > 0$ *such that* $S_{\mathbf{M}}$ *is smooth on* $[0, \epsilon[$, *then there exists a comparison manifold* $\mathbf{M}^*$ *for* $\mathbf{M}$.

Proof: Consider the ioperimetric function $S_{\mathbf{M}}$ of $\mathbf{M}$. $S_{\mathbf{M}}$ might not be smooth, but there exists an $\epsilon > 0$ and a smooth function $L : [0, \mu_{\mathbf{M}}] \longrightarrow]0, \infty[$ with the properties:
1) $L \leq S_{\mathbf{M}}$,
2) $L(x) = S_{\mathbf{M}}(x)$ for all $x \in [0, \epsilon[$,
3) $L(x) = L(\mu_{\mathbf{M}}(\mathbf{M}) - x)$ for all $x \in [0, \frac{1}{2}\mu_{\mathbf{M}}(\mathbf{M})]$.
Now take a d-dimensional sphere $\mathbf{S}^d_a$ of volume $\mu_{\mathbf{M}}(\mathbf{M})$ in $I\!R^{d+1}$.
Denote by $g : T\mathbf{S}^d_a \times T\mathbf{S}^d_a \longrightarrow I\!R$ the Riemannian metric on $\mathbf{S}^d_a$ arising from the embedding of $\mathbf{S}^d_a$ in $\mathbf{R}^{d+1}$. Lets denote by μ^d_a the resulting volume measure. Consider the polar cap $\hat{A}_c$ of volume c on $\mathbf{S}^d_a$.

One has $\partial \hat{A}_c = \{x \in I\!R^{d+1}; x_{d+1} = a(c)\}$ where $a(c)$ is chosen in a suitable way. For $x \in \partial \hat{A}_c$ consider the subspace $T_x \partial \hat{A}_c$ of $T_x \mathbf{S}^d_a$ consisting of the vectors tangential to $\partial \hat{A}_c$ and let $T_x \partial \hat{A}_c^{\perp}$ be its orthogonal complement in $T_x \mathbf{S}^d_a$. Now g_x splits into two parts. Let $g_{x,1}$ be the part describing the Riemannian metric on $T_x \partial \hat{A}_c$ and let $g_{x,2}$ be the part describing the Riemannian metric on $T_x \partial \hat{A}_c^{\perp}$. On $\partial \hat{A}_c$ one obtains a volume measure $\mu_{\partial \hat{A}_c}$ by restricting the metric g to $T\partial \hat{A}_c \times T\partial \hat{A}_c$. Put

$$F(c) := \left(L(c) \left(\mu_{\partial \hat{A}_c}(\partial \hat{A}_c) \right)^{-1} \right)^{\frac{1}{d-1}}$$

and

$$H(c) := \left((L(c))^{-1} \mu_{\partial \hat{A}_c}(\partial \hat{A}_c) \right).$$

Now one can define $\tilde{g}_{x,1} := (F(c))^2 g_{x,1}$ and $\tilde{g}_{x,2} := (H(c))^2 g_{x,2}$ for all $x \in \partial \hat{A}_c$. Then one gets a new Riemannian metric on $\mathbf{S}^d_a$. By the choice of this new metric one gets in any local coordinate system

$$\sqrt{\det(g_x)} = \sqrt{\det(\tilde{g}_x)}.$$

Furthermore, one has in any local coordinate system for $\partial \hat{A}_c$ and for all $x \in \partial \hat{A}_c$

$$(F(c))^{d-1} \sqrt{\det(g_{x,1})} = \sqrt{\det(\tilde{g}_{x,1})}.$$

Let $\tilde{\mu}^d_a$ be the volume measure corresponding to this new metric. With this new metric $\mathbf{S}^d_a$ becomes a comparison manifold for $\mathbf{M}$. One has after choice of local coordinates

$$\begin{aligned} & \tilde{\mu}^d_a(\hat{A}_c) \\ = & \int_{\hat{A}_c} \sqrt{\det(\tilde{g}_x)} dx_1 \wedge ... \wedge dx_d \\ = & \int_{\hat{A}_c} \sqrt{\det(g_x)} dx_1 \wedge ... \wedge dx_d \\ = & \mu^d_a(\hat{A}_c). \end{aligned}$$

Especially one has

$$\tilde{\mu}^d_a(\mathbf{S}^d_a) = \mu^d_a(\mathbf{S}^d_a) = \mu_{\mathbf{M}}(\mathbf{M}).$$

On $\partial \hat{A}_c$ one obtains another volume measure $\tilde{\mu}_{\partial \hat{A}_c}$ by restricting the metric $\tilde{g}$ to $T\partial \hat{A}_c \times T\partial \hat{A}_c$. Now, one has after choice of local coordinates for $\partial \hat{A}_c$

$$\tilde{\mu}_{\partial \hat{A}_c}(\partial \hat{A}_c)$$

$$
\begin{aligned}
&= \int_{\hat{A}_c} \sqrt{\det(\tilde{g}_{x,1})}\, dx_1 \wedge ... \wedge dx_{d-1} \\
&= (F(c))^{d-1} \int_{\hat{A}_c} \sqrt{\det(g_{x,1})}\, dx_1 \wedge ... \wedge dx_{d-1} \\
&= (F(c))^{d-1} \mu_{\partial \hat{A}_c}(\partial \hat{A}_c) \\
&= L(c) \\
&\leq S_{\mathbf{M}}(c).
\end{aligned}
$$

It is obvious that the metric $\tilde{g}$ on $\mathbf{S}^d_a$ is invariant under elements of $SO(d+1)$ fixing the x_{d+1}-axis. Furthermore by property b) in the definition of L it is invariant under reflection at the hyperplane $\{x \in I\!R^{d+1}; x_{d+1} = 0\}$. $\square$

Example 1: The d-dimensional flat torus $\mathbf{T}^d$ satisfies the requirements of the previous theorem: For small volumes the ball is the solution of the isoperimetric problem. This works as long as the boundary is smaller than two. For bigger volumes the solution turns out to be a collar. After further growth of the volume, the solution is realized by the exterior of a ball. Therefore, there exists a comparison manifold for the flat torus $\mathbf{T}^d$.

2.3 The Polar Rearrangement

For a comparison manifold $\mathbf{M}^* = \{x \in I\!R^{d+1}; \|x\| = a\}$ it makes sense to define the north pole $\mathbf{n} := (0, ..., 0, a)$ and the south pole $\mathbf{s} := (0, ..., 0, -a)$.

For a function $f \in \mathcal{B}_b(\mathbf{M})$ let $\mu_{\mathbf{M},f}$ be the image measure on $I\!R$. The distribution function of $\mu_{\mathbf{M},f}$ is denoted by $F_{\mathbf{M},f}$. Define $F^-_{\mathbf{M},f}(y) := \sup F^{-1}_{\mathbf{M},f}([0, y])$. $F^-_{\mathbf{M},f}$ is right continuous and increasing on $[0, \mu_{\mathbf{M}}(\mathbf{M})]$. It is almost a left inverse of the function $F_{\mathbf{M},f}$. For further properties of $F_{\mathbf{M},f}$ consult the Appendix of this thesis.

Definition 3 *Let $\mathbf{M}$ and $\mathbf{N}$ be two compact, connected, orientable d-dimensional Riemannian manifold without boundary and with same volume. Let $\mathbf{M}^*$ be a comparison manifold for $\mathbf{M}$. For $f \in \mathcal{B}_b(\mathbf{N})$ the function*

$$\hat{f} : \mathbf{M}^* \longrightarrow I\!R; x \mapsto F^-_{\mathbf{N},f}(\mu_{\mathbf{M}^*}(B_{d_{\mathbf{M}^*}(x,\mathbf{s})}(\mathbf{s})))$$

will be called the polar rearrangement of f on $\mathbf{M}^*$.

With this definition it makes sense to apply the rearrangement operation to $f \in \mathbf{M}^*$. For larger expressions we will use the following notation for the polar rearrangement: $\widehat{\text{larger expression}}$.

Lemma 2 *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be two compact d-dimensional Riemannian manifolds with same volume. If* $f \in \mathcal{C}(\mathbf{N})$ *then* $\hat{f} \in \mathcal{C}(\mathbf{M}^*)$.

Proof: Since f is continuous every nonempty open set in $[\inf f(x), \sup f(x)]$ has positive measure with respect to μ_f. Therefore, $F_{\mathbf{N},f}$ is strictly increasing on $[\inf f(x), \sup f(x)]$. From this follows that $F^-_{\mathbf{N},f}$ is continuous. Since the function $x \mapsto \mu_{\mathbf{M}^*}(B_{d_{\mathbf{M}^*}(\mathbf{s},x)}(\mathbf{s}))$ is continuous, the lemma is true. $\square$

Let $\mathbf{M}$ and $\mathbf{N}$ be two compact, connected, orientable d-dimensional Riemannian manifolds with same volume. For $f \in \mathcal{B}_b(\mathbf{M})$ the set

$$\mathcal{M}_f(\mathbf{N}) := \{g \in \mathcal{B}_b(\mathbf{N}); \mu_{\mathbf{M},f} = \mu_{\mathbf{N},g}\}$$

will be called valuefold of f in $\mathbf{N}$.

Definition 4 *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be two compact, connected, orientable d-dimensional Riemannian manifolds without boundary and with same finite volume. A transformation* $S : \mathcal{B}_b(\mathbf{M}) \longrightarrow \mathcal{B}_b(\mathbf{N})$ *will be called value-preserving, iff for all* $f \in \mathcal{B}_b(\mathbf{M})$ *one has*

$$S(\mathcal{M}_f(\mathbf{M})) \subset \mathcal{M}_f(\mathbf{N}).$$

Let $\mathbf{M}$ and $\mathbf{N}$ be two orientable Riemannian manifolds with same finite volume. In measure theory a map $s : \mathbf{N} \longrightarrow \mathbf{M}$ is called volume preserving if for all Borel sets $A \subset \mathbf{M}$ one has $\mu_{\mathbf{N}}(s^{-1}(A)) = \mu_{\mathbf{M}}(A)$.

Lemma 3 *Let* $s : \mathbf{N} \longrightarrow \mathbf{M}$ *be a volume preserving transformation. Then the map* $S : \mathcal{B}_b(\mathbf{M}) \longrightarrow \mathcal{B}_b(\mathbf{N}); f \mapsto f \circ s$ *is a value-preserving transformation.*

Proof: For a Borel set $A \subset I\!R$ and a function $f \in \mathcal{B}_b(\mathbf{M})$ one has for all $g \in \mathcal{M}_f(\mathbf{M})$

$$\begin{aligned} & \mu_{\mathbf{N},Sg}(A) = \mu_{\mathbf{N},g\circ s}(A) = \mu_{\mathbf{N}}(s^{-1}g^{-1}(A)) \\ = \; & \mu_{\mathbf{M}}(g^{-1}(A)) = \mu_{\mathbf{M},g}(A) = \mu_{\mathbf{M},f}(A). \end{aligned}$$

This proves the statement of the lemma. $\square$

Lemma 4 *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be two compact d-dimensional Riemannian manifolds with same volume. Then polar rearrangement* $\mathcal{B}_b(\mathbf{N}) \longrightarrow \mathcal{B}_b(\mathbf{M}^*); f \mapsto \hat{f}$ *is value-preserving.*

Proof: Let $F_{\mathbf{M}^*,\hat{f}}$ be the distribution function of $\mu_{\mathbf{M}^*,\hat{f}}$. For any $a \in I\!R$ one has by Lemma 34

$$\{x, F^-_{\mathbf{N},f}(\mu_{\mathbf{M}^*}(B_{d_{\mathbf{M}^*}(x,\mathbf{s})}(\mathbf{s}))) \leq a\} \subset \{x, \mu_{\mathbf{M}^*}(B_{d_{\mathbf{M}^*}(x,\mathbf{s})}(\mathbf{s})) \leq F_{\mathbf{N},f}(a)\}.$$

Further, one has by Lemma 35

$$\{x, \mu_{\mathbf{M}^*}(B_{d_{\mathbf{M}^*}(x,\mathbf{s})}(\mathbf{s})) < F_{\mathbf{N},f}(a)\} \subset \{x, F^-_{\mathbf{N},f}(\mu_{\mathbf{M}^*}(B_{d_{\mathbf{M}^*}(x,\mathbf{s})}(\mathbf{s}))) \leq a\}.$$

Since the set

$$\{x, \mu_{\mathbf{M}^*}(B_{d_{\mathbf{M}^*}(x,\mathbf{s})}(\mathbf{s})) = F_{\mathbf{N},f}(a)\}$$

is a zero set with respect to $\mu_{\mathbf{M}^*}$, we obtain

$$\begin{aligned} & \mu_{\mathbf{M}^*}(\{x, \hat{f}(x) \leq a\}) \\ = \; & \mu_{\mathbf{M}^*}(\{x, F^-_{\mathbf{N},f}(\mu_{\mathbf{M}^*}(B_{d_{\mathbf{M}^*}(x,\mathbf{s})}(\mathbf{s}))) \leq a\}) \\ = \; & \mu_{\mathbf{M}^*}(\{x, \mu_{\mathbf{M}^*}(B_{d_{\mathbf{M}^*}(x,\mathbf{s})}(\mathbf{s})) \leq F_{\mathbf{N},f}(a)\}). \end{aligned}$$

Furthermore, $\{x, \mu_{\mathbf{M}^*}(B_{d_{\mathbf{M}^*}(x,\mathbf{s})}(\mathbf{s})) \leq F_{\mathbf{N},f}(a)\}$ is just the closed ball centred at the south pole of $\mathbf{M}^*$ with volume $F_{\mathbf{N},f}(a)$. Therefore, one has

$$\begin{aligned} & F_{\mathbf{M}^*,\hat{f}}(a) = \mu_{\mathbf{M}^*}(\{x, \hat{f}(x) \leq a\}) \\ = \; & \mu_{\mathbf{M}^*}(\{x, \mu_{\mathbf{M}^*}(B_{d_{\mathbf{M}^*}(x,\mathbf{s})}(\mathbf{s})) \leq F_{\mathbf{N},f}(a)\}) = F_{\mathbf{N},f}(a). \end{aligned}$$

Therefore, one has $\mu_{\mathbf{N},f} = \mu_{\mathbf{M}^*,\hat{f}}$. □

From this follows that the polar rearrangement preserves the $\mathcal{L}_p$-norms:
$\|f\|_p = \|\hat{f}\|_p$.

Let $f \in \mathcal{B}_b(\mathbf{M})$ and $(f_n)_{n\in I\!N}$ be a sequence of Borel measurable bounded functions. In measure theory $(f_n)_{n\in I\!N}$ is called convergent to f in measure, if for all $\epsilon > 0$

$$\mu_{\mathbf{M}}(\{x; |f_n(x) - f(x)| > \epsilon) \longrightarrow 0.$$

Lemma 5 *Let $\mathbf{M}$ and $\mathbf{N}$ be two compact d-dimensional Riemannian manifolds with same volume. Let $f \in \mathcal{B}_b(\mathbf{N})$ and $(f_n)_{n\in I\!N} \subset \mathcal{B}_b(\mathbf{N})$ be a sequence of Borel measurable bounded functions converging in measure to f. Then $(\hat{f}_n)_{n\in I\!N}$ converges to $\hat{f}$ $\mu_{\mathbf{M}^*}$-almost everywhere.*

Proof: From lemma 37 we know that in this situation $F^-_{\mathbf{N},f_n}$ converges to $F^-_{\mathbf{N},f}$ λ-almost everywhere.
For a zero set $K \subset [0, \mu_{\mathbf{M}}(\mathbf{M})]$ the set $\{x; \mu_{\mathbf{M}^*}(B_{d_{\mathbf{M}^*}(x,\mathbf{s})}(\mathbf{s})) \in K\}$ is a $\mu_{\mathbf{M}^*}$-zero set. Therefore, by definition of the polar rearrangement $\hat{f}_n$ converges to $\hat{f}$ $\mu_{\mathbf{M}^*}$-almost everywhere. □

Lemma 6 *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be two compact d-dimensional Riemannian manifolds with same volume. Let* $G \in SO(n+1)$ *be a rotation fixing the* x_{d+1}*-axis. Then one has for all* $f \in \mathcal{B}_b(\mathbf{N})$ *the following properties:*
i) $\hat{f} \circ G = \hat{f}$ *,*
ii) $\hat{f}$ *is decreasing with the distance to* $\mathbf{n}$.
On the other hand if a function $g \in \mathcal{M}_f(\mathbf{M}^*)$ *satisfies the conditions i) and ii) one has* $\hat{f} = g$ μ*-almost sure.*

Proof: Property i) follows from the fact that $B_{d_{\mathbf{M}^*}(G(x),\mathbf{s})}(\mathbf{s}) = B_{d_{\mathbf{M}^*}(x,\mathbf{s})}(\mathbf{s})$. Property ii) follows from the monotonicity of the measure $\mu_{\mathbf{M}^*}$. For the second part let $g \in \mathcal{M}_f(\mathbf{M}^*)$ be a function satisfying i) and ii). By property i) and ii) one has $\{y; g(y) \leq g(x)\} = B_{d_{\mathbf{M}^*}(x,\mathbf{s})}(\mathbf{s})$. Therefore, one has

$$\begin{aligned} \hat{f}(x) &= F^{-}_{\mathbf{M}^*,f}(\mu_{\mathbf{M}^*}(B_{d_{\mathbf{M}^*}(x,\mathbf{s})}(\mathbf{s}))) = F^{-}_{\mathbf{M}^*,g}(\mu_{\mathbf{M}^*}(B_{d_{\mathbf{M}^*}(x,\mathbf{s})}(\mathbf{s}))) \\ &= F^{-}_{\mathbf{M}^*,g}(\mu_{\mathbf{M}^*}(\{y; g(y) \leq g(x)\})) = F^{-}_{\mathbf{M}^*,g}(F_{\mathbf{M}^*,g}(g(x))) = g(x). \end{aligned}$$

□

Lemma 7 *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be two compact d-dimensional Riemannian manifolds with same volume. For all* $f \in \mathcal{B}_b(\mathbf{N})$ *and for all* $t \geq 0$ *one has*

$$\overbrace{T_{\mathbf{M}^*,t}\hat{f}} = T_{\mathbf{M}^*,t}\hat{f}.$$

Proof: It is clear that $T_{\mathbf{M}^*,t}\hat{f}$ satisfies the properties i) and ii) of Lemma 6. Therefore $T_{\mathbf{M}^*,t}\hat{f}$ is a fix point under polar rearrangement. □

Lemma 8 *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be two compact d-dimensional Riemannian manifolds with same volume. For all value-preserving transformations* $S : \mathcal{B}_b(\mathbf{N}) \longrightarrow \mathcal{B}_b(\mathbf{N})$ *one has*

$$\widehat{Sf} = \hat{f}.$$

Proof: This follows directly from the definition of the polar rearrangement since one has $\mu_{\mathbf{N},Sf} = \mu_{\mathbf{N},f}$ and therefore, $F_{\mathbf{N},Sf} = F_{\mathbf{N},f}$. □

Let $\mathbf{M}^* = \{x \in \mathbb{R}^{d+1}; |x| = a\}$ be a comparison manifold for a manifold $\mathbf{M}$.
Put $m : \mathbf{M}^* \longrightarrow \mathbf{M}^*; (x_1, x_2, ..., x_{d+1}) \mapsto (x_1, x_2, ..., -x_{d+1})$ and let
$M : \mathcal{B}_b(\mathbf{M}^*) \longrightarrow \mathcal{B}_b(\mathbf{M}^*); f \mapsto f \circ m$.

Lemma 9 *One has for all* $f \in \mathcal{B}(\mathbf{M}^*)$ *that*

$$-\hat{f} = M(\widehat{-f}).$$

Proof: By construction the function $-M(\widehat{-f})$ is decreasing with the distance to $\mathbf{n}$ and invariant by rotations fixing $\mathbf{n}$. Since m is measure preserving M is value-preserving by lemma 3. Therefore, one has $-M(\widehat{-f}) \in \mathcal{M}_f(\mathbf{M}^*)$. Therefore by Lemma 6 one gets $-\hat{f} = M(\widehat{-f})$. □

Lemma 10 *One has for all $t \geq 0$*

$$M \circ T_{\mathbf{M}^*,t} = T_{\mathbf{M}^*,t} \circ M.$$

Proof: Since $d_{\mathbf{M}^*}(m(x), y) = d_{\mathbf{M}^*}(x, m(y))$, one has by the symmetry property of the comparison manifold

$$p_{\mathbf{M}^*}(t, x, m(y)) = p_{\mathbf{M}^*}(t, m(x), y).$$

From this follows

$$\begin{aligned}
& (T_{\mathbf{M}^*,t} M f)(x) \\
= & \int_{\mathbf{M}^*} p_{\mathbf{M}*}(t, x, y) f(m(y)) d\mu_{\mathbf{M}^*}(y) \\
= & \int_{\mathbf{M}^*} p_{\mathbf{M}^*}(t, x, m(y)) f(y) d\mu_{\mathbf{M}^*}(y) \\
= & \int_{\mathbf{M}^*} p_{\mathbf{M}^*}(t, m(x), y) f(y) d\mu_{\mathbf{M}^*}(y) \\
= & (M T_{\mathbf{M}^*,t} f)(x).
\end{aligned}$$

□

3 Heat Flux Estimate for Infinitesimal Time

In this chapter we compare the heat flux of the heat distributions f and $\hat{f}$ for an infinitesimal time step. The heat flux depends on the isoperimetric function $S_{\mathbf{M}}$ of the manifold $\mathbf{M}$. Therefore, we can control the heat flux on $\mathbf{M}$ by comparing it to the heat flux of the rearranged heat distribution $\hat{f}$ on $\mathbf{M}^*$. For this we need a result about integration on tubular neighbourhoods. This result is proved in the first section. In the second section we prove the following inequality

$$\int_{\{x;f(x)\leq c\}} \triangle f \omega_{\mathbf{M}} \leq \int_{\{x;\hat{f}(x)\leq c\}} \triangle \hat{f} \omega_{\mathbf{M}^*}.$$

This inequality shows that the heat flux in an infinitesimal time step for f is greater than that for $\hat{f}$.

3.1 Integrating on Tubular Neighbourhoods

Let $\mathbf{M}$ be a compact, connected orientable Riemannian manifold without boundary. For a subset K of $\mathbf{M}$ set $K(\epsilon) := \{x \in X; d(x, K) < \epsilon\}$. Further let D be an open sub-manifold of $\mathbf{M}$ with smooth boundary. One can consider ∂D as an oriented Riemannian sub-manifold of $\mathbf{M}$. For a point $x \in \partial D$ let $n_x \in T_x\mathbf{M}$ be the the tangential vector of length one normal to $T_x\partial D$ pointing to the exterior of D.
The following result is the well known result about the existence of a tubular neighbourhood. (For details see [GreHalVan1] p. 138)

Lemma 11 *Let D be an open sub-manifold with non empty smooth boundary ∂D. Then there exist an $\epsilon_D > 0$ and a diffeomorphism $\Phi : \partial D \times]-\epsilon_D, \epsilon_D[\longrightarrow (\partial D)_{\epsilon_D}$, where Φ is given by $\Phi(x,t) := \exp_x(tn_x)$.*

Let $\pi_{\partial D} : \partial D \times]-\epsilon, \epsilon[\longrightarrow \partial D; (y, \delta) \mapsto y$ and
$\pi_{]-\epsilon,\epsilon[} : \partial D \times]-\epsilon, \epsilon[\longrightarrow \partial D; (y, \delta) \mapsto \delta$.

For a continuous function $f : \partial D \longrightarrow [0, \epsilon_D[$ let $\partial D_f := \{x \in D^c; d(y, x) \leq f(y)\}$.
One gets then: $\Phi^{-1}(D_f) = \{(x, y) \in \partial D \times]-\epsilon_D, \epsilon_D[; 0 < y \leq f(x)\}$.

Lemma 12 *Let D be an open sub-manifold with non empty smooth boundary ∂D. There exists a $\mathcal{C}^\infty$-density ρ on $[0, \epsilon_D[\times \partial D$ such that for all continuous functions $f : \partial D \longrightarrow [0, \epsilon_D]$ one has:*

$$\mu_{\mathbf{M}}(\partial D_f) = \int_{\partial D} \int_0^{f(x)} \rho(s, x) ds d\mu_{\partial D}(x). \tag{2}$$

Proof: ∂D is a compact Riemannian sub-manifold of $\mathbf{M}$ with dimension $d-1$. One can chose a finite covering $\{U_i; i \in I\}$ of ∂D by maps $\phi_i : U_i \longrightarrow V_i \subset I\!R^{d-1}$. One can choose a corresponding partition of the unity $\{\psi_i : \partial D \longrightarrow I\!R, \mathrm{supp}(\Psi_i) \subset U_i; i \in I\}$. In this local coordinate system the volume form $\omega_{\partial D}$ is given by $a_i(x_1, ..., x_{d-1}) dx_1 \wedge ... \wedge dx_{d-1}$, where $a_i : V_i \longrightarrow I\!R$ is strictly positive and $\mathcal{C}^\infty$.
With lemma 11 for $\epsilon \leq \epsilon_D$ one can extend the local coordinates for ∂D to local coordinates of ∂D_ϵ

$$\tilde{\phi}_i : \Phi(U_i \times]-\epsilon, \epsilon[) \longrightarrow V_i \times]-\epsilon, \epsilon[; x \mapsto (\phi_i(\pi_{\partial D}(\Phi^{-1}(x))), \pi_{]-\epsilon,\epsilon[}(\Phi^{-1}(x))).$$

In this local coordinate system the volume form $\omega_{\mathbf{M}}$ is given by $b_i(x_1, ..., x_{d-1}, t) dx_1 \wedge ... \wedge dx_{d-1} \wedge dt$ where $b_i : V_i \times]-\epsilon, \epsilon[\longrightarrow I\!R$ is strictly positive and $\mathcal{C}^\infty$. Now one gets

$$\begin{aligned}
& \mu_{\mathbf{M}}(\partial D_f) \\
= & \int_{\partial D_f} \omega_{\mathbf{M}} \\
= & \sum_{i \in I} \int_{\partial D_f \cap \Phi(U_i \times]-\epsilon,\epsilon[)} \psi_i \omega_{\mathbf{M}} \\
= & \sum_{i \in I} \int_{\tilde{\phi}_i(\partial D_f) \cap (V_i \times]-\epsilon,\epsilon[)} (\psi_i b_i)(x_1, ..., x_{d-1}, t) dx_1 \wedge ... \wedge dx_{d-1} \wedge dt \\
= & \sum_{i \in I} \int_{V_i} \int_0^{f(\phi_i^{-1}(x_1,...,x_{d-1}))} (\psi_i b_i)(x_1, ..., x_{d-1}, t) dt dx_1 \wedge ... \wedge dx_{d-1} \\
= & \sum_{i \in I} \int_{V_i} \int_0^{f(\phi_i^{-1}(x_1,...,x_{d-1}))} \frac{\psi_i b_i a_i}{a_i}(x_1, ..., x_{d-1}, t) dt dx_1 \wedge ... \wedge dx_{d-1} \\
= & \sum_{i \in I} \int_{U_i} \int_0^{f(.)} \frac{\psi_i b_i}{a_i}(., t) dt \omega_{\partial_D} \\
= & \int_{\partial D} \int_0^{f(.)} \sum_{i \in I} \frac{b_i}{a_i}(., t) dt \omega_{\partial_D} \\
= & \int_{\partial D} \int_0^{f(x)} \sum_{i \in I} \frac{b_i(x,t)}{a_i(x)} dt d\mu_{\partial_D}(x)
\end{aligned}$$

Since $\sum_{i \in I} \frac{b_i(x,t)}{a_i(x)}$ is $\mathcal{C}^\infty$, the statement of the lemma is true. $\square$

3.2 The Heat Flux Comparison for Infinitesimal Time

Let $\mathbf{M}^*$ be a comparison manifold for $\mathbf{M}$. For a measurable set $D \subset \mathbf{M}$ denote by $\hat{D}$ the ball around the north pole $\mathbf{n}$ of $\mathbf{M}^*$ with $\mu_{\mathbf{M}}(D) = \mu_{\mathbf{M}^*}(\hat{D})$.

Lemma 13 *Let $\mathbf{M}^*$ be a comparison manifold for $\mathbf{M}$. Let D be an open submanifold of $\mathbf{M}$ with smooth boundary. There exist constants $C, \tilde{C} \in I\!R$ and $\gamma > 0$ such that for all continuous $f : \partial D \longrightarrow]0,\gamma]$ and all $\delta \in]0,\gamma]$ one has*

$$\mu_{\mathbf{M}^*}(\partial\hat{D}_\delta)\left(\int_{\partial\hat{D}} \delta^{-1} d\mu_{\partial D}(x) + \tilde{C}\right) \leq \mu_{\mathbf{M}}(\partial D_f)\left(\int_{\partial D} (f(x))^{-1} d\mu_{\partial D}(x) + C\right). \quad (3)$$

Proof: By applying Lemma 12 to $\partial D \in \mathbf{M}$ and $\partial\hat{D} \in \mathbf{M}^*$ one gets $\mathcal{C}^\infty$-densities ρ and $\tilde{\rho}$ with

$$\mu_{\mathbf{M}}(\partial D_f) = \int_{\partial D} \int_0^{f(n)} \rho(s,x) ds d\mu_{\partial D}(x) \quad (4)$$

and

$$\mu_{\mathbf{M}}(\partial\hat{D}_\delta) = \int_{\partial\hat{D}} \int_0^{\delta} \tilde{\rho}(s) ds d\mu_{\partial\hat{D}}(x) \quad (5)$$

for $f \leq \epsilon_D$ and $\delta \leq \epsilon_{\hat{D}}$. For all $x \in \partial D$ there exists by Taylors expansion theorem a $\xi(x) \in [0,t]$ such that

$$\int_0^t \rho(s,x) ds = \rho(0,x)t + \frac{1}{2}(\frac{\partial}{\partial s}\rho)(\xi(x),x)t^2. \quad (6)$$

Further, there exists by Taylors expansion theorem a $\xi \in [0,t]$ such that

$$\int_0^t \rho(s) ds = \rho(0)t + \frac{1}{2}(\frac{\partial}{\partial s}\rho)(\xi)t^2. \quad (7)$$

Since $\mathbf{M}^*$ is a comparison manifold one gets

$$\mu_{\partial\hat{D}}(\partial\hat{D}) \leq \mu_{\partial D}(\partial D). \quad (8)$$

Taking into account that $\tilde{\rho}(0) = 1$ and $\rho(0,x) = 1$ for all $x \in \partial D$ and using 4, 5, 6, 7, Cauchy Schwarz Inequality and 8 one gets

$$\mu_{\mathbf{M}^*}(\partial\hat{D}_\delta) \int_{\partial\hat{D}} \left(\delta + \frac{1}{2!}(\frac{\partial}{\partial x}\tilde{\rho})(\xi)\delta^2\right)^{-1} d\mu_{\partial\hat{D}}(x)$$

$$
\begin{aligned}
&= \int_{\partial\hat{D}}\int_0^{\delta}\tilde{\rho}(s)ds d\mu_{\partial\hat{D}}(x)\int_{\partial\hat{D}}\left(\int_0^{\delta}\tilde{\rho}(s)ds\right)^{-1}d\mu_{\partial\hat{D}}(x)\\
&= \left(\int_{\partial\hat{D}}d\mu_{\partial\hat{D}}(x)\right)^2\\
&\leq \left(\int_{\partial D}d\mu_{\partial D}(x)\right)^2\\
&= \left(\int_{\partial D}\left(\int_0^{f(x)}\rho(s,x)ds\right)^{\frac{1}{2}}\left(\int_0^{f(x)}\rho(s,x)ds\right)^{-\frac{1}{2}}d\mu_{\partial D}(x)\right)^2\\
&\leq \int_{\partial D}\int_0^{f(x)}\rho(s,x)ds d\mu_{\partial D}(x)\int_{\partial D}\left(\int_0^{f(x)}\rho(s,x)ds\right)^{-1}d\mu_{\partial D}(x)\\
&= \mu_{\mathbf{M}}(\partial D_f)\int_{\partial D}\left(f(x)+\frac{1}{2!}(\frac{\partial}{\partial s}\rho)(\xi(x),x)(f(x))^2\right)^{-1}d\mu_{\partial D}(x).
\end{aligned}
$$

Further, one has

$$
\begin{aligned}
&\int_{\partial D}(f(x))^{-1}d\mu_{\partial D}(x)-\int_{\partial D}\left(f(x)+\frac{1}{2!}(\frac{\partial}{\partial s}\rho)(\xi(x),x)(f(x))^2\right)^{-1}d\mu_{\partial D}(x)\\
&= \int_{\partial D}\frac{\frac{1}{2!}(\frac{\partial}{\partial s}\rho)(\xi(x),x)(f(x))^2}{(f(x))^2+\frac{1}{2!}(\frac{\partial}{\partial s}\rho)(\xi(x),x)(f(x))^3}d\mu_{\partial D}(x)\\
&= \int_{\partial D}\frac{\frac{1}{2!}(\frac{\partial}{\partial s}\rho)(\xi(x),x)}{1+\frac{1}{2!}(\frac{\partial}{\partial s}\rho)(\xi(x),x)f(x)}d\mu_{\partial D}(x).
\end{aligned}
$$

The last expression goes to $\int_{\partial D}\frac{1}{2!}(\frac{\partial}{\partial s}\rho)(0,x)d\mu_{\partial D}(x)$ if f goes pointwise to zero. Further, one has

$$
\begin{aligned}
&\int_{\partial\hat{D}}\delta^{-1}d\mu_{\partial D}(x)-\int_{\partial\hat{D}}\left(\delta+\frac{1}{2!}(\frac{\partial}{\partial s}\rho)(\xi)\delta^2\right)^{-1}d\mu_{\partial\hat{D}}(x)\\
&= \int_{\partial\hat{D}}\frac{\frac{1}{2!}(\frac{\partial}{\partial s}\rho)(\xi)\delta^2}{\delta^2+\frac{1}{2!}(\frac{\partial}{\partial s}\rho)(\xi)\delta^3}d\mu_{\partial\hat{D}}(x)\\
&= \int_{\partial\hat{D}}\frac{\frac{1}{2!}(\frac{\partial}{\partial s}\rho)(\xi)}{1+\frac{1}{2!}(\frac{\partial}{\partial s}\rho)(\xi)\delta}d\mu_{\partial\hat{D}}(x).
\end{aligned}
$$

This last expression goes to $\int_{\partial\hat{D}}\frac{1}{2!}(\frac{\partial}{\partial s}\rho)(0)d\mu_{\partial\hat{D}}(x)$ if δ goes to zero. Therefore, one gets for small f and small δ

$$
\mu_{\mathbf{M}}(\partial\hat{D}_\delta)\int_{\partial\hat{D}}\delta^{-1}d\mu_{\partial\hat{D}}(x)+\mu_{\mathbf{M}}(\partial\hat{D}_\delta)\left(\int_{\partial\hat{D}}\frac{1}{2!}(\frac{\partial}{\partial s}\tilde{\rho})(0)d\mu_{\partial\hat{D}}(x)-1\right)
$$

$$\leq \quad \mu_{\mathbf{M}}(\partial D_f)\int_{\partial D}(f(x))^{-1}d\mu_{\partial D}(x) + \mu_{\mathbf{M}}(\partial D_f)\left(\int_{\partial D}\frac{1}{2!}(\frac{\partial}{\partial s}\rho)(0,x)d\mu_{\partial D}(x)+1\right).$$

Since small γ implies small f and small δ this is just the statement of the lemma with $\tilde{C} = \int_{\partial\hat{D}}\frac{1}{2!}(\frac{\partial}{\partial s}\tilde{\rho})(0)d\mu_{\partial\hat{D}}(x) - 1$ and $C = \int_{\partial D}\frac{1}{2!}(\frac{\partial}{\partial s}\rho)(0,x)d\mu_{\partial D}(x)+1$. □
Sard theorem states that the set of all regular values of a smooth map $h : \mathbf{M} \longrightarrow I\!R$ form a dense set in $[\inf h, \sup h]$ (see [GreHalVan] p.136). It follows that the set $\mathcal{S}_h$ of all $c \in [\inf h, \sup h]$ with the property: $h^{-1}(c)$ is a sub-manifold of Co dimension one in $\mathbf{M}$, is dense in $[\inf h, \sup h]$.

Theorem 2 *Let* $\mathbf{M}$ *be a compact, connected, orientable Riemannian manifold without boundary and let* $\mathbf{M}^*$ *be a comparison manifold for* $\mathbf{M}$. *For* $h \in \mathcal{V}(\mathbf{M})$ *one has for all* $c \in \mathcal{S}_h$

$$\int_{\{x;h(x)=c\}} g(\mathrm{grad}(h),n)\omega_{\{x;h(x)=c\}} \leq \int_{\{x;\hat{h}(x)=c\}} g(\mathrm{grad}(\hat{h},n)\omega_{\{x;\hat{h}(x)=c\}}. \tag{9}$$

Proof: Let $(c_m)_{m\in I\!N}$ be a increasing sequence converging against $c \in \mathcal{S}_h$ with the property that $c_m < c$ for all $m \in I\!N$. It is well known that for all $x \in \partial D$ the value $\mathrm{grad}(h)(x)$ is well approximated in the following way

$$\mathrm{grad}(h)(x) = -\lim_{m\longrightarrow\infty}\frac{c-c_m}{d_{\mathbf{M}}(x,\{x;h(x)\leq c_m\})}n_x. \tag{10}$$

One can define $f_m : \{x;h(x)=c\} \longrightarrow [0,\gamma]$ by $f_m(x) := d_{\mathbf{M}}(x,\{x;h(x)\leq c_m\})$ and $\delta_m := d_{\mathbf{M}^*}(\{x;\hat{h}(x)\geq c\},\{x;\hat{h}(x)\leq c_m\})$. If $\gamma > 0$ is chosen according to Lemma 13, it is obvious that there exists an $m_0 \in \mathbf{N}$ such that for all $m \geq m_0$: $\max(\|f_m\|_{sup},\delta_m) \leq \gamma$.
Since h and $\hat{h}$ are continuous, one has $\{x;h(x)=c\} = \partial\{x;h(x)\geq c\}$ and $\{x;\hat{h}(x)=c\} = \partial\{x;\hat{h}(x)\geq c\}$. Therefore, since $\hat{h} \in \mathcal{M}_h$ one has

$$\begin{aligned} & \mu_{\mathbf{M}^*}(\partial\{x;\hat{h}(x)\geq c\}_{\delta_m}) = \mu_{\mathbf{M}^*}(\{x;c\geq\hat{h}(x)\geq c_m\}) \\ = \; & \mu_{\mathbf{M},h}(]c_m,c]) = \mu_{\mathbf{M}}(\{x;c\geq h(x)\geq c_m\}) = \mu_{\mathbf{M}}(\partial\{x;h(x)\geq c\}_{f_m}). \end{aligned}$$

From this one gets by Lemma 13

$$\begin{aligned} & \int_{\{x;h(x)=c\}} g(\mathrm{grad}(h),n)\omega_{\{x;h(x)=c\}} \\ = \; & -\lim_{m\longrightarrow\infty}(c-c_m)\int_{\{x;h(x)=c\}} f_m^{-1}d\mu_{\{x;h(x)=c\}} \\ = \; & -\lim_{m\longrightarrow\infty}(c-c_m)(\int_{\{x;h(x)=c\}} f_m^{-1}d\mu_{\{x;h(x)=c\}} + C) \end{aligned}$$

$$
\begin{aligned}
&\geq -\lim_{n\longrightarrow\infty}(c-c_m)\Big(\int_{\{x;\hat{h}(x)=c\}}\delta_m^{-1}\mu_{\{x;\hat{h}(x)=c\}}+\tilde{C}\Big)\\
&= -\lim_{m\longrightarrow\infty}(c-c_m)\Big(\int_{\{x;\hat{h}(x)=c\}}\delta_m^{-1}\mu_{\{x;\hat{h}(x)=c\}}\Big)\\
&= \int_{\{x;\hat{h}(x)=c\}} g(\mathrm{grad}(\hat{h}),n)\omega_{\{x;\hat{h}(x)=c\}}.
\end{aligned}
$$

This proves the theorem. □

Theorem 3 *Let* $\mathbf{M}$ *be a compact, connected, orientable Riemannian manifold without boundary and let* $\mathbf{M}^*$ *be a comparison manifold for* $\mathbf{M}$*. Then one has for all* $h\in\mathcal{V}(\mathbf{M})$ *and all* $c\in[\inf(h),\sup(h)]$

$$
\int_{\{x;h(x)\geq c\}}\triangle h(y)d\mu_{\mathbf{M}}(y)\leq\int_{\{x;\hat{h}(x)\geq c\}}\triangle\hat{h}(y)d\mu_{\mathbf{M}^*}(y).
$$

Proof: For a given $c\in\mathbf{S}_h^c$ there exists a sequence $d_n\in\mathbf{S}_h$ with $\lim d_n=c$. By a straight forward application dominated convergence theorem one can prove

$$
\int_{\{x;h(x)\geq d_n\}}\triangle h(y)d\mu_{\mathbf{M}}(y)\longrightarrow\int_{\{x;h(x)\geq c\}}\triangle h(y)d\mu_{\mathbf{M}}(y)
$$

and

$$
\int_{\{x;\hat{h}(x)\geq d_n\}}\triangle\hat{h}(y)d\mu_{\mathbf{M}}(y)\longrightarrow\int_{\{x;\hat{h}(x)\geq c\}}\triangle\hat{h}(y)d\mu_{\mathbf{M}}(y).
$$

Therefore it is sufficient to prove the statement of the theorem for all $c\in\mathcal{S}_h$. By theorem 2 and by Stokes theorem one has for all $c\in\mathcal{S}_h$

$$
\begin{aligned}
&\int_{\{x;h(x)\geq c\}}\triangle h\omega_{\mathbf{M}}\\
=&\int_{\partial\{x;h(x)\geq c\}} g(\mathrm{grad}(h),n)\omega_{\partial\{x;h(x)\geq c\}}\\
\leq&\int_{\partial\{x;\hat{h}(x)\geq c\}} g(\mathrm{grad}(\hat{h}),n)\omega_{\partial\{x;\hat{h}(x)\geq c\}}\\
=&\int_{\{x;\hat{h}(x)\geq c\}}\triangle\hat{h}\omega_{\mathbf{M}^*}.
\end{aligned}
$$

This proves the theorem. □

4 The Heat Flux Estimate for finite Time

In the previous section, we compared the heat flux of a heat distribution to the heat flux of the rearranged heat distribution in an infinitesimal time step. It proved to be smaller for the situation on the comparison manifold. In this section, we extend this result to a time step of positive duration. For this we need to investigate the heat content function

$$V :]0, \infty[\longrightarrow I\!R; t \mapsto \sup_{\mu_{\mathbf{M}}(A)=c} \int_A T_{\mathbf{M},t} f(x) d\mu_{\mathbf{M}}(x).$$

It will be necessary to prove that the heat content function is continuously differentiable in its time variable. In order to show this, we need properties of the sets maximising the heat content. These properties are proved in the first subsection. In the second subsection, this result is used to show that the heat content function is continuously differentiable in the time variable. This is the property, which is used to extend the infinitesimal heat flux comparison to a finite time heat flux comparison. As a final inequality we get

$$\sup_{\mu_{\mathbf{M}}(A)=c} \int_A T_{\mathbf{M},t} f(x) d\mu_{\mathbf{M}}(x) \leq \int_{\hat{A}_c} T_{\mathbf{M}^*,t} \hat{f}(x) d\mu_{\mathbf{M}^*}(x).$$

This estimate signifies that the decay of the heat content is faster for f than it is for $\hat{f}$. This fact is used in the next section to understand the behaviour of the heat flow under perturbations.

4.1 About Sets with Maximal Heat Content

Lemma 14 *Let $f \in \mathcal{V}(\mathbf{M})$ and $t \in [0, \infty[$. There exists a Borel set $A_{t,c} \subset \mathbf{M}$ with $\mu_{\mathbf{M}}(A_{t,c}) = c$ such that*

$$\sup_{\mu_{\mathbf{M}}(A)=c} \int_A T_{\mathbf{M},t} f(x) d\mu_{\mathbf{M}}(x) = \int_{A_{t,c}} T_{\mathbf{M},t} f(x) d\mu_{\mathbf{M}}(x).$$

Further, there exists a continuous function $k : [0, \infty[\longrightarrow I\!R$ such that

$$A_{t,c} := \{x \in \mathbf{M}; T_t f(x) \geq k_t\}.$$

Proof: For $c = 0$ the statement of the lemma is clear, since one can take for $A_{t,c}$ the empty set. For $c = \mu_{\mathbf{M}}(\mathbf{M})$ the statement of the lemma is clear too, since one can take for $A_{t,c}$ the whole manifold $\mathbf{M}$. So for the rest of the proof let

$0 < c < \mu_{\mathbf{M}}(\mathbf{M})$.
Define

$$A_{t,c} := \{x \in \mathbf{M}; \mu_{\mathbf{M}}(y; T_{\mathbf{M},t}f(y) \geq T_{\mathbf{M},t}f(x)) \leq c\}.$$

By Lemma 1 the set $A_{t,c}$ has measure c. For the second statement let $k_t := \inf\{T_{\mathbf{M},t}f(x); x \in A_{t,c}\}$. Clearly one has: $A_{t,c} = \{x \in \mathbf{M}; T_t f(x) \geq k_t\}$.
The function k is continuous. This can be seen in the following way:
Assume that k is not continuous in $t \geq 0$.
Then there would exist an $\epsilon > 0$ and there would exist either a sequence $(s_n)_{n\in I\!N}$ converging to t such that $k_{s_n} > t + \epsilon$ for all $n \in I\!N$ or a sequence $(s_n)_{n\in I\!N}$ converging to t such that $k_{s_n} < t - \epsilon$ for all $n \in I\!N$.
Now lets first consider the first case: If $x \in A_{s_n,c}$ for infinitely many $n \in I\!N$, then by the strong continuity of $T_{\mathbf{M}}$ one would have $x \in \{x; T_{\mathbf{M},t}f(x) \geq k_t + \epsilon\}$.
By the continuity of $T_{\mathbf{M},t}f$ one has since $c > 0$

$$\begin{aligned}
& \mu_{\mathbf{M}}(A_{t,c} \cap \limsup A_{s_n,c}) \\
= \; & \mu_{\mathbf{M}}(\{x \in \mathbf{M}; T_{\mathbf{M},t}f(x) \geq k_t + \epsilon\}) \\
< \; & \mu_{\mathbf{M}}(\{x \in \mathbf{M}; T_{\mathbf{M},t}f(x) \geq k_t\}) \\
= \; & \mu_{\mathbf{M}}(A_{t,c}).
\end{aligned}$$

Therefore, one has

$$\mu_{\mathbf{M}}(A_{t,c} \setminus \limsup A_{s_n,c}) > 0.$$

Now,

$$\begin{aligned}
& \mu_{\mathbf{M}}(A_{t,c} \setminus \limsup A_{s_n,c}) \\
= \; & \mu_{\mathbf{M}}(\liminf A_{t,c} \setminus A_{s_n,c}) \\
= \; & \mu_{\mathbf{M}}(\bigcup_{n\in I\!N} \bigcap_{m\geq n} A_{t,c} \setminus A_{s_m,c}) \\
\leq \; & \sum_{n\in I\!N} \mu_{\mathbf{M}}(\bigcap_{m\geq n} A_{t,c} \setminus A_{s_m,c}) \\
= \; & \sum_{n\in I\!N} \mu_{\mathbf{M}}(A_{t,c} \setminus \bigcup_{m\geq n} A_{s_m,c}).
\end{aligned}$$

Therefore, there must exist a $n_0 \in I\!N$ such that

$$\mu_{\mathbf{M}}(A_{t,c} \setminus \bigcup_{n\geq n_0} A_{s_n,c}) > 0. \tag{11}$$

Now assume that $\liminf \mu_{\mathbf{M}}(A_{s_n,c} \setminus A_{t,c}) = 0$.
There would exist a subsequence $s_{n(k)}$ with the property

$$\lim_{k\longrightarrow\infty} \mu_{\mathbf{M}}(A_{s_{n(k)},c} \setminus A_{t,c}) = 0.$$

From this would follow since $\mu_{\mathbf{M}}(A_{t,c}) = \mu_{\mathbf{M}}(A_{s_n,c})$ for all $n \in I\!N$

$$\lim_{k \longrightarrow \infty} \mu_{\mathbf{M}}(A_{t,c} \setminus A_{s_{n(k)},c}) = 0.$$

This would contradict (1) since $A_{t,c} \setminus \bigcup_{m \geq n_0} A_{s_m,c} \subset A_{t,c} \setminus A_{s_{n(k)},c}$ for all $n(k) \geq n_0$. Therefore, one must have $\liminf \mu_{\mathbf{M}}(A_{s_n,c} \setminus A_{t,c}) > 0$.
Since $\mathbf{M}$ has finite volume, we now know that $\mu_{\mathbf{M}}(\limsup A_{s_n,c} \setminus A_{t,c}) > 0$. Now for points in $A_{t,c} \setminus \limsup A_{s_n,c}$ we know that $T_{\mathbf{M},t}f(x) \geq k_t$ and we know for the points in $\limsup A_{s_n,c} \setminus A_{t,c}$ that $T_{\mathbf{M},t}f(x) \geq k_t + \epsilon$. Therefore, we get a contradiction, since $A_{t,c} = \{x \in \mathbf{M}; T_{\mathbf{M},t}f(x) \geq k_t\}$.
From this we conclude, that there can not exist a sequence s_n converging to t with $k_{s_n} \geq k_t + \epsilon$.

Now lets consider the case of the existence of a sequence s_n converging to t with $k_{s_n} \leq k_t - \epsilon$.
If $x \in A^c_{s_n,c}$ for infinitely many $n \in I\!N$, then by the strong continuity of $T_{\mathbf{M}}$ one would have $x \in \{x; T_{\mathbf{M},t}f(x) \leq k_t - \epsilon\}$.
By the continuity of $T_{\mathbf{M},t}f$ one has since $c < \mu_{\mathbf{M}}(\mathbf{M})$

$$\begin{aligned}
& \mu_{\mathbf{M}}(A^c_{t,c} \cap \limsup A^c_{s_n,c}) \\
= \;& \mu_{\mathbf{M}}(\{x \in \mathbf{M}; T_{\mathbf{M},t}f(x) \leq k_t - \epsilon\}) \\
< \;& \mu_{\mathbf{M}}(\{x \in \mathbf{M}; T_{\mathbf{M},t}f(x) < k_t\}) \\
= \;& \mu_{\mathbf{M}}(A^c_{t,c}).
\end{aligned}$$

Therefore, one has

$$\mu_{\mathbf{M}}(A^c_{t,c} \setminus \limsup A^c_{s_n,c}) > 0.$$

Now,

$$\begin{aligned}
& \mu_{\mathbf{M}}(A^c_{t,c} \setminus \limsup A^c_{s_n,c}) \\
= \;& \mu_{\mathbf{M}}(\liminf A^c_{t,c} \setminus A^c_{s_n,c}) \\
= \;& \mu_{\mathbf{M}}(\bigcup_{n \in I\!N} \bigcap_{m \geq n} A^c_{t,c} \setminus A^c_{s_m,c}) \\
\leq \;& \sum_{n \in I\!N} \mu_{\mathbf{M}}(\bigcap_{m \geq n} A^c_{t,c} \setminus A^c_{s_m,c}) \\
= \;& \sum_{n \in I\!N} \mu_{\mathbf{M}}(A^c_{t,c} \setminus \bigcup_{m \geq n} A^c_{s_m,c}).
\end{aligned}$$

Therefore, there must exist a $n_0 \in I\!N$ such that

$$\mu_{\mathbf{M}}(A^c_{t,c} \setminus \bigcup_{n \geq n_0} A^c_{s_n,c}) > 0. \tag{12}$$

Now assume that $\liminf \mu_{\mathbf{M}}(A^c_{s_n,c} \setminus A^c_{t,c}) = 0$.
There would exist a subsequence $s_{n(k)}$ with the property

$$\lim_{k \longrightarrow \infty} \mu_{\mathbf{M}}(A^c_{s_{n(k)},c} \setminus A^c_{t,c}) = 0.$$

From this would follow since $\mu_{\mathbf{M}}(A^c_{t,c}) = \mu_{\mathbf{M}}(A^c_{s_n,c})$ for all $n \in \mathbb{N}$

$$\lim_{k \longrightarrow \infty} \mu_{\mathbf{M}}(A^c_{t,c} \setminus A^c_{s_{n(k)},c}) = 0.$$

This would contradict (1) since $A^c_{t,c} \setminus \bigcup_{m \geq n_0} A^c_{s_m,c} \subset A^c_{t,c} \setminus A^c_{s_{n(k)},c}$ for all $n(k) \geq n_0$.
Therefore, one must have $\liminf \mu_{\mathbf{M}}(A^c_{s_n,c} \setminus A^c_{t,c}) > 0$.
Since $\mathbf{M}$ has finite volume, we now know that $\mu_{\mathbf{M}}(\limsup A^c_{s_n,c} \setminus A^c_{t,c}) > 0$. Now, for points in $A^c_{t,c} \setminus \limsup A^c_{s_n,c}$ we know that $T_{\mathbf{M},t}f(x) < k_t$ and we know for the points in $\limsup A_{s_n,c} \setminus A_{t,c}$ that $T_{\mathbf{M},t}f(x) \leq k_t - \epsilon$. Therefore, we get a contradiction, since $A^c_{t,c} = \{x \in \mathbf{M}; T_{\mathbf{M},t}f(x) < k_t\}$.
From this we conclude, that there can not exist a sequence s_n converging to t with $k_{s_n} \leq k_t - \epsilon$.
So we have proven the continuity of k. □

Lemma 15 *For $f \in \mathcal{V}(\mathbf{M})$ one has: $\mathbf{1}_{A_{t,c}}$ converges $\mu_{\mathbf{M}}$-almost everywhere to $\mathbf{1}_{A_{t,c}}$ if s converges to t.*

Proof: Now assume that there exists a set $B \subset A_{t,c}$ with $\mu_{\mathbf{M}}(B) > 0$ such that for all $x \in B$ and for all $\delta > 0$ there exists a $s \in]t-\delta, t+\delta[$ with $T_{\mathbf{M},s}f(x) \leq k_s$. By the strong continuity of the semigroup $T_{\mathbf{M}}$ and the continuity of k one would have for all $x \in B$ that $T_{\mathbf{M},t}f(x) \leq k_t$. On the other hand, since $B \subset A_{t,c}$, one has for all $x \in B$ $T_{\mathbf{M},t}f(x) \geq k_t$. Therefore, our assumption would lead to the following contradiction

$$\mu_{\mathbf{M}}(\{x \in \mathbf{M}; T_{\mathbf{M},t}f(x) = k_t) \geq \mu_{\mathbf{M}}(B) > 0.$$

This is impossible by Lemma 1.
Now assume that there exists a set $C \subset A^c_{t,c}$ with $\mu_{\mathbf{M}}(C) > 0$ such that for all $x \in C$ and for all $\delta > 0$ there exists a $s \in]t-\delta, t+\delta[$ with $T_{\mathbf{M},s}f(x) \geq k_s$. By the strong continuity of the semigroup $T_{\mathbf{M}}$ and the continuity of k one would have for all $x \in C$: $T_{\mathbf{M},t}f(x) \geq k_t$. On the other hand, since $C \subset A^c_{t,c}$, one has for all $x \in C$ that $T_{\mathbf{M},t}f(x) \leq k_t$. Therefore, our assumption would lead to the following contradiction

$$\mu_{\mathbf{M}}(\{x \in \mathbf{M}; T_{\mathbf{M},t}f(x) = k_t) \geq \mu_{\mathbf{M}}(C) > 0.$$

This is impossible by Lemma 1.
Therefore, $\mathbf{1}_{A_{t,c}}$ converges $\mu_{\mathbf{M}}$-almost everywhere to $\mathbf{1}_{A_{t,c}}$ if s converges to t. □

4.2 The Differentiation of the Heat Content Function

Lemma 16 *Let $f \in \mathcal{V}(\mathbf{M})$ and $c \in [0, \mu_{\mathbf{M}}(\mathbf{M})]$. The function*

$$V :]0, \infty[\longrightarrow I\!R; t \mapsto \sup_{\mu_{\mathbf{M}}(A)=c} \int_A T_{\mathbf{M},t} f(x) d\mu_{\mathbf{M}}(x)$$

is differentiable. One has

$$(\partial_t V)(t) = \int_{A_{t,c}} \triangle T_{\mathbf{M},t} f(x) d\mu_{\mathbf{M}}(x).$$

Proof: For fixed t_0 set $R = \inf_{x \in A_{t_0,c}} T_{\mathbf{M},t_0} f(x)$. one has for all $x \in A_{t_0,c} \setminus A_{t_0+t,c}$

$$\begin{aligned} & T_{\mathbf{M},t_0+t} f(x) \\ = \; & T_{\mathbf{M},t_0} f(x) + t \triangle T_{\mathbf{M},t_0} f(x) + O(t^2) \\ \geq \; & R + t \triangle T_{\mathbf{M},t_0} f(x) + O(t^2) \\ \geq \; & R - t \| \triangle T_{\mathbf{M},t_0} f \|_{\text{sup}} + O(t^2). \end{aligned}$$

By construction of $A_{t,c}$ one has for all $x \in A_{t_0+t,c} \setminus A_{t_0,c}$

$$\begin{aligned} & T_{\mathbf{M},t_0+t} f(x) \\ = \; & T_{\mathbf{M},t_0} f(x) + t \triangle T_{t_0} f(x) + O(t^2) \\ \leq \; & R + t \triangle T_{\mathbf{M},t_0} f(x) + O(t^2) \\ \leq \; & R + t \| \triangle T_{\mathbf{M},t_0} f \|_{sup} + O(t^2). \end{aligned}$$

By construction of $A_{t,c}$ one has

$$\sup\{T_{\mathbf{M},t_0+t} f(x); x \in A_{t_0,c} \setminus A_{t_0+t,c}\} \leq \inf\{T_{\mathbf{M},t_0+t} f(x); x \in A_{t_0+t,c} \setminus A_{t_0,c}\}$$

and

$$\mu_{\mathbf{M}}(A_{t_0,c} \setminus A_{t_0+t,c}) = \mu_{\mathbf{M}}(A_{t_0+t,c} \setminus A_{t_0,c}).$$

Therefore, one gets from the above inequalities

$$\begin{aligned} & \mu_{\mathbf{M}}(A_{t_0,c} \setminus A_{t_0+t,c})(O(t^2) - 2t \| \triangle T_{\mathbf{M},t_0} f \|_{sup}) \\ = \; & \mu_{\mathbf{M}}(A_{t_0,c} \setminus A_{t_0+t,c})(R - t \| \triangle T_{\mathbf{M},t_0} f \|_{sup} + O(t^2)) \\ & -\mu_{\mathbf{M}}(A_{t_0+t,c} \setminus A_{t_0,c})(R + t \| \triangle T_{\mathbf{M},t_0} f \|_{sup} + O(t^2)) \\ \leq \; & \int_{A_{t_0,c} \setminus A_{t_0+t,c}} T_{\mathbf{M},t_0+t} f(x) d\mu_{\mathbf{M}}(x) - \int_{A_{t_0+t,c} \setminus A_{t_0,c}} T_{\mathbf{M},t_0+t} f(x) d\mu_{\mathbf{M}}(x) \\ \leq \; & 0. \end{aligned}$$

Now by the result of Lemma 15 one knows that for $t \longrightarrow 0$

$$\mu_{\mathbf{M}}(A_{t_0,c} \setminus A_{t_0+t,c}) \longrightarrow 0.$$

This implies that for all $\epsilon > 0$ one can find a suitable $t' > 0$ such that for all $t \in]0, t'[$ one has

$$\begin{aligned}
& \frac{1}{t}\left|\int_{A_{t_0,c}} T_{\mathbf{M},t_0} f(x) d\mu_{\mathbf{M}}(x) - \int_{A_{t_0+t,c}} T_{\mathbf{M},t_0+t} f(x) d\mu_{\mathbf{M}}(x) - t\int_{A_{t_0,c}} \triangle T_{\mathbf{M},t_0} f(x) d\mu_{\mathbf{M}}(x)\right| \\
\leq\ & \frac{1}{t}\left|\int_{A_{t_0,c}} T_{\mathbf{M},t_0} f(x) d\mu_{\mathbf{M}}(x) - \int_{A_{t_0,c}} T_{\mathbf{M},t_0+t} f(x) d\mu_{\mathbf{M}}(x) - t\int_{A_{t_0,c}} \triangle T_{\mathbf{M},t_0} f(x) d\mu_{\mathbf{M}}(x)\right| \\
& +\frac{1}{t}\left|\int_{A_{t_0,c}} T_{\mathbf{M},t_0+t} f(x) d\mu_{\mathbf{M}}(x) - \int_{A_{t_0+t,c}} T_{\mathbf{M},t_0+t} f(x) d\mu_{\mathbf{M}}(x)\right| \\
\leq\ & \epsilon + \frac{1}{t}\left|\int_{A_{t_0,c}\setminus A_{t_0+t,c}} T_{\mathbf{M},t_0+t} f(x) d\mu_{\mathbf{M}}(x) - \int_{A_{t_0+t,c}\setminus A_{t_0,c}} T_{\mathbf{M},t_0+t} f(x) d\mu_{\mathbf{M}}(x)\right| \\
\leq\ & 2\epsilon.
\end{aligned}$$

$\square$

Lemma 17 *Let $f \in \mathcal{V}(\mathbf{M})$. The function*

$$(t, c) \mapsto \int_{A_{t,c}} \triangle T_{\mathbf{M},t} f(x) d\mu_{\mathbf{M}}(x).$$

is continuous.

Proof: Since $\mathbf{1}_{A_{t,\tilde{c}}}$ converges pointwise to $\mathbf{1}_{A_{t,c}}$ if $\tilde{c}$ converges to c one gets the continuity in c by the dominated convergence theorem. Further, by Lemma 15 one knows that $\mathbf{1}_{A_{t',c}}$ converges $\mu_{\mathbf{M}}$-almost everywhere to $\mathbf{1}_{A_{t,c}}$ if t' converges to t. Therefore by the strong continuity of $T_{\mathbf{M}}$ one gets the $\mu_{\mathbf{M}}$-almost everywhere convergence of $\mathbf{1}_{A_{t',c}} \triangle T_{\mathbf{M},t'} f$ to $\mathbf{1}_{A_{t,c}} \triangle T_{\mathbf{M},t} f$. Now by dominated convergence theorem one gets the continuity in t of the given function. $\square$

4.3 The Heat Flux Comparison for Finite Time

Lemma 18 *Let $\mathbf{M}$ be a compact, connected, orientable Riemannian manifold without boundary and let $\mathbf{M}^*$ be a rotation manifold for $\mathbf{M}$. Let $f, g \in \mathcal{B}_b(\mathbf{M})$ and $K > 0$ with*

$$\sup_{\mu_{\mathbf{M}}(A)=c} \int_A f(x) d\mu_{\mathbf{M}}(x) \leq K + \int_{\hat{A}_c} \hat{g}(x) d\mu_{\mathbf{M}}(x)$$

for all $c \in [0, \mu_{\mathbf{M}}]$. *Then one has for all* $t \geq 0$

$$\int_{\hat{A}_c} T_{\mathbf{M}^*,t}\hat{f}(x)d\mu_{\mathbf{M}^*}(x) \leq K + \int_{\hat{A}_c} T_{\mathbf{M}^*,t}\hat{g}(x)d\mu_{\mathbf{M}^*}(x)$$

Proof: As was pointed out in the introduction the fundamental solution of the Laplace-Beltrami operator satisfies $p_{\mathbf{M}^*}(t,x,y) = p_{\mathbf{M}^*}(t,y,x)$ for all $t > 0$, $x, y \in \mathbf{M}^*$. Therefore, one has for all polar caps $\hat{A}_c$

$$\begin{aligned}
& \int_{\hat{A}_c} T_{\mathbf{M}^*,t}\hat{f}(x)d\mu_{\mathbf{M}^*}(x) \\
= & \int_{\hat{A}_c}\int_{\mathbf{M}^*} p_{\mathbf{M}^*}(t,x,y)\hat{f}(y)d\mu_{\mathbf{M}^*}(y)d\mu_{\mathbf{M}^*}(x) \\
= & \int_{\mathbf{M}^*}\int_{\hat{A}_c} p_{\mathbf{M}^*}(t,x,y)\hat{f}(y)d\mu_{\mathbf{M}^*}(x)d\mu_{\mathbf{M}^*}(y) \\
= & \int_{\mathbf{M}^*}(T_{\mathbf{M}^*,t}\mathbf{1}_{\hat{A}_c})(y)\hat{f}(y)d\mu_{\mathbf{M}^*}(y) \\
= & \int_0^1 \int_{\{T_{\mathbf{M}^*,t}\mathbf{1}_{\hat{A}_c} \geq s\}} \hat{f}(y)d\mu_{\mathbf{M}^*}(y)ds \\
\leq & \int_0^1 \left(\int_{\{T_{\mathbf{M}^*,t}\mathbf{1}_{\hat{A}_c} \geq s\}} \hat{g}(y)d\mu_{\mathbf{M}^*}(y) + K\right) ds \\
\leq & \int_{\mathbf{M}^*}(T_{\mathbf{M}^*,t}\mathbf{1}_{\hat{A}_c})(y)\hat{g}(y)d\mu_{\mathbf{M}^*}(y) + K \\
= & \int_{\hat{A}_c} T_{\mathbf{M}^*,t}\hat{g}(x)d\mu_{\mathbf{M}^*}(x) + K.
\end{aligned}$$

□

Theorem 4 *Let* $\mathbf{M}$ *be a compact, connected, orientable Riemannian manifold without boundary and let* $\mathbf{M}^*$ *be a rotation manifold for* $\mathbf{M}$. *Then one has for all* $f \in \mathcal{C}(\mathbf{M})$ *and all* $c \in [0, \mu_{\mathbf{M}}(\mathbf{M})]$

$$\sup_{\mu_{\mathbf{M}}(A)=c} \int_A T_{\mathbf{M},t}f(x)d\mu_{\mathbf{M}}(x) \leq \int_{\hat{A}_c} T_{\mathbf{M}^*,t}\hat{f}(x)d\mu_{\mathbf{M}^*}(x),$$

where $\hat{A}_c$ *is the ball around the north pole of* $\mathbf{M}^*$ *with volume c.*

Proof: First assume that $f \in \mathcal{V}(\mathbf{M})$. Define

$$K'_\epsilon(s) := \sup_{c\in[0,\mu_{\mathbf{M}}(\mathbf{M})]} \sup_{\delta_1,\delta_2\in[0,\epsilon]} \left(\int_{A_{s,c}} \triangle T_{\mathbf{M},s+\delta_1}f(x)d\mu_{\mathbf{M}}(x) - \int_{\hat{A}_c} \triangle \widehat{T_{\mathbf{M},s+\delta_2}f}(x)d\mu_{\mathbf{M}^*}(x)\right)$$

and $K_\epsilon(t) := \max(0, K'_\epsilon(t))$. We will prove by induction that for all $m \in I\!N$ and all $c \in [0, \mu_{\mathbf{M}}(\mathbf{M})]$

$$
\begin{aligned}
&\sup_{\mu_{\mathbf{M}}(A)} \int_A T_{\mathbf{M},m\frac{t}{n}} f(x) d\mu_{\mathbf{M}^*}(x) \\
\leq\ & \int_{\hat{A}_c} T_{\mathbf{M}^*,m\frac{t}{n}} \hat{f}(x) d\mu_{\mathbf{M}^*}(x) + \sum_{i=1}^{m} \frac{t}{n} K_{\frac{t}{n}} \left((i-1)\frac{t}{n} \right).
\end{aligned}
$$

The induction start for $m = 0$ is clear.
Now let the statement be true up to $m-1$.
By the mean value theorem and Lemma 16 there exist $\delta_1, \delta_2 \in [0, \frac{t}{n}]$ such that

$$
\begin{aligned}
& \sup_{\mu_{\mathbf{M}}(A)=c} \int_A T_{\mathbf{M},m\frac{t}{n}} f(x) d\mu_{\mathbf{M}}(x) - \int_{\hat{A}_c} T_{\mathbf{M}^*,\frac{t}{n}} \overbrace{T_{\mathbf{M},(m-1)\frac{t}{n}} f}(x) d\mu_{\mathbf{M}^*}(x) \\
= \ & \sup_{\mu_{\mathbf{M}}(A)=c} \int_A T_{\mathbf{M},(m-1)\frac{t}{n}} f(x) d\mu_{\mathbf{M}}(x) - \int_{\hat{A}_c} \overbrace{T_{\mathbf{M},(m-1)\frac{t}{n}} f}(x) d\mu_{\mathbf{M}^*}(x) \\
& + \frac{t}{n} \left(\int_{A_{(m-1)\frac{t}{n},c}} \triangle T_{\mathbf{M},(m-1)\frac{t}{n}+\delta_1} f(x) d\mu_{\mathbf{M}}(x) - \int_{\hat{A}_c} \triangle \overbrace{T_{\mathbf{M},(m-1)\frac{t}{n}+\delta_2} f}(x) d\mu_{\mathbf{M}^*}(x) \right) \\
\leq \ & 0 + \frac{t}{n} K_{\frac{t}{n}} \left((m-1)\frac{t}{n} \right).
\end{aligned}
$$

Therefore, one has

$$
\begin{aligned}
&\sup_{\mu_{\mathbf{M}}(A)=c} \int_A T_{\mathbf{M},m\frac{t}{n}} f(x) d\mu_{\mathbf{M}}(x) \\
\leq\ & \int_{\hat{A}_c} T_{\mathbf{M}^*,\frac{t}{n}} \overbrace{T_{\mathbf{M},(m-1)\frac{t}{n}} f}(x) d\mu_{\mathbf{M}^*}(x) + \frac{t}{n} K_{\frac{t}{n}} \left((m-1)\frac{t}{n} \right).
\end{aligned}
$$

Furthermore, one knows by the induction hypothesis and Lemma 18

$$
\begin{aligned}
&\int_{\hat{A}_c} T_{\mathbf{M}^*,\frac{t}{n}} \overbrace{T_{\mathbf{M},(m-1)\frac{t}{n}} f}(x) d\mu_{\mathbf{M}^*}(x) \\
\leq\ & \int_{\hat{A}_c} T_{\mathbf{M}^*,\frac{t}{n}} T_{\mathbf{M},(m-1)\frac{t}{n}} \hat{f}(x) d\mu_{\mathbf{M}^*}(x) + \sum_{i=1}^{m-1} \frac{t}{n} K_{\frac{t}{n}} \left((i-1)\frac{t}{n} \right).
\end{aligned}
$$

Therefore, one gets

$$
\begin{aligned}
&\sup_{\mu_{\mathbf{M}}(A)=c} \int_A T_{\mathbf{M},m\frac{t}{n}} f(x) d\mu_{\mathbf{M}}(x) \\
\leq\ & \int_{\hat{A}_c} T_{\mathbf{M}^*,\frac{t}{n}} T_{\mathbf{M},(m-1)\frac{t}{n}} \hat{f}(x) d\mu_{\mathbf{M}^*}(x)
\end{aligned}
$$

$$+\sum_{i=1}^{m-1}\frac{t}{n}K_{\frac{t}{n}}\left((i-1)\frac{t}{n}\right)+\frac{t}{n}K_{\frac{t}{n}}\left((m-1)\frac{t}{n}\right)$$

$$= \int_{\hat{A}_c} T_{\mathbf{M}^*,\frac{t}{n}}T_{\mathbf{M},(m-1)\frac{t}{n}}\hat{f}(x)d\mu_{\mathbf{M}^*}(x)+\sum_{i=1}^{m}\frac{t}{n}K_{\frac{t}{n}}\left((i-1)\frac{t}{n}\right).$$

This proofs the induction step. One has

$$\sum_{i=1}^{n}\frac{t}{n}K_{\frac{t}{n}}\left((i-1)\frac{t}{n}\right)=\int_0^t\sum_{i=1}^{n}H_n(s)ds.$$

Where

$$H_n(s):=K_{\frac{t}{n}}\left((i-1)\frac{t}{n}\right)\mathbf{1}_{[(i-1)\frac{t}{n},i\frac{t}{n}[}(s).$$

Now one knows by Lemma 17 that for $\epsilon\longrightarrow 0$ the functions K'_ϵ converge uniformly on $[0,t]$ to the function

$$K'(s):=\sup_{c\in[0,\mu_{\mathbf{M}}(\mathbf{M})]}\int_{A_{s,c}}\triangle T_{\mathbf{M},s}f(x)d\mu_{\mathbf{M}}(x)-\int_{\hat{A}_c}\triangle\widehat{T_{\mathbf{M},s}f}(x)d\mu_{\mathbf{M}^*}(x).$$

By theorem 3 one knows that K' is always less or equal to zero. Therefore, K_ϵ converges uniformly on $[0,t]$ to 0 if $\epsilon\longrightarrow 0$. From this follows that H_n converges pointwise on $[0,t]$ to 0. An application of the bounded convergence theorem proves the theorem if $f\in\mathcal{V}(\mathbf{M})$.

Now if $f\in\mathcal{C}(\mathbf{M})$, there exists a sequence of functions $f_n\in\mathcal{V}(\mathbf{M})$ converging in $\mathcal{L}^2(\mathbf{M})$ to f. From this one gets the pointwise convergence of $T_{\mathbf{M},t}f_n$ to $T_{\mathbf{M},t}f$. On the other side one knows by the result of the Appendix that the functions $\hat{f}_n$ converge to $\hat{f}$ pointwise. Therefore $T_{\mathbf{M}^*,t}\hat{f}_n$ converge pointwise to $T_{\mathbf{M}^*,t}\hat{f}$. Now for an arbitrary Borel set A with $\mu_{\mathbf{M}}(A)=c$ one knows by dominated convergence theorem that

$$\int_A T_{\mathbf{M},t}f_n(x)d\mu_{\mathbf{M}}(x)\longrightarrow\int_A T_{\mathbf{M},t}f(x)d\mu_{\mathbf{M}}(x)$$

and

$$\int_{\hat{A}_c}T_{\mathbf{M}^*,t}f_n(x)d\mu_{\mathbf{M}^*}(x)\longrightarrow\int_{\hat{A}_c}T_{\mathbf{M}^*,t}f(x)d\mu_{\mathbf{M}^*}(x).$$

Further, one knows that for all $n\in I\!N$

$$\int_A T_{\mathbf{M},t}f_n(x)d\mu_{\mathbf{M}}(x)\leq\int_{\hat{A}_c}T_{\mathbf{M}^*,t}f_n(x)d\mu_{\mathbf{M}^*}(x).$$

Therefore, one gets for all A with $\mu_{\mathbf{M}}(A)=c$

$$\int_A T_{\mathbf{M},t}f(x)d\mu_{\mathbf{M}}(x)\leq\int_{\hat{A}_c}T_{\mathbf{M}^*,t}f(x)d\mu_{\mathbf{M}^*}(x).$$

This leads to the conclusion of the theorem. □

5 The Perturbation of the Heat Flow

In this chapter, we analyse the behaviour of the heat flow under special perturbations. The perturbations considered here are value-preserving transformations of the heat distribution. The results will be generalised to time continuous perturbation by divergence-free vector fields. It is natural to consider divergence-free vector fields here, since the associated flows are value-preserving.

5.1 Comparison for Perturbed Heat Flow

Definition 5 *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be two compact, orientable, d-dimensional Riemannian manifolds without boundary and with same volume. Further, let* $f \in \mathcal{B}_b(\mathbf{M})$ *and* $g \in \mathcal{B}_b(\mathbf{N})$. f *will be called more concentrated than* g *(*$f \gg g$*) iff for all* $c \in [0, \mu_{\mathbf{M}}(\mathbf{M})]$ *one has*

$$\sup_{\mu_{\mathbf{M}}(A)=c} \int_A f(x)d\mu_{\mathbf{M}}(x) \geq \sup_{\mu_{\mathbf{M}}(B)=c} \int_B g(x)d\mu_{\mathbf{N}}(x).$$

Lemma 19 *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be two compact, orientable, d-dimensional Riemannian manifolds without boundary and with same volume. Further, let* $\mathbf{M}^*$ *be a comparison manifold for* $\mathbf{M}$, $f \in \mathcal{B}_b(\mathbf{M})$ *and* $g \in \mathcal{B}_b(\mathbf{N})$*. If* f *is more concentrated than* g*, then one has for all* $t \geq 0$

$$\sup_{x \in \mathbf{M}^*} T_{\mathbf{M}^*,t}\hat{f}(x) \leq \sup_{x \in \mathbf{M}^*} T_{\mathbf{M}^*,t}\hat{g}(x).$$

Proof: Let $x \in \mathbf{M}$ and $\epsilon > 0$. By Lemma 7 one has

$$\sup_{x \in \mathbf{M}^*} (T_{\mathbf{M}^*,t}\hat{g})(x) = (T_{\mathbf{M}^*,t}\hat{g})(\mathbf{n}).$$

Put $K_s := \mu_{\mathbf{M}^*}(\{y \in \mathbf{M}^*, p_{\mathbf{M}^*}(t, \mathbf{n}, y) \geq s\})$.
Then, one has

$$\begin{aligned}
& (T_{\mathbf{M}^*,t}\hat{g})(\mathbf{n}) \\
= & \int_{\mathbf{M}^*} p_{\mathbf{M}^*}(t, \mathbf{n}, y)\hat{g}(y)d\mu_{\mathbf{M}^*}(y) \\
= & \int_0^\infty \int_{\{y, p_{\mathbf{M}^*}(t,\mathbf{n},y) \geq s\}} \hat{g}(y)d\mu_{\mathbf{M}^*}(y)ds \\
= & \int_0^\infty \sup_{\mu_{\mathbf{M}^*}(A)=K_s} \int_A g(y)d\mu_{\mathbf{N}}(y)ds \\
\leq & \int_0^\infty \sup_{\mu_{\mathbf{M}^*}(A)=K_s} \int_A f(y)d\mu_{\mathbf{M}}(y)ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \int_{\{y, p_{\mathbf{M}^*}(t,\mathbf{n},y)\geq s\}} \hat{f}(y) d\mu_{\mathbf{M}^*}(y) ds \\
&\leq \int_{\mathbf{M}^*} p_{\mathbf{M}^*}(t,\mathbf{n},y) \hat{f}(y) d\mu_{\mathbf{M}^*}(y) \\
&= (T_{\mathbf{M}^*,t}\hat{f})(\mathbf{n}).
\end{aligned}
$$

This proves the lemma. □

Theorem 5 *Let* $\mathbf{M}$ *be a compact, connected, orientable Riemannian manifold without boundary and let* $\mathbf{M}^*$ *be a comparison manifold for* $\mathbf{M}$.
Further let $S_n : \mathcal{C}(\mathbf{M}) \longrightarrow \mathcal{C}(\mathbf{M})$ *be a sequence of value-preserving transformations and* $(t_n)_{n\in\mathbf{N}}$ *a sequence of positive real numbers. Define* $K_n := \prod_i^n T_{\mathbf{M},t_i} S_i$
Then for all $n \in I\!N$ *and for all* $f \in \mathcal{C}(\mathbf{M})$ *one has*

$$\overbrace{K_n f} \ll T_{\mathbf{M}^*,\sum_i^n t_i} \hat{f},$$

$$-\overbrace{K_n f} \ll -T_{\mathbf{M}^*,\sum_i^n t_i} \hat{f}.$$

Proof: One may prove the first part of the Theorem by induction. The statement is certainly true for $n=0$. Theorem 4 shows that for all $f \in \mathcal{C}(\mathbf{M})$

$$\overbrace{T_{\mathbf{M},t} f} \ll T_{\mathbf{M}^*,t} \hat{f}.$$

Therefore, one obtains with Lemma 8

$$\overbrace{K_{n+1} f} = \overbrace{T_{\mathbf{M},t_{n+1}} S_{n+1} K_n f} \ll T_{\mathbf{M}^*,t_{n+1}} \overbrace{S_{n+1} K_n f} = T_{\mathbf{M}^*,t_{n+1}} \overbrace{K_n f}.$$

Now, one sees using the induction hypothesis and Lemma 18

$$T_{\mathbf{M}^*,t_{n+1}} \overbrace{K_n f} \ll T_{\mathbf{M}^*,t_{n+1}} T_{\mathbf{M}^*,\sum_i^n t_i} \hat{f} = T_{\mathbf{M}^*,\sum_i^{(n+1)} t_i} \hat{f}.$$

This proves the first statement of the theorem. For the second statement one has by Lemma 9

$$-\overbrace{K_{n+1} f} = M(\overbrace{-K_{n+1} f}) = M(\overbrace{K_{n+1}(-f)})$$

Since M is value-preserving, one gets by the first inequality of the theorem and Lemma 10

$$M(\overbrace{K_{n+1}(-f)}) \ll M(T_{\mathbf{M}^*,\sum_i^{n+1} t_i} (\overbrace{-f})) = T_{\mathbf{M}^*,\sum_i^{n+1} t_i} M(\overbrace{-f})).$$

Using Lemma 9 again we get:

$$T_{\mathbf{M}^*,\sum_i^{n+1} t_i} M(\widehat{-f})) = -T_{\mathbf{M}^*,\sum_i^{n+1} t_i} \hat{f}.$$

This proves the second statement of the theorem. $\square$

Corollary 1 *Under the same assumptions as in Theorem 5 one has for all $n \in \mathbb{N}$, all $c \in [0, \mu_{\mathbf{M}}(\mathbf{M})]$ and all $f \in \mathcal{C}(\mathbf{M})$*

$$\sup_{\mu_{\mathbf{M}}(A)=c} \int_A K_n f(x) d\mu_{\mathbf{M}}(x) \leq \int_{\hat{A}_c} T_{\mathbf{M}^*,\sum_i^n t_i} \hat{f}(x) d\mu_{\mathbf{M}^*}(x),$$

$$\inf_{\mu_{\mathbf{M}}(A)=c} \int_A K_n f(x) d\mu_{\mathbf{M}}(x) \geq \int_{\check{A}_c} T_{\mathbf{M}^*,\sum_i^n t_i} \hat{f}(x) d\mu_{\mathbf{M}^*}(x).$$

Here $\check{A}_c$ denotes the ball of volume c around the south pole $\mathbf{s}$ and $\hat{A}_c$ denotes the ball of volume c around the north pole.

Proof: For the first inequality one has by the previous theorem

$$\begin{aligned}
& \sup_{\mu_{\mathbf{M}}(A)=c} \int_A K_n f(x) d\mu_{\mathbf{M}}(x) \\
\leq & \sup_{\mu_{\mathbf{M}^*}(A)=c} \int_A T_{\mathbf{M}^*,\sum_i^n t_i} \hat{f}(x) d\mu_{\mathbf{M}^*}(x) \\
\leq & \int_{\hat{A}_c} T_{\mathbf{M}^*,\sum_i^n t_i} \hat{f}(x) d\mu_{\mathbf{M}^*}(x).
\end{aligned}$$

For the second inequality one has by the previous theorem

$$\sup_{\mu_{\mathbf{M}}(A)=c} \int_A (-K_n f(x)) d\mu_{\mathbf{M}}(x) \leq \sup_{\mu_{\mathbf{M}^*}(A)=c} \int_A (-T_{\mathbf{M}^*,\sum_i^n t_i} \hat{f}(x)) d\mu_{\mathbf{M}^*}(x).$$

Therefore one gets

$$-\inf_{\mu_{\mathbf{M}}(A)=c} \int_A K_n f(x) d\mu_{\mathbf{M}}(x) \leq -\inf_{\mu_{\mathbf{M}^*}(A)=c} \int_A T_{\mathbf{M}^*,\sum_i^n t_i} \hat{f}(x) d\mu_{\mathbf{M}^*}(x).$$

And this gives

$$\begin{aligned}
& \inf_{\mu_{\mathbf{M}}(A)=c} \int_A K_n f(x) d\mu_{\mathbf{M}}(x) \\
\geq & \inf_{\mu_{\mathbf{M}^*}(A)=c} \int_A T_{\mathbf{M}^*,\sum_i^n t_i} \hat{f}(x) d\mu_{\mathbf{M}^*}(x) \\
\geq & \int_{\check{A}_c} T_{\mathbf{M}^*,\sum_i^n t_i} \hat{f}(x) d\mu_{\mathbf{M}^*}(x).
\end{aligned}$$

$\square$

Corollary 2 *Under the same assumptions as in Theorem 5 one has for all $n \in I\!N$ and all $f \in \mathcal{C}(\mathbf{M})$*

$$\sup_{x\in\mathbf{M}} K_n f(x) \le T_{\mathbf{M}^*,\sum_i^n t_i} \hat{f}(\mathbf{n}),$$

$$\inf_{x\in\mathbf{M}} K_n f(x) \ge T_{\mathbf{M}^*,\sum_i^n t_i} \hat{f}(\mathbf{s}).$$

Proof: For $x \in \mathbf{M}$, let $B_r(x)$ be the ball of radius $r \ge 0$ in $\mathbf{M}$.
Put $c(r,x) := \mu_{\mathbf{M}}(B_r(x))$. For the first inequality one has for all $x \in \mathbf{M}$

$$\begin{aligned} & K_n f(x) \\ = \ & \lim_{r\longrightarrow 0} \frac{1}{c(r,x)} \int_{B_r(x)} K_n f(y) d\mu_{\mathbf{M}}(y) \\ \le \ & \lim_{r\longrightarrow 0} \frac{1}{c(r,x)} \int_{\hat{A}_{c(r,x)}} K_n f(y) d\mu_{\mathbf{M}}(y) \\ = \ & T_{\mathbf{M}^*,\sum_i^n t_i} \hat{f}(\mathbf{n}). \end{aligned}$$

For the second inequality one has for all $x \in \mathbf{M}$

$$\begin{aligned} & K_n f(x) \\ = \ & \lim_{r\longrightarrow 0} \frac{1}{c(r,x)} \int_{B_r(x)} K_n f(y) d\mu_{\mathbf{M}}(y) \\ \ge \ & \lim_{r\longrightarrow 0} \frac{1}{c(r,x)} \int_{\check{A}_{c(r,x)}} K_n f(y) d\mu_{\mathbf{M}}(y) \\ = \ & T_{\mathbf{M}^*,\sum_i^n t_i} \hat{f}(\mathbf{s}). \end{aligned}$$

□

Corollary 3 *Under the same assumptions as in Theorem 5 one has for all $n \in I\!N$ and all $f \in \mathcal{C}(\mathbf{M})$*

$$\|\mathbf{K}_n f\|_\infty \le \|T_{\mathbf{M}^*,\sum_i^n t_i} \hat{f}\|_\infty.$$

Proof: This follows directly from the previous Corollary. □

5.2 The Comparison for Diffusions

In this subsection we want to concentrate on diffusions with divergence-free vector fields. Let b be a divergence-free vector field on $\mathbf{M}$. By Stokes theorem we know that the flow x_t generated by b on $\mathbf{M}$ preserves the Riemannian volume. This will be the key to the comparison results in this subsection.

Theorem 6 *Let* $\mathbf{M}$ *be a compact, connected, orientable Riemannian manifold without boundary and let* $\mathbf{M}^*$ *be a rotation manifold for* $\mathbf{M}$. *Then one has for all divergence-free vector fields* b *on* $\mathbf{M}$ *and all* $f \in \mathcal{C}(\mathbf{M})$

$$\sup_{\mu_{\mathbf{M}}(A)} \int_A K_{\mathbf{M},b,t} f(x) d\mu_{\mathbf{M}}(x) \leq \int_{\hat{A}_c} T_{\mathbf{M}^*,t} \hat{f}(x) d\mu_{\mathbf{M}^*}(x),$$

$$\inf_{\mu_{\mathbf{M}}(A)} \int_A K_{\mathbf{M},b,t} f(x) d\mu_{\mathbf{M}}(x) \geq \int_{\check{A}_c} T_{\mathbf{M}^*,t} \hat{f}(x) d\mu_{\mathbf{M}^*}(x).$$

Proof: For all smooth vector fields b on $\mathbf{M}$ the semi-groups $K_{\mathbf{M},b}$ are strongly continuous on $\mathcal{C}(\mathbf{M})$. Further for all smooth vector fields b on $\mathbf{M}$ the semi-groups $S_{\mathbf{M},b}$ generated by the flow associated to b are strongly continuous on $\mathcal{C}(\mathbf{M})$. Therefore, by Trotters product formula for every $f \in \mathcal{C}(\mathbf{M})$ and for all $\epsilon > 0$ there exists a $n_0 \in I\!N$ such that for all $n \geq n_0$

$$\left\| K_{\mathbf{M},b,t} f - (\prod_{i=1}^{n} T_{\mathbf{M},\frac{t}{n}} S_{\mathbf{M},b,\frac{t}{n}}) f \right\|_{\sup} \leq \epsilon$$

(see [EthKur] p.33). Now for an arbitrary $A \subset \mathbf{M}$ by dominated convergence theorem there exists for all $\epsilon > 0$ a $n_0 \in I\!N$ such that for all $n \geq n_0$

$$\left| \int_A K_{\mathbf{M},b,t} f(x) d\mu_{\mathbf{M}} - \int_A (\prod_{i=1}^{n} T_{\mathbf{M},\frac{t}{n}} S_{\mathbf{M},b,\frac{t}{n}}) f(x) d\mu_{\mathbf{M}} \right| \leq \epsilon.$$

Therefore, one gets for all $n \geq n_0$

$$\begin{aligned} & -\epsilon + \int_A (\prod_{i=1}^{n} T_{\mathbf{M},\frac{t}{n}} S_{\mathbf{M},b,\frac{t}{n}}) f(x) d\mu_{\mathbf{M}} \\ \leq & \int_A K_{\mathbf{M},b,t} f(x) d\mu_{\mathbf{M}} \\ \leq & \ \epsilon + \int_A (\prod_{i=1}^{n} T_{\mathbf{M},\frac{t}{n}} S_{\mathbf{M},b,\frac{t}{n}}) f(x) d\mu_{\mathbf{M}}. \end{aligned}$$

From this follows by Corollary 1 for $A \subset \mathbf{M}$

$$-\epsilon + \int_{\check{A}} T_{\mathbf{M}^*,t} \hat{f}(x) d\mu_{\mathbf{M}^*} \leq \int_A K_{\mathbf{M},b,t} f(x) d\mu_{\mathbf{M}} \leq \epsilon + \int_{\hat{A}} T_{\mathbf{M}^*,t} \hat{f}(x) d\mu_{\mathbf{M}^*}.$$

From this follows if ϵ converges to zero for all $A \subset \mathbf{M}$:

$$\int_{\check{A}} T_{\mathbf{M}^*,t} \hat{f}(x) d\mu_{\mathbf{M}^*} \leq \int_A K_{\mathbf{M},b,t} f(x) d\mu_{\mathbf{M}} \leq \int_{\hat{A}} T_{\mathbf{M}^*,t} \hat{f}(x) d\mu_{\mathbf{M}^*}.$$

This is just the statement of the theorem. $\square$

Theorem 7 *Let $\mathbf{M}$ be a compact, connected, orientable Riemannian manifold without boundary and let $\mathbf{M}^*$ be a rotation manifold for $\mathbf{M}$. Then for all divergence-free vector field b on $\mathbf{M}$ and for all $c \in [0, \mu_{\mathbf{M}}(\mathbf{M})]$ one has the following inequalities*

$$\sup_{\mu_{\mathbf{M}}(A)=c} \sup_{\mu_{\mathbf{M}}(B)=b} \int_A \int_B q_{\mathbf{M},b}(t,x,y) d\mu_{\mathbf{M}}(y) d\mu_{\mathbf{M}}(x)$$
$$\leq \int_{\hat{A}_c} \int_{\hat{B}(\mathbf{n})_b} p_{\mathbf{M}^*}(t,x,y) d\mu_{\mathbf{M}^*}(y) d\mu_{\mathbf{M}^*}(x),$$

$$\inf_{\mu_{\mathbf{M}}(A)=c} \inf_{\mu_{\mathbf{M}}(B)=b} \int_A \int_B q_{\mathbf{M},b}(t,x,y) d\mu_{\mathbf{M}}(y) d\mu_{\mathbf{M}}(x)$$
$$\geq \int_{\check{A}_c} \int_{\hat{B}(\mathbf{n})_b} p_{\mathbf{M}^*}(t,x,y) d\mu_{\mathbf{M}^*}(y) d\mu_{\mathbf{M}^*}(x).$$

Proof: Let B be a Borel set in $\mathbf{M}$ with $\mu_{\mathbf{M}}(B) = b$. $\mathcal{C}(\mathbf{M})$ is dense in $\mathcal{L}^1(\mathbf{M})$. Therefore, there exists a sequence of continuous functions $(f_{B,n})_{n\in\mathbf{N}}$ with $\lim_{n\to\infty} \|f_{B,n} - \mathbf{1}_B\|_1 = 0$. By Lemma 5 one has that $\hat{f}_{B,n}$ converges μ-almost sure to $\mathbf{1}_{\hat{B}(\mathbf{n})_b}$. Now, one gets the following inequalities from 6

$$\begin{aligned}
& \int_A \int_B q_{\mathbf{M},b}(t,x,y) d\mu_{\mathbf{M}}(y) d\mu_{\mathbf{M}}(x) \\
= & \lim_{k\longrightarrow\infty} \int_A \int_{\mathbf{M}} f_{B,n}(y) q_{\mathbf{M},b}(t,x,y) d\mu_{\mathbf{M}}(y) d\mu_{\mathbf{M}}(x) \\
= & \lim_{k\longrightarrow\infty} \int_A K_{\mathbf{M},b,t} f_{B,n}(x) d\mu_{\mathbf{M}}(x) \\
\leq & \lim_{k\longrightarrow\infty} \int_{\hat{A}_c} T_{\mathbf{M}^*,t} \hat{f}_{B,n}(x) d\mu_{\mathbf{M}^*}(x) \\
= & \int_{\hat{A}_c} \int_{\hat{B}(\mathbf{n})_b} p_{\mathbf{M}^*}(t,x,y) d\mu_{\mathbf{M}^*}(y) d\mu_{\mathbf{M}^*}(x).
\end{aligned}$$

On the other hand, one has

$$\begin{aligned}
& \int_A \int_B q_{\mathbf{M},b}(t,x,y) d\mu_{\mathbf{M}}(y) d\mu_{\mathbf{M}}(x) \\
= & \lim_{k\longrightarrow\infty} \int_A \int_{\mathbf{M}} f_{B,n}(y) q_{\mathbf{M},b}(t,x,y) d\mu_{\mathbf{M}}(y) d\mu_{\mathbf{M}}(x) \\
= & \lim_{k\longrightarrow\infty} \int_A K_{\mathbf{M},b,t} f_{B,n}(x) d\mu_{\mathbf{M}}(x) \\
\geq & \lim_{k\longrightarrow\infty} \int_{\check{A}_c} T_{\mathbf{M}^*,t} \hat{f}_{B,n}(x) d\mu_{\mathbf{M}^*}(x) \\
= & \int_{\check{A}_c} \int_{\hat{B}(\mathbf{n})_b} p_{\mathbf{M}^*}(t,x,y) d\mu_{\mathbf{M}^*}(y) d\mu_{\mathbf{M}^*}(x).
\end{aligned}$$

□

Theorem 8 *Let* $\mathbf{M}$ *be a compact, connected, orientable Riemannian manifold without boundary and let* $\mathbf{M}^*$ *be a rotation manifold for* $\mathbf{M}$*. Then for all divergence-free vector field* b *on* $\mathbf{M}$ *and for all* $c \in [0, \mu_{\mathbf{M}}(\mathbf{M})]$ *one has the following inequalities*

$$\sup_{x\in\mathbf{M}} \sup_{\mu_{\mathbf{M}}(A)=c} \int_A q_{\mathbf{M},b}(t,x,y)d\mu_{\mathbf{M}}(y) \le \int_{\hat{A}_c} p_{\mathbf{M}^*}(t,\mathbf{n},y)d\mu_{\mathbf{M}^*}(y),$$

$$\inf_{x\in\mathbf{M}} \inf_{\mu_{\mathbf{M}}(A)=c} \int_A q_{\mathbf{M},b}(t,x,y)d\mu_{\mathbf{M}}(y) \ge \int_{\hat{A}_c} p_{\mathbf{M}^*}(t,\mathbf{s},y)d\mu_{\mathbf{M}^*}(y).$$

Proof: Let $B_r(y)$ be the ball of radius r in $\mathbf{M}$. Put $c(r,y) := \mu_{\mathbf{M}}(B_r(y))$. Let $A \subset \mathbf{M}$ with $\mu_{\mathbf{M}}(A) = c$. Then, one has for all $x \in \mathbf{M}$ by Theorem 7

$$\begin{aligned}
& \int_A q_{\mathbf{M},b}(t,x,y)d\mu_{\mathbf{M}}(y) \\
= \ & \lim_{r\to 0} \frac{1}{c(r,x)} \int_{B_r(x)} \int_A q_{\mathbf{M},b}(t,z,y)d\mu_{\mathbf{M}}(y)d\mu_{\mathbf{M}}(z) \\
\le \ & \lim_{r\to 0} \frac{1}{c(r,x)} \int_{\hat{B}(\mathbf{n})_{c(r,x)}} \int_{\hat{A}_c} p_{\mathbf{M}^*}(t,z,y)d\mu_{\mathbf{M}^*}(y)d\mu_{\mathbf{M}^*}(z) \\
= \ & \int_{\hat{A}_c} p_{\mathbf{M}^*}(t,\mathbf{n},y)d\mu_{\mathbf{M}^*}(y).
\end{aligned}$$

On the other hand, one has for all $x \in \mathbf{M}$ by Theorem 7

$$\begin{aligned}
& \int_A q_{\mathbf{M},b}(t,x,y)d\mu_{\mathbf{M}}(y) \\
= \ & \lim_{r\to 0} \frac{1}{c(r,x)} \int_{B_r(x)} \int_A q_{\mathbf{M},b}(t,z,y)d\mu_{\mathbf{M}}(y)d\mu_{\mathbf{M}}(z) \\
\ge \ & \lim_{r\to 0} \frac{1}{c(r,x)} \int_{\hat{B}(\mathbf{s})_{c(r,x)}} \int_{\hat{A}_c} p_{\mathbf{M}^*}(t,z,y)d\mu_{\mathbf{M}^*}(y)d\mu_{\mathbf{M}^*}(z) \\
= \ & \int_{\hat{A}_c} p_{\mathbf{M}^*}(t,\mathbf{s},y)d\mu_{\mathbf{M}^*}(y).
\end{aligned}$$

$\square$

Corollary 4 *Under the same assumptions as in Theorem 8 one has for all divergence-free vector fields* b *on* $\mathbf{M}$ *and all* $f \in \mathcal{B}_b(\mathbf{M})$

$$\sup_{x\in\mathbf{M}} K_{\mathbf{M},b,t}f(x) \le T_{\mathbf{M}^*,t}\hat{f}(\mathbf{n}),$$

$$\inf_{x\in\mathbf{M}} K_{\mathbf{M},b,t}f(x) \ge T_{\mathbf{M}^*,t}\hat{f}(\mathbf{s}).$$

Proof: For an arbitrary $x \in \mathbf{M}$ one has by Theorem 8

$$\begin{aligned}
& K_{\mathbf{M},b,t} f(x) \\
= & \int_{\mathbf{M}} q_{\mathbf{M},b}(t,x,y) f(y) d\mu_{\mathbf{M}}(y) \\
= & \int_0^\infty \int_{\{x; f+\inf f \geq s\}} q_{\mathbf{M},b}(t,x,y) d\mu_{\mathbf{M}}(y) - \inf f \\
\leq & \int_0^\infty \int_{\{x; \hat{f}+\inf \hat{f} \geq s\}} p_{\mathbf{M}^*}(t,\mathbf{n},y) d\mu_{\mathbf{M}^*}(y) - \inf \hat{f} \\
= & \int_{\mathbf{M}^*} p_{\mathbf{M}^*}(t,\mathbf{n},y) f(y) d\mu_{\mathbf{M}^*}(y) \\
= & T_{\mathbf{M}^*,t} \hat{f}(\mathbf{n}),
\end{aligned}$$

and

$$\begin{aligned}
& K_{\mathbf{M},b,t} f(x) \\
= & \int_{\mathbf{M}} q_{\mathbf{M},b}(t,x,y) f(y) d\mu_{\mathbf{M}}(y) \\
= & \int_0^\infty \int_{\{x; f+\inf f \geq s\}} q_{\mathbf{M},b}(t,x,y) d\mu_{\mathbf{M}}(y) - \inf f \\
\geq & \int_0^\infty \int_{\{x; \hat{f}+\inf \hat{f} \geq s\}} p_{\mathbf{M}^*}(t,\mathbf{s},y) d\mu_{\mathbf{M}^*}(y) - \inf \hat{f} \\
= & \int_{\mathbf{M}^*} p_{\mathbf{M}^*}(t,\mathbf{s},y) f(y) d\mu_{\mathbf{M}^*}(y) \\
= & T_{\mathbf{M}^*,t} \hat{f}(\mathbf{s}).
\end{aligned}$$

□

Theorem 9 *Let $\mathbf{M}$ be a compact, connected, orientable Riemannian manifold without boundary and let $\mathbf{M}^*$ be a rotation manifold for $\mathbf{M}$. Then for all divergence-free vector field b on $\mathbf{M}$ one has the following inequalities*

$$\sup_{x,y \in \mathbf{M}} q_{\mathbf{M},b}(t,x,y) \leq p_{\mathbf{M}^*}(t,\mathbf{n},\mathbf{n}),$$

$$\inf_{x,y \in \mathbf{M}} q_{\mathbf{M},b}(t,x,y) \geq p_{\mathbf{M}^*}(t,\mathbf{n},\mathbf{s}).$$

Proof: Let $B_r(y)$ be the ball of radius r in $\mathbf{M}$. Put $c(r,y) := \mu_{\mathbf{M}}(B_r(y))$. Then one has for all $x, y \in \mathbf{M}$

$$q_{\mathbf{M},b}(t,x,y)$$

$$
\begin{aligned}
&= \lim_{r\to 0}\frac{1}{c(r,y)}\int_{B_r(y)} q_{\mathbf{M},b}(t,x,z)d\mu_{\mathbf{M}}(z)\\
&\le \lim_{r\to 0}\frac{1}{c(r,y)}\int_{\hat{A}_{c(r)}} p_{\mathbf{M}^*}(t,\mathbf{n},z)d\mu_{\mathbf{M}^*}(z)\\
&= p_{\mathbf{M}^*}(t,\mathbf{n},\mathbf{n}).
\end{aligned}
$$

On the other hand one has for all $x, y \in \mathbf{M}$

$$
\begin{aligned}
& q_{\mathbf{M},b}(t,x,y)\\
&= \lim_{r\to 0}\frac{1}{c(r,y)}\int_{B_r(y)} q_{\mathbf{M},b}(t,x,z)d\mu_{\mathbf{M}}(z)\\
&\ge \lim_{r\to 0}\frac{1}{c(r,y)}\int_{\hat{A}_{c(r)}} p_{\mathbf{M}^*}(t,\mathbf{s},z)d\mu_{\mathbf{M}^*}(z)\\
&= \inf_{y\in\mathbf{M}^*} p_{\mathbf{M}^*}(t,\mathbf{s},\mathbf{n}).
\end{aligned}
$$

□

The following results follow directly from the previous theorems.

Corollary 5 *Under the same assumptions as in Theorem 8 one has for all divergence-free vector fields b on $\mathbf{M}$ and all $f \in \mathcal{B}_b(\mathbf{M})$*

$$\|K_{b,\mathbf{M},t} f\|_\infty \le \|T_{\mathbf{M}^*,t}\hat{f}\|_\infty.$$

Proof: One has for all $x \in \mathbf{M}$ by corollary 4

$$T_{\mathbf{M}^*,t}\hat{f}(\mathbf{s}) \le \inf_{x\in\mathbf{M}} K_{\mathbf{M},b,t} f(x) \le \sup_{x\in\mathbf{M}} K_{\mathbf{M},b,t} f(x) \le T_{\mathbf{M}^*,t}\hat{f}(\mathbf{n}).$$

□

Corollary 6 *Under the same assumptions as Theorem in 6 there exists a $C > 0$ and a $\lambda > 0$ such that for all vector fields b on $\mathbf{M}$ with $\operatorname{div}(b) = 0$ and for all $f \in \mathcal{B}_b(\mathbf{M})$ with $\int_{\mathbf{M}} f d\mu = 0$ one has*

$$\|K_{b,\mathbf{M},t} f\|_\infty \le C\|f\|_\infty \exp(-\lambda t).$$

Proof: By a classical result from ergodic theory (see [BenLioPap] p.365) there exist a $C > 0$ and a $\lambda > 0$ such that for all $f \in \mathcal{B}_b(\mathbf{M}^*)$ with $\int_{\mathbf{M}^*} f d\mu_{\mathbf{M}^*} = 0$ and for all $t \ge 0$ one has

$$\|T_{\mathbf{M}^*,t} f\|_\infty \le C\|f\|_\infty \exp(-\lambda t).$$

Further, one has $\int_{\mathbf{M}} f d\mu = \int_{\mathbf{M}^*} \hat{f} d\mu$. Therefore one gets by Corollary 5

$$\|K_{\mathbf{M},b,t} f\|_\infty \le \|T_{\mathbf{M}^*,t}\hat{f}\|_\infty \le C\|\hat{f}\|_\infty \exp(-\lambda t) = C\|f\|_\infty \exp(-\lambda t).$$

□

Corollary 7 *Under the same assumptions as in Theorem 6 there exists a* $\tilde{C} > 0$ *such that for all vector fields* b *with* $div(b) = 0$ *and for all* $f \in \mathcal{B}(\mathbf{S}^d)$ *with* $\int_{\mathbf{M}} f d\mu = 0$ *and for all solutions of the Poisson equations* $A_b u = f$ *one has*

$$\|u\|_\infty \le \tilde{C}\|f\|_\infty.$$

Proof: By a classical result from functional analysis the solution of the Poisson equation $A_b u = f$ is given by $u(x) = \int_0^\infty K_{\mathbf{M},b,t} f(x) dt$ if $\int_{\mathbf{M}} f d\mu_{\mathbf{M}} = 0$. Therefore by corollary 6 one gets

$$\begin{aligned}
\|u\|_\infty &= \|\int_0^\infty K_{\mathbf{M},b,t} f(.) dt\|_\infty \\
&\le \int_0^\infty \|K_{\mathbf{M},b,t} f(.)\|_\infty dt \\
&\le \int_0^\infty C\exp(-\lambda t)\|f\|_\infty dt \\
&= \frac{C}{\lambda}\|f\|_\infty.
\end{aligned}$$

□

6 The Example of the Sphere

The following part of this thesis is devoted to the situation on the sphere. In this section we want to introduce the concept of polar rearrangement for the sphere. Since the sphere is in some sense its own comparison manifold, the results of this section are special cases of the results in the second section. We first introduce some simplified notations.

6.1 Notations for the Sphere

Let $\mathbf{S}_a^d$ be the d-dimensional sphere of radius a. There is a canonical embedding of the sphere in $\mathbb{R}^{d+1}$ $\mathbf{S}_a^d = \{x \in \mathbb{R}^{d+1}, \|x\| = a\}$. In polar coordinates $\theta_1, ..., \theta_d$, one has $x_1 = a\cos\theta_1$, $x_i = a\cos\theta_i \prod_{j=1}^{i-1} \sin\theta_j$ for $i = 2, 3, ..., d$ and $x_{d+1} = a\prod_{j=1}^{d} \sin\theta_j$. The domain of the θ_i is $[0, \pi]$ for all $i \le d-1$ and $[0, 2\pi]$ for θ_d. Let μ_a^d be the surface measure of $\mathbf{S}_a^d$. One knows that

$$\mu_a^d(\mathbf{S}_a^d) = \frac{\sqrt{\pi}^{d+1} a^d (d+1)}{\Gamma(\frac{d+1}{2})},$$

where Γ denotes the Γ-function (see [Mad] p.243). For $|b| \le a$ put

$$K_b := \{x \in \mathbf{S}_a^d, x_1 = b\}.$$

The sets K_b define a fibration of the sphere $\mathbf{S}_a^d$. $\mu^{d-1}_{\sqrt{a^2-b^2}}$ is then naturally defined on K_b since K_b can be identified with $\mathbf{S}^{d-1}_{\sqrt{a^2-b^2}}$.
For a set $A \in \mathbf{S}_a^d$ put $A_y := \{x \in \mathbb{R}^{d+1}; x_1 = y\}$. Further, put

$$\tilde{A} := \{(\theta_1, \theta_2, ..., \theta_d) \in [0,\pi]^{d-1} \times [0, 2\pi]; a(\cos\theta_1, ..., \cos\theta_d \prod_{j=1}^{d-1} \sin\theta_j, \prod_{j=1}^{d} \sin\theta_j) \in A\}$$

and

$$\tilde{A}_\psi := \{(\theta_2, ..., \theta_d) \in [0,\pi]^{d-1} \times [0, 2\pi]; a(\cos\psi, ..., \cos\theta_d \prod_{j=1}^{d-1} \sin\theta_j, \prod_{j=1}^{d} \sin\theta_j) \in A\}.$$

From this follows

$$\mu_a^d(A) = \int_{\tilde{A}} a^d \prod_{j=1}^{d} \sin(\theta_j)^{d-j} d\theta_j$$

$$= \quad a^d \int_0^\pi \sin(\theta_1)^{d-1} \int_{\tilde{A}_{\theta_1}} \prod_{j=2}^{d} \sin(\theta_j)^{d-j} d\theta_j d\theta_1$$

$$
\begin{aligned}
&= a\int_0^\pi \mu^{d-1}_{a\sin\theta_1}(A_{a\cos\theta_1})d\theta_1 \\
&= \int_{-a}^{a} \frac{a}{\sqrt{a^2-x_1^2}}\mu^{d-1}_{\sqrt{a^2-x_1^2}}(A_{x_1})dx_1.
\end{aligned}
$$

The point $\mathbf{n} = (0, ..., 0, a) \in I\!R^{d+1}$ will be called the north pole of $\mathbf{S}^d_a$. The point $\mathbf{s} = (0, ..., 0, -a) \in \mathbf{R}^{d+1}$ will be called the south pole of $\mathbf{S}^d_a$. Let $\mathcal{B}_b(\mathbf{S}^d)$ be the space of bounded measurable real valued functions on $\mathbf{S}^d$ with $\mathcal{L}_\infty$-semi-norm. Let $\mathcal{C}(\mathbf{S}^d_a)$ be the space of continuous functions on $\mathbf{S}^d_a$ with supremum-norm. $\mathcal{C}(\mathbf{S}^d_a)$ is a Banach space. Denote by $\mathcal{C}^\infty(\mathbf{S}^d_a)$ the space of all infinite differentiable real valued functions on $\mathbf{S}_a$. Let $\triangle$ be the Laplace-Beltrami operator on $\mathbf{S}^d_a$. The heat equation on $\mathbf{S}^d$ with initial condition $f \in \mathcal{C}(\mathbf{S}^d_a)$ is given by

$$
\begin{aligned}
&\triangle u(x,t) = \partial_t u(x,t), \\
&u(x,0) = f(x).
\end{aligned}
$$

There exist fundamental solutions

$$
p :]0,\infty[\times \mathbf{S}^d_a \times \mathbf{S}^d_a \longrightarrow]0,\infty[
$$

with the properties:

$$
\begin{aligned}
&\triangle_x p(t,x,y) = \partial_t p(t,x,y), \\
&\lim_{t\to 0} p(t,x,y) = \delta_x(y),
\end{aligned}
$$

where the limit is taken in the sense of distributions. Since the fundamental solutions are smooth (see [GetBerVer] p.86), they can be considered as Feller transition probabilities (see [Tai] p.329). Therefore, they generate a strongly continuous semigroup T on $\mathcal{C}(\mathbf{S}^d_a)$ by the formula

$$
T_t f(x) = \int_{\mathbf{S}^d_a} p(t,x,y) f(y) d\mu(y)
$$

(see [Tai] p.333).
Let b be a smooth vector field on $\mathbf{S}^d_a$. $A_b := \triangle + b$ is an elliptic differential operator of second order on $\mathcal{C}^\infty(\mathbf{S}^d_a)$. Consider the generalised heat equation on $\mathbf{S}^d_a$ with initial condition f given by

$$
\begin{aligned}
&\triangle u(x,t) + bu(x,t) = \partial_t u(x,t), \\
&u(x,0) = f(x).
\end{aligned}
$$

There exist fundamental solutions

$$q_b :]0,\infty[\times \mathbf{S}_a^d \times \mathbf{S}_a^d \longrightarrow]0,\infty[$$

with the properties:

$$\begin{aligned} &A_{b,x} q_b(t,x,y) = \partial_t q_b(t,x,y), \\ &\lim_{t\to 0} q_b(t,x,y) = \delta_x(y), \end{aligned}$$

where the limit is taken in the space of distributions on $\mathbf{S}_a^d$ (see [GetBerVer] p.75). Since the fundamental solutions of the generalised heat equation are smooth (see [GetBerVer] p.86), they can be considered as Feller transition probabilities (see [Tai] p.329). Therefore, they generate a strongly continuous semigroup K_b on $\mathcal{C}(\mathbf{M})$ by the formula

$$K_b f(x) = \int_{\mathbf{S}_a^d} q_b(t,x,y) f(y) d\mu(y).$$

(see [Tai] p.333). Notice that $K_0 = T$.

6.2 The Polar Rearrangement

For the rest of this section we denote μ_a^d by μ. For $f \in \mathcal{B}(\mathbf{S}_a^d)$ we denote by μ_f the image measure of μ on the real line: $\mu_f(A) = \mu(f^{-1}(A))$.
For $f \in \mathcal{B}_b(\mathbf{S}_a^d)$ we call the set $\mathcal{M}_f := \{g \in \mathcal{B}_b(\mathbf{S}_a^d), \mu_f = \mu_g\}$ the valuefold of f.

Definition 6 *A transformation $S : \mathcal{B}_b(\mathbf{S}_a^d) \longrightarrow \mathcal{B}_b(\mathbf{S}_a^d)$ with the property $S(\mathcal{M}_f) \subset \mathcal{M}_f$ for all $f \in \mathcal{B}_b(\mathbf{S}_a^d)$ will be called value-preserving.*

In measure theory a map $s : \mathbf{S}_a^d \longrightarrow \mathbf{S}_a^d$ is called measure preserving, iff

$$\mu(s^{-1}(A)) = \mu(A).$$

Lemma 20 *Let $s : \mathbf{S}_a^d \longrightarrow \mathbf{S}_a^d$ be a transformation preserving the measure μ. The map $S : \mathcal{B}_b(\mathbf{S}_a^d) \longrightarrow \mathcal{B}_b(\mathbf{S}_a^d); f \mapsto f \circ s$ is then value-preserving.*

Proof: For a Borel set $A \in \mathbb{R}$ and a function $f \in \mathcal{B}_b(\mathbf{S}_a^d)$ one has

$$\mu_{Sf}(A) = \mu_{f\circ s}(A) = \mu(s^{-1}f^{-1}(A)) = \mu(f^{-1}(A)) = \mu_f(A).$$

This proves the statement of the Lemma. $\square$

Let $F_f : \mathbf{R} \longrightarrow [0,\infty]$ be the distribution function of μ_f. One can define the following function:

$$F_f^- : [0, \mu(\mathbf{S}_a^d)] \longrightarrow I\!R; x \mapsto \sup(F^{-1}([0,x])).$$

The function F_f^- is right continuous an increasing. Some properties of this construction can be found in the Appendix. The function F_f^- can be used to construct a rearrangement of the function f on the d- dimensional sphere $\mathbf{S}_a^d$.

Definition 7 *For $f \in \mathcal{B}_b(\mathbf{S}_a^d)$ the function*

$$\hat{f} : \mathbf{S}_a^d \longrightarrow I\!R; x \mapsto F_f^-(\mu(B_{d(x,\mathbf{s})}(\mathbf{s})))$$

will be called the polar rearrangement of the function f on $\mathbf{S}_a^d$.

For larger expressions we want to use the following notation for the polar rearrangement: $\widehat{\text{larger expression}}$.

Lemma 21 *Polar rearrangement on $\mathbf{S}_a^d$ is value-preserving.*

Proof: Let $F_{\hat{f}}$ be the distribution function of $\mu_{\hat{f}}$. By Lemma 34 one has

$$\{x, F_f^-(\mu(B_{d(x,\mathbf{s})}(\mathbf{s}))) \leq a\} \subset \{x, \mu(B_{d(x,\mathbf{s})}(\mathbf{s})) \leq F_f(a)\}.$$

Furthermore, one has: by Lemma 35

$$\{x, \mu(B_{d(x,\mathbf{s})}(\mathbf{s})) < F_f(a)\} \subset \{x, F_f^-(\mu(B_{d(x,\mathbf{s})}(\mathbf{s}))) \leq a\}.$$

Since the set $\{x, \mu(B_{d(x,\mathbf{s})}(\mathbf{s})) = F_f(a)\}$ is a zero set with respect to μ we see

$$\begin{aligned} & \mu(\{x, \hat{f}(x) \leq a\}) \\ = \; & \mu(\{x, F_f^-(\mu(B_{d(x,\mathbf{s})}(\mathbf{s}))) \leq a\}) \\ = \; & \mu(\{x, \mu(B_{d(x,\mathbf{s})}(\mathbf{s})) \leq F_f(a)\}). \end{aligned}$$

Furthermore, one can see that $\{x, \mu(B_{d(x,\mathbf{s})}(\mathbf{s})) \leq F_f(a)\}$ is a ball around the south pole $\mathbf{s}$ of $\mathbf{S}_a^d$ with volume $F_f(a)$. Therefore, one has

$$F_{\hat{f}}(a) = \mu(\{x, \hat{f}(x) \leq a\}) = \mu(\{x, \mu(B_{d(x,\mathbf{s})}(\mathbf{s})) \leq F_f(a)\}) = F_f(a).$$

Therefore one has $\mu_f = \mu_{\hat{f}}$. □

From this follows that $\mathcal{L}_p$ norms are preserved by polar rearrangement:
$\|f\|_p = \|\hat{f}\|_p$ for all $f \in \mathcal{B}_b(\mathbf{S}_a^d)$.

Lemma 22 *If $f \in \mathcal{C}(\mathbf{S}_a^d)$ then $\hat{f} \in \mathcal{C}(\mathbf{S}_a^d)$.*

Proof: Since f is continuous every nonempty open set in $[\inf f(x), \sup f(x)]$ has positive measure with respect to μ_f. Therefore F_f is strictly increasing. From this follows by Lemma 36 that F_f^- is continuous. Since the function $x \mapsto \mu(B_{d(\mathbf{s},x)}(\mathbf{s}))$ is continuous the lemma is true. □

Let $f \in \mathcal{B}_b(\mathbf{S}_a^d)$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence of Borel measurable bounded functions. In measure theory f_n is called convergent to f in measure, iff for all $\epsilon > 0$

$$\mu_a^d(\{x; |f_n(x) - f(x)| > \epsilon) \longrightarrow 0.$$

Lemma 23 *Let $f \in \mathcal{B}_b(\mathbf{S}_a^d)$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence of Borel measurable bounded functions converging in measure to f. Then $(\hat{f}_n)_{n\in\mathbb{N}}$ converges to $\hat{f}$ μ_a^d-almost everywhere.*

Proof: From Lemma 37 we know that in this situation $F_{f_n}^-$ converges to F_f^- λ-almost everywhere. Furthermore, for a zero set $K \subset [0, \mu(\mathbf{S}_a^d)]$ the set $\{x; \mu(B_{d(x,\mathbf{s})}(\mathbf{s}) \in K\}$ is a μ-zero set. It then follows from the definition of the polar rearrangement, that $\hat{f}_n$ converges to $\hat{f}$ μ-almost everywhere. □

Lemma 24 *Let $G \in SO(n+1)$ be a rotation fixing the x_{d+1}-axis. Then one has for all $f \in \mathcal{B}_b(\mathbf{S}_a^d)$ the following properties:*
i) $\hat{f} \circ G = \hat{f}$,
ii) Further $\hat{f}$ is decreasing with the distance to $\mathbf{n}$.
On the other hand if a function $g \in \mathcal{M}_f$ satisfies the conditions i) and ii) one has: $\hat{f} = g$ μ-almost sure.

Proof: The properties i) follows from the fact that $B_{d(G(x),\mathbf{s})}(\mathbf{s}) = B_{d(x,\mathbf{s})}(\mathbf{s})$. Property ii) of $\hat{f}$ follows from the monotonicity of the measure μ. For the second part let $g \in \mathcal{M}_f$ be a function satisfying i) and ii). By property i) and ii) one has $\{y; g(y) \le g(x)\} = B_{d(x,\mathbf{s})}(\mathbf{s})$. And therefore one gets

$$\begin{aligned} \hat{f}(x) &= F_f^-(\mu(B_{d(x,\mathbf{s})}(\mathbf{s}))) = F_g^-(\mu(B_{d(x,\mathbf{s})}(\mathbf{s}))) \\ &= F_g^-(\mu(\{y; g(y) \le g(x)\})) = F_g^-(F_g(g(x))) = g(x). \end{aligned}$$

□

Lemma 25 *Let $p : \mathbf{S}_a^d \times \mathbf{S}_a^d \times]0,\infty[\longrightarrow [0,\infty[$ be the fundamental solutions of the heat equation on $\mathbf{S}_a^d$. Then one has for all $x \in \mathbf{S}_a^d$ and $t \in]0,\infty[$*

$$\widehat{p(t,x,.)} = p(t,\mathbf{n},.).$$

Proof: Since rotations are isometries of the sphere, one has $p(t, \mathbf{n}, .) \in \mathcal{M}_{p(t,x,.)}$. Furthermore, $p(t, \mathbf{n}, .)$ has the properties i) and ii) of Lemma 24. Therefore, the statement follows from Lemma 24. $\square$

Lemma 26 *For all $f \in \mathcal{B}(\mathbf{S}_a^d)$ and for all $t \geq 0$ one has*

$$\widehat{T_t \hat{f}} = T_t \hat{f}.$$

Proof: It is clear that $T_t \hat{f}$ satisfies the properties i) and ii) of Lemma 24. Therefore, $T_t \hat{f}$ is a fix point under polar rearrangement. $\square$

Lemma 27 *For all value-preserving transformations $S : \mathcal{B}_b(\mathbf{S}_a^d) \longrightarrow \mathcal{B}_b(\mathbf{S}_a^d)$ one has*

$$\widehat{Sf} = \hat{f}.$$

Proof: This follows directly from the definition of the polar rearrangement since $F_{Sf} = F_f$. $\square$

For $\mathbf{S}_a^d = \{x \in I\!R^{d+1}; \|x\| = a\}$
put $m : \mathbf{S}_a^d \longrightarrow \mathbf{S}_a^d; (x_1, x_2, ..., x_{d+1}) \mapsto (x_1, x_2, ..., -x_{d+1})$, and let
$M : \mathcal{B}_b(\mathbf{S}_a^d) \longrightarrow \mathcal{B}_b(\mathbf{S}_a^d); f \mapsto f \circ m$.

Lemma 28 *One has for all $f \in \mathcal{B}(\mathbf{S}_a^d)$*

$$-\hat{f} = M(\widehat{-f}).$$

Proof: By construction the function $-M(\widehat{-f})$ is increasing with the distance to $\mathbf{n}$ and invariant by rotations fixing the x_{d+1}-axis. Since m is measure preserving M is value preserving by Lemma 20. This implies that: $-M(\widehat{-f}) \in \mathcal{M}_f$. Therefore, by Lemma 24 one gets $-\hat{f} = M(\widehat{-f})$. $\square$

Lemma 29 *One has for all $t \geq 0$*

$$M \circ T_t = T_t \circ M.$$

Proof: Since $d(m(x), y) = d(x, m(y))$, one has by the symmetries of the sphere

$$p(t, m(x), y) = p(t, x, m(y)).$$

From this follows

$$
\begin{aligned}
& (T_t M f)(x) \\
= & \int_{\mathbf{S}_a^d} p(t,x,y) f(m(y)) d\mu(y) \\
= & \int_{\mathbf{S}_a^d} p(t,x,m(y)) f(y) d\mu(y) \\
= & \int_{\mathbf{S}_a^d} p(t,m(x),y) f(y) d\mu(y) \\
= & (M T_t f)(x).
\end{aligned}
$$

□

7 Set Rearrangements on the Sphere

In this section, we will prove the following concentration inequality:

$$\begin{aligned} &\sup_{\mu_a^d(B)=c}\ \sup_{\mu_a^d(A)=b} \mu_a^d \times \mu_a^d(A \times B \cap \{(x,y); d(x,y) \le r\}) \\ \le\ & \mu_a^d \times \mu_a^d(\hat{A} \times \hat{B} \cap \{(x,y); d(x,y) \le r\}), \end{aligned} \tag{13}$$

where $\hat{A}$ is the ball around the north pole with volume b and $\hat{B}$ is the ball around the north pole with volume c. This inequality will be used in the following section to understand the fundamental solutions of the heat equation on $\mathbf{S}_a^d$. In order to prove the above inequality, we define two rearrangements of a Borel set A. The first one denoted by TA will be called the northern rearrangement and the second denoted by SA will be called the western rearrangement. Both rearrangements are measure preserving and do increase the functional

$$A \times B \mapsto \mu_a^d \times \mu_a^d(A \times B \cap \{(x,y); d(x,y) \le r\}).$$

Furthermore, we will see that $\mu_a^d((ST)^n A \bigtriangleup \hat{A})$ converges to zero as n converges to infinity. Both facts will play a crucial part in the proof of the concentration inequality.

7.1 The Rearrangement of Sets

Let $K_b := \{x \in \mathbf{S}_a^d, x_1 = b\}$ where $|b| \le a$. The sets K_b define a fibration of the sphere $\mathbf{S}_a^d$. The measure $\mu^{d-1}_{\sqrt{a^2-b^2}}$ is then naturally defined on K_b since K_b can be identified with $\mathbf{S}^{d-1}_{\sqrt{a^2-b^2}}$. For a Borel set $A \subset \mathbf{S}_a^d$ put $A_y := \{x \in A; x_1 = y\}$.

One has for all Borel sets $A \in \mathbf{S}_a^d$

$$\mu_a^d(A) = \int_{-a}^{a} \frac{a}{\sqrt{a^2 - x_1^2}} \mu^{d-1}_{\sqrt{a^2-x_1^2}}(A_{x_1}) dx_1. \tag{14}$$

Call $n_b := (b, 0, ..., \sqrt{a^2 - b^2})$ the north pole of K_b.
Let $A \in \mathbf{S}_a^d$ be an arbitrary Borel set.
Then one can look for the ball $B_{r_A(b)}(n_b)$ in K_b around n_b with

$$\mu^{d-1}_{\sqrt{a^2-b^2}}(B_{r_A(b)}(n_b)) = \mu^{d-1}_{\sqrt{a^2-b^2}}(A \cap K_b).$$

One can see with formula 14, that the set $TA := \bigcup_{|b| \le a} B_{r_A(b)}(n_b)$ is certainly a measurable subset of $\mathbf{S}_a^d$ with $\mu_a^d(TA) = \mu_a^d(A)$.
We call TA the northern rearrangement of A.

Definition 8 *For $d \geq 2$ Borel set $A \in \mathbf{S}_a^d$ will be called in northern shape iff there exists a function $f : [-a, a] \longrightarrow [-a, a]$ such that*

$$A = \{x \in \mathbf{S}_a^d; x_{d+1} \geq f(x_1)\}.$$

It is obvious, that TA is in northern shape for all Borel sets A.

Let $\hat{A}$ be the ball around the north pole $\mathbf{n}$ with $\mu_a^d(\hat{A}) = \mu_a^d(A)$. There exists a suitable $c \in [-a, a]$ such that $\hat{A} = \{x \in \mathbf{S}_a^d; x_{d+1} \geq c\}$.

Now lets consider another fibration of the sphere $\mathbf{S}_a^d$.
Define $H_b := \{x \in \mathbf{S}_a^d; x_{d+1} = b\}$ for $b \in [-a, a]$. As before, H_b can be identified with $\mathbf{S}^{d-1}_{\sqrt{a^2-b^2}}$. Therefore, it is endowed with the measure $\mu^{d-1}_{\sqrt{a^2-b^2}}$. Call $w_b := (0, \sqrt{a^2 - b^2}, ..., b)$ the west pole of H_b.
Let $A \in \mathbf{S}_a^d$ be an arbitrary Borel set.
Then one can look for the ball $B_{r_A(b)}(w_b)$ in H_b around w_b with

$$\mu^{d-1}_{\sqrt{a^2-b^2}}(B_{r_A(b)}(w_b)) = \mu^{d-1}_{\sqrt{a^2-b^2}}(A \cap H_b).$$

One can see with formula 14, that the set $SA := \bigcup_{|b| \leq a} B_{r_A(b)}(n_b)$ is certainly a measurable subset of $\mathbf{S}_a^d$ with $\mu_a(SA) = \mu_a(A)$.
We call SA the western rearrangement of A.

Definition 9 *For $d \geq 2$ a Borel set $A \in \mathbf{S}_a^d$ will be called in western shape iff there exists a function $f : [-a, a] \longrightarrow [-a, a]$ such that*

$$A = \{x \in \mathbf{S}_a^d; x_2 \geq f(x_{d+1})\}.$$

Obviously, SA is in western shape.

Further, one can see that $\mu_a^d(\hat{A} \triangle SA) = \mu_a^d(\hat{A} \triangle A)$ since moving points in H_b do not change their distance to $\mathbf{n}$.

Lets denote by $\mathcal{W}_{p,q}$ the collection of all Borel sets $A \in \mathbf{S}_a^d$ in western shape satisfying $\mu_a^d(A) = p$ and $\mu_a^d(A \triangle \hat{A}) \geq q$.

For all $A \in \mathcal{W}_{p,q}$ define the following functions:

$$g_A : [-a, a] \longrightarrow [0, \infty[; b \mapsto \mu^{d-1}_{\sqrt{a^2-b^2}}(K_b \cap A \backslash \hat{A}),$$

$$h_A : [-a,a] \longrightarrow [0,\infty[; b \mapsto \mu^{d-1}_{\sqrt{a^2-b^2}}(K_b \cap \hat{A}\backslash A).$$

The following formulas are true for any $A \in \mathcal{V}_{p,q}$:

$$\int_{-a}^{a} \frac{a}{\sqrt{a^2-b^2}} h_A(b)db + \int_{-a}^{a} \frac{a}{\sqrt{a^2-b^2}} g_A(b)db = \mu_a^d(A \bigtriangleup \hat{A}) \geq q, \tag{15}$$

$$\int_{-a}^{a} \frac{a}{\sqrt{a^2-b^2}} h_A(b)db - \int_{-a}^{a} \frac{a}{\sqrt{a^2-b^2}} g_A(b)db = 0. \tag{16}$$

Adding up 15 and 16 one obtains

$$\int_{-a}^{a} \frac{a}{\sqrt{a^2-b^2}} h_A(b)db \geq \frac{1}{2}q \tag{17}$$

and by taking differences one gets

$$\int_{-a}^{a} \frac{a}{\sqrt{a^2-b^2}} g_A(b)db \geq \frac{1}{2}q. \tag{18}$$

Since g_A and h_A are symmetric, one sees that

$$\int_{0}^{a} \frac{a}{\sqrt{a^2-b^2}} h_A(b)db \geq \frac{1}{4}q \tag{19}$$

and

$$\int_{-a}^{a} \frac{a}{\sqrt{a^2-b^2}} g_A(b)db \geq \frac{1}{2}q. \tag{20}$$

Further, one has by construction of the northern rearrangement

$$\mu_a^d(A \bigtriangleup \hat{A}) - \mu_a^d(TA \bigtriangleup \hat{A}) = \int_{-a}^{a} \frac{a}{\sqrt{a^2-b^2}} \min(g_A(b), h_A(b))db. \tag{21}$$

Lemma 30 *For $d \geq 2$ let $p \in]0, \mu_a^d(\mathbf{S}_a^d)]$ and $q \in]0,p]$ be fixed. There exists a $\beta \in]0,a]$ such that for all $b \in [0,\beta]$ there exist an $\epsilon(b) > 0$ with the properties:*

$$\inf_{A \in \mathcal{W}_{p,q}} g_A(b) \geq \epsilon(b),$$

$$\inf_{A \in \mathcal{W}_{p,q}} h_A(b) \geq \epsilon(b).$$

Proof: Let β be implicitly defined by

$$\int_0^\beta \frac{a}{\sqrt{a^2-b^2}}\mu^{d-1}_{\sqrt{a^2-b^2}}(\mathbf{S}^{d-1}_{\sqrt{a^2-b^2}})db = \frac{1}{8}q.$$

The existence of β follows from the formula 19 since $h_A(b) \leq \mu^{d-1}_{\sqrt{a^2-b^2}}(\mathbf{S}^{d-1}_{\sqrt{a^2-b^2}})$. Lets first consider the case $d \geq 3$:

$$\begin{aligned}
&h_A(b) = \mu^{d-1}_{\sqrt{a^2-b^2}}(K_b \cap \hat{A}\backslash A)\\
&= \mu^{d-1}_{\sqrt{a^2-b^2}}(\{x \in \mathbf{S}^d_a; x_1 = b, x_2 \leq f_A(x_{d+1}), x_{d+1} \geq c\})\\
&= \int_c^{\sqrt{a^2-b^2}} \frac{\sqrt{a^2-b^2}}{\sqrt{a^2-b^2-u^2}}\mu^{d-2}_{\sqrt{a^2-b^2-u^2}}(\{x \in \mathbf{S}^d_a; x_1 = b, x_2 \leq f_A(x_{d+1}), x_{d+1} = u\})du\\
&= \int_c^{\sqrt{a^2-b^2}} \int_{-\sqrt{a^2-b^2-u^2}}^{f_A(u)} \frac{\sqrt{a^2-b^2}}{\sqrt{a^2-b^2-u^2-v^2}}\mu^{d-3}_{\sqrt{a^2-b^2-u^2-v^2}}(\mathbf{S}^{d-3}_{\sqrt{a^2-b^2-u^2-v^2}})dvdu\\
&= \int_c^{\sqrt{a^2-b^2}} \int_{-\sqrt{a^2-b^2-u^2}}^{f_A(u)} \frac{\sqrt{a^2-b^2}}{\sqrt{a^2-b^2-u^2-v^2}} \frac{\sqrt{\pi}^{d-2}\sqrt{a^2-b^2-u^2-v^2}^{d-3}(d-2)}{\Gamma(\frac{d-2}{2})}dvdu\\
&= \frac{\sqrt{a^2-b^2}\sqrt{\pi}^{d-2}(d-2)}{\Gamma(\frac{d-2}{2})}\int_c^{\sqrt{a^2-b^2}} \int_{-\sqrt{a^2-b^2-u^2}}^{f_A(u)} \sqrt{a^2-b^2-u^2-v^2}^{d-4}dvdu.
\end{aligned}$$

Now if for a fixed $b \in [0, \beta]$ one had a sequence of sets $A_n = \{x \in \mathbf{S}^d_a; x_2 \geq f_n(x_{d+1})\} \in \mathcal{W}_{p,q}$ with $\lim_{n\to\infty} h_{A_n}(b) = 0$, the sequence

$$K_n(u) := \int_{-\sqrt{a^2-b^2-u^2}}^{f_n(u)} \sqrt{a^2-b^2-u^2-v^2}^{d-4}dv$$

would converge in $\mathcal{L}^1([c, \sqrt{a^2-b^2}], \lambda)$ to zero. There would exist a subsequence converging λ-almost everywhere. So without loss of generality we can assume that there would exist a sequence $A_n \in \mathcal{W}_{p,q}$ such that $K_n(u) \longrightarrow 0$ for λ-almost all $u \in [c, \sqrt{a^2-b^2}]$. This would imply

$$\max(f_n(u), -\sqrt{a^2-b^2-u^2}) \longrightarrow -\sqrt{a^2-b^2-u^2}$$

for λ-almost all $u \in [c, \sqrt{a^2-b^2}]$. From this would then follow the convergence of h_{A_n} to zero on $[b, a]$. But this would give the following contradiction:

$$\begin{aligned}
&\frac{1}{4}q \leq \int_0^a \frac{a}{\sqrt{a^2-s^2}}h_{A_n}(s)ds\\
&= \int_0^b \frac{a}{\sqrt{a^2-s^2}}h_{A_n}(s)ds + \int_b^a \frac{a}{\sqrt{a^2-s^2}}h_{A_n}(s)ds
\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^\beta \frac{a}{\sqrt{a^2-s^2}} h_{A_n}(s)ds + \int_b^a \frac{a}{\sqrt{a^2-s^2}} h_{A_n}(s)ds \\
&\leq \frac{1}{8}q + \int_b^a \frac{a}{\sqrt{a^2-s^2}} h_{A_n}(s)ds \longrightarrow \frac{1}{8}q.
\end{aligned}$$

Therefore, there exists for each $b \in [0,\beta]$ an $\epsilon(b) > 0$ such that

$$\inf_{A \in \mathcal{W}_{p,q}} h_A(b) \geq \epsilon_1(b).$$

With the same proof one can show that there exists for every $b \in [0,\beta]$ an $\epsilon_2(b) > 0$ such that

$$\inf_{A \in \mathcal{W}_{p,q}} g_A(b) \geq \epsilon_1(b).$$

By taking $\epsilon(b) := \min(\epsilon_1(b), \epsilon_2(b))$ one gets the statement of the lemma for $d \geq 3$. Now lets consider the case $d = 2$. One has

$$\begin{aligned}
& h_A(b) = \mu^1_{\sqrt{a^2-b^2}}(K_b \cap \hat{A} \backslash A) \\
&= \mu^1_{\sqrt{a^2-b^2}}(\{x \in \mathbf{S}^2_a; x_1 = b, x_2 \leq f_A(x_3), x_3 \geq c\}) \\
&= \int_c^{\sqrt{a^2-b^2}} \frac{\sqrt{a^2-b^2}}{\sqrt{a^2-b^2-u^2}} \mu^0_{\sqrt{a^2-b^2-u^2}}(\{x \in \mathbf{S}^2_a; x_1 = b, x_3 = u, x_2 \leq f_A(u)\})du \\
&= \int_c^{\sqrt{a^2-b^2}} \frac{\sqrt{a^2-b^2}}{\sqrt{a^2-b^2-u^2}} \mu^0_{\sqrt{a^2-b^2-u^2}}(\{x \in \mathbf{S}^2_a; x_1 = b, x_3 = u, \sqrt{a^2-b^2-u^2} \leq f_A(u)\})du \\
&\quad + \int_c^{\sqrt{a^2-b^2}} \frac{\sqrt{a^2-b^2}}{\sqrt{a^2-b^2-u^2}} \mu^0_{\sqrt{a^2-b^2-u^2}}(\{x \in \mathbf{S}^2_a; x_1 = b, x_3 = u, -\sqrt{a^2-b^2-u^2} \leq f_A(u)\})du \\
&= \int_c^{\sqrt{a^2-b^2}} \frac{\sqrt{a^2-b^2}}{\sqrt{a^2-b^2-u^2}} \left(\mathbf{1}_{\{r; \sqrt{a^2-b^2-r^2} \leq f_A(r)\}}(u) + \mathbf{1}_{\{r; -\sqrt{a^2-b^2-r^2} \leq f_A(r)\}}(u)\right) du.
\end{aligned}$$

If for a fixed $b \in [0,\beta]$ one had a sequence of sets $A_n = \{x \in \mathbf{S}^2_a; x_2 \geq f_n(x_3)\} \in \mathcal{W}_{p,q}$ with $\lim_{n\to 0} h_{A_n}(b) = 0$, this would imply the $\mathcal{L}^1([c, \sqrt{a^2-b^2}], \lambda)$-convergence to zero of the functions

$$K_n(r) := \mathbf{1}_{\{r; \sqrt{a^2-b^2-r^2} \leq f_n(r)\}}(r) + \mathbf{1}_{\{r; -\sqrt{a^2-b^2-r^2} \leq f_n(r)\}}(r).$$

There would exist a subsequence converging λ-almost surely on $[c, \sqrt{a^2-b^2}]$. Therefor without loss of generality we could assume the existence of a sequence of sets $A_n \in \mathcal{W}_{p,q}$ with

$$K_n(r) \longrightarrow 0$$

for λ-almost all $r \in [c, \sqrt{a^2-b^2}]$. This implies

$$\max(f_n(r), -\sqrt{a^2-b^2-r^2}) \longrightarrow -\sqrt{a^2-b^2-r^2}$$

for λ-almost all $r \in [c, \sqrt{a^2 - b^2}]$. One would get the convergence of $h_{A_n}(b)$ to zero for every $b \in [b, a]$. But this would give the following contradiction:

$$\begin{aligned}
\frac{1}{4}q &= \int_0^a \frac{a}{\sqrt{a^2 - s^2}} h_{A_n}(s) ds \\
&= \int_0^b \frac{a}{\sqrt{a^2 - s^2}} h_{A_n}(s) ds + \int_b^a \frac{a}{\sqrt{a^2 - s^2}} h_{A_n}(s) ds \\
&< \int_0^\beta \frac{a}{\sqrt{a^2 - s^2}} h_{A_n}(s) ds + \int_b^a \frac{a}{\sqrt{a^2 - s^2}} h_{A_n}(s) ds \\
&\leq \frac{1}{8}q + \int_b^a \frac{a}{\sqrt{a^2 - s^2}} h_{A_n}(s) ds \longrightarrow \frac{1}{8}q.
\end{aligned}$$

Therefor there exists for all $b \in [0, \beta]$ an $\epsilon_1(b) > 0$ such that

$$\inf_{A \in \mathcal{S}_{p,q}} h_A(b) \geq \epsilon_1(b).$$

With the same proof one can show that there exists for every $b \in [0, \beta]$ an $\epsilon_2(b) > 0$ such that

$$\inf_{A \in \mathcal{S}_{p,q}} g_A(b) \geq \epsilon_2(b).$$

By taking $\epsilon(b) := \min(\epsilon_1(b), \epsilon_2(b))$ one gets the statement of the lemma for $d \geq 2$. Therefore, the lemma is proved. $\square$

Corollary 8 *Let $p \in]0, \mu_a^d(\mathbf{S}_a^d)]$ and $q \in]0, p]$, then there exists an $\epsilon > 0$ such that for all Borel sets $A \in \mathbf{S}_a^d$ in western shape with the properties $\mu_a^d(A) = p$ and $\mu_a^d(A \bigtriangleup \hat{A}) \geq q$ one has $\mu_a^d(TA \bigtriangleup \hat{A}) \leq \mu_a^d(A \bigtriangleup \hat{A}) - \epsilon$.*

Proof: By formula 21 one has

$$\mu_a^d(A \bigtriangleup \hat{A}) - \mu_a^d(TA \bigtriangleup \hat{A}) = \int_{-a}^a \frac{a}{\sqrt{a^2 - b^2}} \min(g_A(b), h_A(b)) db.$$

Lemma 30 implies for all $b \in [0, \beta]$ the existence of an $\epsilon(b) > 0$ such that

$$\inf_{A \in \mathcal{S}_{p,q}} \min(h_A(b), g_A(b)) \geq \epsilon(b).$$

From this follows

$$\epsilon := \inf_{A \in \mathcal{V}_{p,q}} \int_{-a}^a \frac{a}{\sqrt{a^2 - b^2}} \min(h_A(b), g_A(b)) db > 0.$$

This is just the statement of the lemma. $\square$

Corollary 9 *The sequence $\mu_a^d((TS)^n A \bigtriangleup \hat{A})$ converges to zero if n goes to infinity.*

Proof: This is a straight forward consequence of Corollary 8 and the fact, that $\mu_a^d(\hat{A} \bigtriangleup SA) = \mu_a^d(\hat{A} \bigtriangleup A)$. $\square$

7.2 The Concentration Inequality

For $u \in [0, \mu_a^d(\mathbf{S}_a^d)]$ we denote by $\hat{A}_u$ the ball of volume u in $\mathbf{S}_a^d$ around the north pole $\mathbf{n}$.

Lemma 31 *Let $\mathbf{S}_a^1 = \{x \in I\!R^2; \|x\|_2 = a\}$. For $0 \leq b, c \leq \mu_a^1(\mathbf{S}_a^1)$ and $r \geq 0$ one has*

$$\begin{aligned} &\sup_{\mu_a^1(A)=b, \mu_b^1(B)=c} (\mu_a^1 \times \mu_a^1)(A \times B \cap \{(x,y), \|x-y\|_2 \leq r\}) \\ &\leq (\mu_a^1 \times \mu_a^1)(\hat{A} \times \hat{B} \cap \{(x,y), \|x-y\|_2 \leq r\}). \end{aligned}$$

Here $\hat{A}$ is the ball around the north pole $\mathbf{n}$ with volume b and $\hat{B}$ is the volume around the north pole with volume c.

Proof: For the proof lets identify $\mathbf{S}_a^1$ with $[-\pi a, \pi a[$. One has

$$\begin{aligned} &(\mu_a^1 \times \mu_a^1)(A \times B \cap \{(x,y); \|x-y\|_2 \leq r\}) \\ = &\int_A \mu_a^1(B \cap \{(x,y); \|x-y\|_2 \leq r\}) d\mu_a^1(x). \end{aligned}$$

Define

$$\begin{aligned} g_B(x) &:= \mu_a^1(B \cap \{(x,y); \|x-y\|_2 \leq r\}), \\ A_B^1(b) &:= \{x \in \mathbf{S}_a^1; \mu_a^1(\{y \in \mathbf{S}_a^1; g_A(y) > g_A(x)\}) \leq b\}, \\ A_B^2(b) &:= \{x \in \mathbf{S}_a^1; \mu_a^1(\{y \in \mathbf{S}_a^1; g_A(y) \geq g_A(x)\}) \leq b\}. \end{aligned}$$

Since $\mu_a^1(A_B^1(b)) \geq b$, $\mu_a^1(A_B^2(b)) \leq b$ and $A_B^2(b) \subset A_B^1(b)$ one can choose a set $A_B(b)$ with $A_B^2(b) \subset A_B(b) \subset A_B^1(b)$ and $\mu_a^1(A_B(b)) = b$.
Now, it is clear that

$$\begin{aligned} &\sup_{\mu_a^1(A)=b} \int_A \mu_a^1(B \cap \{(x,y); \|x-y\|_2 \leq r\}) d\mu_a^1(x) \\ = &\int_{A_B} \mu_a^1(B \cap \{(x,y); \|x-y\|_2 \leq r\}) d\mu_a^1(x). \end{aligned}$$

It is sufficient to prove the statement of the lemma when B is a finite union of open intervals

$$B = \bigcup_{j=1}^n I_j = \bigcup_{j=1}^n]\alpha_j, \beta_j[.$$

Lets compare

$$f_B(b) := \int_{A_B(b)} \mu_a^1(B \cap \{(x,y); \|x-y\|_2 \leq r\}) d\mu_a^1(x)$$

with

$$f_{\hat{B}}(b) := \int_{A_{\hat{B}}(b)} \mu_a^1(\hat{B} \cap \{(x,y); \|x-y\|_2 \le r\}) d\mu_a^1(x).$$

f_B and $f_{\hat{B}}$ are both differentiable. Further one has

$$f'_B(b) = g_B(\partial A_B(b))$$

and

$$f'_{\hat{B}}(b) = g_{\hat{B}}(\partial A_{\hat{B}}(b)).$$

Therefore, one has

$$f_{\hat{B}}(b) = \int_0^b g_{\hat{B}}(\partial A_{\hat{B}}(t))dt$$

and

$$f_B(b) = \int_0^b g_B(\partial A_B(t))dt.$$

i) There exists a $b' \in [0, 2a\pi]$ with $g_B(\partial A_B(b)) \le g_{\hat{B}}(\partial A_{\hat{B}}(b))$ for all $b \le b'$ and $g_{\hat{B}}(\partial A_{\hat{B}}(b)) \le g_B(\partial A_B(b))$ for all $b \ge b'$.
This can be seen by considering the shape of $g_{\hat{B}}$ and taking into account that the functions g_B and $g_{\hat{B}}$ are piecewise differentiable and their derivative takes only values in the set: $\{1, -1, 0\}$.
Now assume that there exists a $b \in [0, 2a\pi]$ with $f_B(b) > f_{\hat{B}}(b)$. One has for all $b \le b'$

$$f_{\hat{B}}(b) = \int_0^b g_{\hat{B}}(\partial A_{\hat{B}}(t))dt \ge \int_0^b g_B(\partial A_B(t))dt = f_B(b).$$

Therefore, $b > b'$. But, then would follow

$$\begin{aligned}
& 2r\mu_a^1(B) \\
= \ & f_{\hat{B}}(2a\pi) \\
= \ & f_{\hat{B}}(b) + \int_b^{2a\pi} g_{\hat{B}}(\partial A_{\hat{B}}(t))dt \\
< \ & f_B(b) + \int_b^{2a\pi} g_B(\partial A_B(t))dt \\
= \ & 2r\mu_a^1(B).
\end{aligned}$$

This would be a contradiction. Therefore, one has for all $b \in [0, 2a\pi]$
$f_B(b) \le f_{\hat{B}}(b)$. This is just the statement of the lemma. □

Corollary 10 *Let* $\mathbf{S}_a^1 = \{x \in I\!R^2; \|x\|_2 = a\}$ *and* $\mathbf{S}_b^1 = \{x \in I\!R^2; \|x\|_2 = b\}$. *For* $0 \leq u \leq \mu_a^1(\mathbf{S}_a^1)$, $0 \leq v \leq \mu_b^1(\mathbf{S}_b^1)$ *and* $r \geq 0$ *one has*

$$\begin{aligned}
&\sup_{\mu_a^1(A)=u, \mu_b^1(B)=v} (\mu_a^1 \times \mu_b^1)(A \times B \cap \{(x,y), \|x-y\|_2 \leq r\}) \\
&\leq (\mu_a^1 \times \mu_b^1)(\hat{A} \times \hat{B} \cap \{(x,y), \|x-y\|_2 \leq r\}).
\end{aligned}$$

Here $\hat{A}$ *is the ball around the north pole in* $\mathbf{S}_a^1$ *with volume* u *and* $\hat{B}$ *is the ball around the north pole in* $\mathbf{S}_b^1$ *with volume* v.

Proof: On $\mathbf{S}_a^1$ there exists a measure $\tilde{\mu}_a^1$ with the property that the map

$$\Pi_a^b : I\!R^2 \longrightarrow R^2; x \mapsto \frac{a}{b} x$$

restricted to $\mathbf{S}_b^1$ is a measure preserving map from $\mathbf{S}_b^1$ to $\mathbf{S}_a^1$. One has: $\tilde{\mu}_a^1 = \frac{a}{b}\mu_a^1$. Furthermore, there exists for a $r \geq 0$ a $r'(r) \geq 0$ such that for all $x \in \mathbf{S}_a^1$

$$\Pi(\{y \in \mathbf{S}_b^1; \|x-y\|_2 \leq r\}) = \{y \in \mathbf{S}_a^1; \|x-y\|_2 \leq r'(r)\}.$$

Now, for all $B \in \mathbf{S}_b^1$ with $\mu_b^1(B) = v$ one has $\mu_a^1(\Pi(B)) = v\frac{b}{a}$. Therefore, for all $A \in \mathbf{S}_a^1$ with $\mu_a^1(A) = u$ one obtains with Lemma 31

$$\begin{aligned}
& (\mu_a^1 \times \mu_b^1)(A \times B \cap \{(x,y), \|x-y\|_2 \leq r\}) \\
= & \int_A \mu_b^1(B \cap \{y \in \mathbf{S}_b^1; \|x-y\|_2 \leq r\}) d\mu_a^1(x) \\
= & \int_A \tilde{\mu}_a^1(\Pi(B) \cap \{y \in \mathbf{S}_a^1; \|x-y\|_2 \leq r'(r)\}) d\mu_a^1(x) \\
= & \frac{a}{b}\mu_a^1 \times \mu_a^1(A \times \Pi(B) \cap \{(x,y); \|x-y\|_2 \leq r'(r)\}) \\
\leq & \frac{a}{b}\mu_a^1 \times \mu_a^1(\hat{A} \times \hat{D} \cap \{(x,y); \|x-y\|_2 \leq r'(r)\}) \\
= & \int_{\hat{A}} \tilde{\mu}_a^1(\hat{D} \cap \{y; \|x-y\|_2 \leq r'(r)\}) d\mu_a^1(x) \\
= & \int_{\hat{A}} \mu_b^1(\Pi^{-1}(\hat{D}) \cap \{y; \|x-y\|_2 \leq r\}) d\mu_a^1(x) \\
= & (\mu_a^1 \times \mu_b^1)(\hat{A} \times \hat{B} \cap \{(x,y), \|x-y\|_2 \leq r\}),
\end{aligned}$$

where $\hat{D}$ is the ball around the north pole in $\mathbf{S}_a^1$ with μ_a^1-volume $v\frac{b}{a}$. This proves the corollary. □

Now we are in position to prove the main theorem of this section.

Theorem 10 *Let $\mathbf{S}_a^d = \{x \in I\!R^{d+1}; \|x\|_{d+1} = a\}$ $\mathbf{S}_b^d = \{x \in I\!R^{d+1}; \|x\|_{d+1} = b\}$. For $0 \le u \le \mu(\mathbf{S}_a^d)$, $0 \le v \le \mu(\mathbf{S}_b^d)$ and $r > 0$ one has*

$$
\begin{aligned}
& \sup_{\mu(A)=u,\mu(B)=v} (\mu_a \times \mu_b)(A \times B \cap \{(x,y), \|x-y\|_{d+1} \le r\}) \\
\le \quad & (\mu_a \times \mu_b)(\hat{A} \times \hat{B} \cap \{(x,y), \|x-y\|_{d+1} \le r\}).
\end{aligned}
$$

Here $\hat{A}$ is the ball around the north pole in $\mathbf{S}_a^d$ with volume u and $\hat{B}$ is the ball around the north pole in $\mathbf{S}_b^d$ with volume v.

Proof: This can be proved by induction. The induction start is just given by corollary 10. If the theorem is true in dimension $d-1$ for all $a, b > 0$, $r > 0$ $0 \le k \le \mu^{d-1}(\mathbf{S}_a^{d-1})$, and $0 \le l \le \mu(\mathbf{S}_b^{d-1})$ then one can show that the northern rearrangements TA, TB of two given Borel set $A, B \in \mathbf{S}_a^d$ increases the functional considered in the statement of the theorem. For this observe that for $r(s,t) = \sqrt{(t-s)^2}$ one has

$$
\begin{aligned}
& \left\{x \in \mathbf{S}_a^d, y \in \mathbf{S}_b^d; \|x-y\|_{d+1} \le r\right\}_{s,t} \\
:= & \left\{x \in \mathbf{S}_a^d, y \in \mathbf{S}_b^d; x_{d+1} = s, y_{d+1} = t, \|x-y\|_{d+1} \le r\right\} \\
= & \left\{x \in \mathbf{S}_a^d, y \in \mathbf{S}_b^d; x_1 = s, y_1 = t, \|x-y\|_{d+1} \le r\right\} \\
= & \left\{x \in \mathbf{S}_a^d, y \in \mathbf{S}_b^d; x_1 = s, y_1 = t, \sqrt{(x_1-y_1)^2 + \ldots + (x_{d+1}-y_{d+1})^2} \le r\right\} \\
= & \left\{x \in \mathbf{S}_a^d, y \in \mathbf{S}_b^d; x_1 = s, y_1 = t, \sqrt{(x_2-y_2)^2 + \ldots + (x_{d+1}-y_{d+1})^2} \le \sqrt{r^2-(t-s)^2}\right\} \\
= & \left\{(s,x')(t,y'); x' \in \mathbf{S}^{d-1}_{\sqrt{a^2-s^2}}, y' \in \mathbf{S}^{d-1}_{\sqrt{b^2-t^2}}, \|x'-y'\|_d \le r(s,t)\right\}.
\end{aligned}
$$

Therefore, one has for all $A \in \mathbf{S}_a^d$ and all $B \in \mathbf{S}_b^d$

$$
\begin{aligned}
& \mu_a^d \times \mu_b^d(A \times B \cap \{\|x-y\|_{d+1} \le r\}) \\
= & \int_{\mathbf{S}_a^d} \int_{\mathbf{S}_b^d} \mathbf{1}_{A \times B \cap \{\|x-y\|_{d+1} \le r\}}(\xi,\eta) d\mu_a^d(\xi) d\mu_b^d(\eta) \\
= & \int_{-a}^{a} \int_{-b}^{b} \frac{a}{\sqrt{a^2-s^2}} \frac{b}{\sqrt{b^2-t^2}} \int_{A_s} \int_{B_t} \mathbf{1}_{\{\|x-y\|_{d+1} \le r\}_{s,t}}(\xi',\eta') d\mu^{d-1}_{\sqrt{a^2-s^2}}(\xi') d\mu^{d-1}_{\sqrt{b^2-t^2}}(\eta') dsdt \\
= & \int_{-a}^{a} \int_{-b}^{b} \frac{a}{\sqrt{a^2-s^2}} \frac{b}{\sqrt{b^2-t^2}} \int_{A_s} \int_{B_t} \mathbf{1}_{\{\|x'-y'\|_d \le r(s,t)\}}(\xi',\eta') d\mu^{d-1}_{\sqrt{a^2-s^2}}(\xi') d\mu^{d-1}_{\sqrt{b^2-t^2}}(\eta') dsdt \\
= & \int_{-a}^{a} \int_{-b}^{b} \frac{a}{\sqrt{a^2-s^2}} \frac{b}{\sqrt{b^2-t^2}} \mu^{d-1}_{\sqrt{a^2-s^2}} \times \mu^{d-1}_{\sqrt{b^2-t^2}}(A_s \times B_t \cap \{\|x'-y'\|_d \le r(s,t)\}) dsdt \\
\le & \int_{-a}^{a} \int_{-b}^{b} \frac{a}{\sqrt{a^2-s^2}} \frac{b}{\sqrt{b^2-t^2}} \mu^{d-1}_{\sqrt{a^2-s^2}} \times \mu^{d-1}_{\sqrt{b^2-t^2}}(\hat{A}_s \times \hat{B}_t \cap \{\|x'-y'\|_d \le r(s,t)\}) dsdt
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-a}^{a}\int_{-b}^{b}\frac{a}{\sqrt{a^2-s^2}}\frac{b}{\sqrt{b^2-t^2}}\int_{\hat{A}_s}\int_{\hat{B}_t}\mathbf{1}_{\{\|x-y\|_{d+1}\le r\}_{s,t}}(\xi',\eta')\,d\mu^{d-1}_{\sqrt{a^2-s^2}}(\xi')\,d\mu^{d-1}_{\sqrt{b^2-t^2}}(\eta')\,ds\,dt \\
&= \int_{TA}\int_{TB}\mathbf{1}_{\{\|x-y\|_{d+1}\le r\}}(\xi,\eta)\,d\mu_a^d(\xi)\,d\mu_b^d(\eta) \\
&= \mu_a^d \times \mu_b^d(TA\times TB \cap \{\|x-y\|_{d+1}\le r\}).
\end{aligned}
$$

Since the same inequality is true for the western rearrangement one gets for all Borel sets $A \subset \mathbf{S}_a^d$, $B \subset \mathbf{S}_b^d$ and all $n \in I\!N$

$$
\begin{aligned}
& \mu_a^d \times \mu_b^d(A\times B \cap \{\|x-y\|_{d+1}\le r\}) \\
\le\ & \mu_a^d \times \mu_b^d((ST)^n A\times (ST)^n B \cap \{\|x-y\|_{d+1}\le r\})
\end{aligned}
$$

Now the statement of the theorem follows since by corollary 9 the right side of the previous inequality converges to

$$
\mu_a^d \times \mu_b^d(\hat{A}\times \hat{B} \cap \{\|x-y\|_{d+1}\le r\}).
$$

$\square$

8 The Heat Flow on the Sphere

In this section, we analyse the behaviour of the heat flow on the sphere. For this, additional information about the fundamental solutions of the heat equation on $\mathbf{S}_a^d$ will be required. It turns out that the fundamental solutions satisfy the following integral equality:

$$\sup_{\mu(A)=c,\mu(B)=b} \int_A \int_B p(t,x,y)d\mu(x)d\mu(y) = \int_{\hat{A}_c} \int_{\hat{B}_b} p(t,x,y)d\mu(x)d\mu(y),$$

where $\hat{A}_c$ resp. $\hat{B}_b$ are balls around the north pole $\mathbf{n}$ with volume c resp. b. This inequality will be useful for the heat flow comparison of a heat distribution f and its polar rearrangement $\hat{f}$. It turns out that the heat flux of the rearranged heat distribution $\hat{f}$ is slower than the heat flux of the original heat distribution f. This will be established in Theorem 13. In the following subsection, we prove that the comparison still holds if we disturb the heat flow by value-preserving maps. These results are stated and proved in Theorem 14 and its corollary. In the last subsection we will see that the comparison still holds for time continuous perturbation of the heat equation by divergence-free vector fields. The main result consists of the following equalities involving the fundamental solutions of the generalised heat equation $p_b(t,x,y)$:

$$\sup_{\mathrm{div}(b)=0} \sup_{\mu(B)=b} \sup_{\mu(A)=c} \int_B \int_A p_b(t,x,y)d\mu(y)d\mu(x) = \int_{\hat{B}_b} \int_{\hat{A}_c} p(t,x,y)d\mu(y)d\mu(x),$$

$$\inf_{\mathrm{div}(b)=0} \inf_{\mu(B)=b} \inf_{\mu(A)=c} \int_B \int_A p_b(t,x,y)d\mu(y)d\mu(x) = \int_{\check{B}_b} \int_{\hat{A}_c} p(t,x,y)d\mu(y)d\mu(x).$$

These equalities will be used to compare the long-time behaviour of diffusions on the sphere with the long-time behaviour of Brownian motion on the sphere. From these equations, we get estimates for the fundamental solutions of the following form

$$\sup_{\mathrm{div}(b)=0} \sup_{x,y\in\mathbf{S}_a^d} p_b(t,x,y) = p(t,\mathbf{n},\mathbf{n}),$$
$$\inf_{\mathrm{div}(b)=0} \inf_{x,y\in\mathbf{S}_a^d} p_b(t,x,y) = p(t,\mathbf{s},\mathbf{n}).$$

8.1 Inequalities for the Fundamental Solutions

Definition 10 *Let $f,g \in \mathcal{B}_b(\mathbf{S}_a^d)$, f will be called more concentrated than g ($f \gg g$) iff for all $c \geq 0$ one has*

$$\sup_{\mu(A)=c} \int_A f d\mu \geq \sup_{\mu(B)=c} \int_B g d\mu.$$

Lemma 32 *Let $f, g \in \mathcal{B}_b(\mathbf{S}_a^d)$, if f is more concentrated than g, then for all $t \geq 0$: $T_t\hat{f}$ is more concentrated as $T_t\hat{g}$.*

Proof: For $c \geq 0$ and $t \geq 0$ one has, since $p(t,x,y) = p(t,y,x)$ for all $x, y \in \mathbf{S}_a^d$,

$$
\begin{aligned}
& \int_{\hat{A}_c} T_t\hat{f}(x)d\mu(x) \\
= & \int_{\hat{A}_c}\int_{\mathbf{S}_a^d} p(t,x,y)\hat{g}(y)d\mu(y)d\mu(x) \\
= & \int_{\mathbf{S}_a^d} T_t\mathbf{1}_{\hat{A}_c}(y)\hat{g}(y)d\mu(y) \\
= & \int_0^\infty \int_{\{y, T_t\mathbf{1}_{\hat{A}_c} \geq s\}} \hat{g}(y)d\mu(y)ds \\
\leq & \int_0^\infty \int_{\{y, T_t\mathbf{1}_{\hat{A}_c} \geq s\}} \hat{f}(y)d\mu(y)ds \\
= & \int_{\mathbf{S}_a^d} T_t\mathbf{1}_{\hat{A}_c}(y)\hat{f}(y)d\mu(y) \\
= & \int_{\hat{A}_c}\int_{\mathbf{S}_a^d} p(t,x,y)\hat{g}(y)d\mu(y)d\mu(x) \\
= & \int_{\hat{A}_c} T_t\hat{f}(x)d\mu(x)
\end{aligned}
$$

$\square$

Theorem 11 *Let $0 \leq c, b \leq \mu(\mathbf{S}_a^d)$, $r \geq 0$, then one has*

$$
\begin{aligned}
& \sup_{\mu(A)=c,\mu(B)=b} (\mu \times \mu)(A \times B \cap \{(x,y); d(x,y) \leq r\}) \\
& = (\mu \times \mu)(\hat{A}_c \times \hat{B}_b \cap \{(x,y); d(x,y) \leq r\}),
\end{aligned}
$$

where $\hat{A}_c$ is the ball around $\mathbf{n}$ with volume c, and $\hat{B}_b$ is the ball around $\mathbf{n}$ with volume b.

Proof: With $\mathbf{S}_a^d = \{x \in I\!R^{d+1}; \|x\| = a\}$ there exists for all $r \geq 0$ a $r'(r) \geq 0$ with the property $\{(x,y); d(x,y) \leq r\} = \{(x,y); \|x-y\|_{d+1} \leq r'(r)\}$. Now the statement of the theorem is a special case of Theorem 10. $\square$

Theorem 12 *Let $0 \leq c, b \leq \mu(\mathbf{S}_a^d)$, then one has for all $t \geq 0$*

$$
\sup_{\mu(A)=c,\mu(B)=b} \int_A\int_B p(t,x,y)d\mu(x)d\mu(y) = \int_{\hat{A}_c}\int_{\hat{B}_b} p(t,x,y)d\mu(x)d\mu(y),
$$

where $\hat{A}_c$ is the ball around $\mathbf{n}$ with volume c, and let $\hat{B}_b$ be the ball around $\mathbf{n}$ with volume b

Proof: Let $\mu \times \mu$ be the product measure on $\mathbf{S}_a^d \times \mathbf{S}_a^d$. For fixed $t \geq 0$ and for all $s \geq 0$ there exists a $r(s) \geq 0$ such that: $\{p(t,x,y) \geq s\} = \{d(x,y) \leq r(s)\}$. With Theorem 11 one can prove

$$
\begin{aligned}
& \int_A \int_B p(t,x,y) d\mu(x) d\mu(y) \\
= & \int_{A \times B} p(t,x,y) d(\mu \times \mu)(x,y) \\
\leq & \int_0^\infty (\mu \times \mu)(A \times B \cap \{(x,y); p(t,x,y) \geq s\}) ds \\
\leq & \int_0^\infty (\mu \times \mu)(\hat{A} \times \hat{B} \cap \{(x,y); p(t,x,y) \geq s\}) ds \\
\leq & \int_{\hat{A} \times \hat{B}} p(t,x,y) d(\mu \times \mu)(x,y) \\
= & \int_{\hat{A}} \int_{\hat{B}} p(t,x,y) d\mu(x) d\mu(y)
\end{aligned}
$$

This proves the theorem. □

Theorem 13 *For all $f \in \mathcal{B}_b(\mathbf{S}_a^d)$ and all $t \geq 0$ one has $T_t \hat{f} \gg \widehat{T_t f}$.*

Proof: Let $r \geq 0$ and $\epsilon > 0$. With the help of the Theorem 12 one proves for all $A \subset \mathbf{S}_a^d$ with $\mu(A) = r$

$$
\begin{aligned}
& \int_A T_t f(x) d\mu(x) \\
= & \int_A \int_{\mathbf{S}_a^d} p(t,x,y) f(y) d\mu(y) d\mu(x) \\
= & \left(\int_A \int_{\mathbf{S}_a^d} p(t,x,y)(f(y) + \inf f) d\mu(y) d\mu(x) \right) - r \inf f \\
\leq & \left(\int_0^\infty \int_A \int_{\{f + \inf f \geq s\}} p(t,x,y) d\mu(y) d\mu(x) ds \right) - r \inf f \\
\leq & \left(\int_0^\infty \int_{\hat{A}} \int_{\{\hat{f} + \inf \hat{f} \geq s\}} p(t,x,y) d\mu(y) d\mu(x) ds \right) - r \inf \hat{f} \\
\leq & \int_{\hat{A}} \int_{\mathbf{S}_a^d} p(t,x,y)(\hat{f}(y) + \inf \hat{f}) d\mu(y) d\mu(x) - r \inf \hat{f} \\
= & \int_{\hat{A}} \int_{\mathbf{S}_a^d} p(t,x,y) \hat{f}(y) d\mu(y) d\mu(x) \\
= & \int_A T_t \hat{f}(x) d\mu(x).
\end{aligned}
$$

This implies

$$\widehat{T_t \hat{f}} \gg \widehat{T_t f}\,.$$

Now the statement of the Theorem follows from Lemma 26. □

Corollary 11 *For $f \in \mathcal{B}_b(\mathbf{S}_a^d)$ and for all $g \in \mathcal{M}_f$ one has $\hat{f} \gg g$.*

Proof: This follows directly from Theorem 13. □

8.2 The Perturbation of the spherical Heat Flow

Theorem 14 *Let $S_n : \mathcal{B}_b(\mathbf{S}_a^d) \longrightarrow \mathcal{B}_b(\mathbf{S}_a^d)$ be a sequence of value-preserving transformations and let $(t_n)_{n\in I\!N}$ be a sequence of positive real numbers.*
Define $K_n := \prod_i^n T_{t_i} S_i$. Then one has for all $n \in I\!N$ and for all $f \in \mathcal{B}_b(\mathbf{S}_a^d)$

$$\widehat{K_n f} \ll T_{\sum_i^n t_i} \hat{f},$$

$$-\widehat{K_n f} \ll -T_{\sum_i^n t_i} \hat{f}.$$

Proof: Lets proceed by induction for the first inequality: For $n = 0$ the statement is just trivial.
Lets assume now, that the statement is true up to $n \in I\!N$.
One has with Theorem 13 and Lemma 27

$$\widehat{K_{n+1} f} = \widehat{T_{t_{n+1}} S_{n+1} K_n f} \ll T_{t_{n+1}} \widehat{S_{n+1} K_n f} = T_{t_{n+1}} \widehat{K_n f}.$$

Now, by the induction hypothesis and and Lemma 19

$$T_{t_{n+1}} \widehat{K_n f} \ll T_{t_{n+1}} T_{\sum_i^n t_i} \hat{f} = T_{\sum_i^{(n+1)} t_i} \hat{f}.$$

This finishes our induction. For the second statement one has by Lemma 28

$$-\widehat{K_{n+1} f} = M(\widehat{-K_{n+1} f}) = M(\widehat{K_{n+1}(-f)}).$$

Since M is value-preserving one gets by the first inequality and Lemma 29

$$M(\widehat{K_{n+1}(-f)}) \ll M(T_{\sum_i^{n+1} t_i}(\widehat{-f})) = T_{\sum_i^{n+1} t_i} M(\widehat{-f}).$$

Using Lemma 28 again we get

$$T_{\sum_i^{n+1} t_i} M(\widehat{-f}) = -T_{\sum_i^{n+1} t_i} \hat{f}.$$

This proves the second statement of the theorem. □

Corollary 12 *Under the same assumptions as in Theorem 14 one has for all $n \in I\!N$ and all $f \in \mathcal{B}_b(\mathbf{S}_a^d)$*

$$\sup_{\mu(A)=c} \int_A K_n f(x) d\mu(x) \leq \int_{\hat{A}_c} T_{\sum_i^n t_i} \hat{f}(x) d\mu(x),$$

$$\inf_{\mu(A)=c} \int_A K_n f(x) d\mu(x) \geq \int_{\check{A}_c} T_{\sum_i^n t_i} \hat{f}(x) d\mu(x).$$

Here, $\check{A}_c$ denotes the ball of volume c around the south pole $\mathbf{s}$ and $\check{A}_c$ denotes the ball of volume c around the north pole.

Proof: For the first inequality one has

$$\sup_{\mu(A)=c} \int_A K_n f(x) d\mu(x) \leq \sup_{\mu(A)=c} \int_A T_{\sum_i^n t_i} \hat{f}(x) d\mu(x) \leq \int_{\hat{A}_c} T_{\sum_i^n t_i} \hat{f}(x) d\mu(x)$$

For the second inequality one has

$$\sup_{\mu(A)=c} \int_A (-K_n f(x)) d\mu(x) \leq \sup_{\mu(A)=c} \int_A (-T_{\sum_i^n t_i} \hat{f}(x)) d\mu(x)$$

Therefore, one gets

$$-\inf_{\mu(A)=c} \int_A K_n f(x) d\mu(x) \leq -\inf_{\mu(A)=c} \int_A T_{\sum_i^n t_i} \hat{f}(x) d\mu(x)$$

And this gives

$$\inf_{\mu(A)=c} \int_A K_n f(x) d\mu(x) \geq \inf_{\mu(A)=c} \int_A T_{\sum_i^n t_i} \hat{f}(x) d\mu(x) \geq \int_{\check{A}_c} T_{\sum_i^n t_i} \hat{f}(x) d\mu(x).$$

□

Corollary 13 *Under the same assumptions as in Theorem 14 one has for all $n \in I\!N$ and all $f \in \mathcal{B}_b(\mathbf{S}_a^d)$*

$$\sup_{x \in \mathbf{S}_a^d} K_n f(x) \leq T_{\sum_i^n t_i} \hat{f}(\mathbf{n}),$$

$$\inf_{x \in \mathbf{S}_a^d} K_n f(x) \geq T_{\sum_i^n t_i} \hat{f}(\mathbf{s}).$$

Proof: Let $B_r(x)$ be the ball around x of radius $r \geq 0$. Put $c(r,x) := \mu(B_r(x))$. For the first inequality one has:

$$\begin{aligned}
& K_n f(x) \\
= & \lim_{r\to 0} \frac{1}{c(r,x)} \int_{B_r(x)} K_n f(y) d\mu(y) \\
\leq & \lim_{r\to 0} \frac{1}{c(r,x)} \int_{\hat{A}_c} K_n f(y) d\mu(y) \\
= & T_{\sum_i^n t_i} \hat{f}(\mathbf{n})
\end{aligned}$$

For the second inequality one has:

$$\begin{aligned}
& K_n f(x) \\
= & \lim_{r\to 0} \frac{1}{c(r,x)} \int_{B_r(x)} K_n f(y) d\mu(y) \\
\geq & \lim_{r\to 0} \frac{1}{c(r,x)} \int_{\check{A}_c} K_n f(y) d\mu(y) \\
= & T_{\sum_i^n t_i} \hat{f}(\mathbf{s})
\end{aligned}$$

□

Corollary 14 *Under the same assumptions as in Theorem 14 one has for all $n \in I\!N$ and all $f \in \mathcal{B}_b(\mathbf{S}_a^d)$*

$$\|K_n f\|_\infty \leq \|T_{\sum_i^n t_i} \hat{f}\|_\infty.$$

Proof: This follows immediately from the previous corollary. □

8.3 The Comparison for Diffusions on the Sphere

Theorem 15 *For all smooth vector fields b on $\mathbf{S}_a^d$ with $\operatorname{div}(b) = 0$ and all $t > 0$ and all $f \in \mathcal{C}(\mathbf{S}_a^d)$ one has*

$$\sup_{\mu(A)=c} \int_A K_{b,t} f(x) d\mu(x) \leq \int_{\hat{A}_c} T_t \hat{f}(x) d\mu(x),$$

$$\inf_{\mu(A)=c} \int_A K_{b,t} f(x) d\mu(x) \geq \int_{\check{A}_c} T_t \hat{f}(x) d\mu(x).$$

Proof: For all smooth vector fields b on $\mathbf{S}_a^d$ the semi-groups K_b are strongly continuous on $\mathcal{C}(\mathbf{S}_a^d)$. Further for all smooth vector fields b on $\mathbf{S}_a^d$ the semi-groups S_b generated by the flow associated to b are strongly continuous on $\mathcal{C}(\mathbf{S}_a^d)$. Therefore, one gets by Trotters product formula for every $f \in \mathcal{C}(\mathbf{S}_a^d)$ and for all $\epsilon > 0$ the exists a $n_0 \in I\!N$ such that for all $n \geq n_0$

$$\left\| K_{b,t} f - \left(\prod_{i=1}^{n} T_{\frac{t}{n}} S_{b,\frac{t}{n}}\right) f \right\|_{\sup} \leq \epsilon$$

(see [EthKur] p.33). Now for an arbitrary $A \subset \mathbf{S}_a^d$ by dominated convergence theorem there exists for all $\epsilon > 0$ a $n_0 \in I\!N$ such that for all $n \geq n_0$

$$\left\| \int_A K_{b,t} f d\mu - \int_A \left(\prod_{i=1}^{n} T_{\frac{t}{n}} S_{b,\frac{t}{n}}\right) f d\mu \right\|_{\sup} \leq \epsilon.$$

Therefore, one gets for all $n \geq n_0$

$$-\epsilon + \int_A \left(\prod_{i=1}^{n} T_{\frac{t}{n}} S_{b,\frac{t}{n}}\right) f(x) d\mu \leq \int_A K_{b,t} f(x) d\mu \leq \epsilon + \int_A \left(\prod_{i=1}^{n} T_{\frac{t}{n}} S_{b,\frac{t}{n}}\right) f(x) d\mu.$$

From this follows by Theorem 14 for all $x \in \mathbf{S}_a^d$

$$-\epsilon + \int_{\breve{A}} T_t \hat{f}(x) d\mu \leq \int_A K_{b,t} f(x) d\mu \leq \epsilon + \int_{\hat{A}_c} T_t \hat{f}(x) d\mu.$$

From this follows if ϵ converges to zero for all $A \subset \mathbf{S}_a^d$:

$$\int_{\breve{A}} T_t \hat{f}(x) d\mu \leq \int_A K_{b,t} f(x) d\mu \leq \int_{\hat{A}_c} T_t \hat{f}(x) d\mu.$$

This is just the statement of the theorem. □

Theorem 16 *For all smooth vector fields b on $\mathbf{S}_a^d$ with* $\mathrm{div}(b) = 0$ *and all $t > 0$ one has for all $c, b \in [0, \mu(\mathbf{S}_a^d)]$*

$$\sup_{\mu(A)=c} \sup_{\mu(B)=b} \int_A \int_B p_b(t,x,y) d\mu(y) d\mu(x) \leq \int_{\hat{A}_c} \int_{\hat{B}(\mathbf{n})_b} p(t,x,y) d\mu(y) d\mu(x),$$

$$\inf_{\mu(A)=c} \inf_{\mu(B)=b} \int_A \int_B p_b(t,x,y) d\mu(y) d\mu(x) \geq \int_{\breve{A}_c} \int_{\hat{B}(\mathbf{n})_b} p(t,x,y) d\mu(y) d\mu(x).$$

Proof: Let B be a Borel set with $\mu(B) = b$. The space $\mathcal{C}(\mathbf{S}_a^d)$ is dense in $\mathcal{L}^1(\mathbf{S}_a^d)$. Therefore, there exists a sequence of continuous functions $(f_{B,n})_{n\in I\!N}$ with

$$\lim_{n\longrightarrow\infty} \|f_{B,n} - \mathbf{1}_B\|_1 = 0.$$

And by Lemma 23 one has that $\hat{f}_{B,n}$ converges μ-almost sure to $\mathbf{1}_{\hat{B}(\mathbf{n})_b}$. Now, one gets by Theorem 15

$$\begin{aligned}
&\int_A \int_B p_b(t,x,y) d\mu(y) d\mu(x) \\
= &\lim_{n\longrightarrow\infty} \int_A \int_{\mathbf{S}_a^d} f_{B,n}(y) p_b(t,x,y) d\mu(y) d\mu(x) \\
= &\lim_{k\longrightarrow\infty} \int_A K_{b,t} f_{B,n}(x) d\mu(x) \\
\leq &\lim_{k\longrightarrow\infty} \int_{\hat{A}_c} T_t \hat{f}_{B,n}(x) d\mu(x) \\
= &\int_{\hat{A}_c} \int_{\hat{B}(\mathbf{s})_b} p(t,x,y) d\mu(y) d\mu(x).
\end{aligned}$$

On the other hand, one has for all $x \in \mathbf{S}_a^d$

$$\begin{aligned}
&\int_A \int_B p_b(t,x,y) d\mu(y) d\mu(x) \\
= &\lim_{n\longrightarrow\infty} \int_A \int_{\mathbf{S}_a^d} f_{B,n}(y) p_b(t,x,y) d\mu(y) d\mu(x) \\
= &\lim_{k\longrightarrow\infty} \int_A K_{b,t} f_{B,n}(x) d\mu(x) \\
\geq &\lim_{k\longrightarrow\infty} \int_{\check{A}_c} T_t \hat{f}_{B,n}(x) d\mu(x) \\
= &\int_{\check{A}_c} \int_{\hat{B}(\mathbf{s})_b} p(t,x,y) d\mu(y) d\mu(x).
\end{aligned}$$

□

Theorem 17 *For all smooth vector fields b on $\mathbf{S}_a^d$ with $\operatorname{div}(b) = 0$ and all $t > 0$ one has*

$$\sup_{x\in\mathbf{S}_a^d} \sup_{\mu(A)=c} \int_A p_b(t,x,y) d\mu(y) \leq \int_{\hat{A}_c} p(t,\mathbf{n},y) d\mu(y),$$

$$\inf_{x\in\mathbf{S}_a^d} \inf_{\mu(A)=c} \int_A p_b(t,x,y) d\mu(y) \geq \int_{\hat{A}_c} p(t,\mathbf{s},y) d\mu(y).$$

Proof: Let $B_r(y)$ be the ball of radius $r \geq 0$ around x in $\mathbf{S}_a^d$. For all $A \subset \mathbf{S}_a^d$ with $\mu(A) = c$ and for all $x \in \mathbf{S}_a^d$ one has with Theorem 16

$$\begin{aligned}
& \int_A p_b(t,x,y)d\mu(y) \\
= & \lim_{r\to 0} \frac{1}{c(x,r)} \int_{B(r,x)} \int_A p_b(t,z,y)d\mu(y)d\mu(z) \\
\leq & \lim_{r\to 0} \frac{1}{c(x,r)} \int_{\hat{A}_{c(r,x)}} \int_{\hat{A}_c} p(t,z,y)d\mu(y)d\mu(z) \\
= & \int_{\hat{A}_c} p(t,\mathbf{n},y)d\mu(y)
\end{aligned}$$

and

$$\begin{aligned}
& \int_A p_b(t,x,y)d\mu(y) \\
= & \lim_{r\to 0} \frac{1}{c(x,r)} \int_{B(r,x)} \int_A p_b(t,z,y)d\mu(y)d\mu(z) \\
\geq & \lim_{r\to 0} \frac{1}{c(x,r)} \int_{\hat{B}(\mathbf{s})_{c(r,x)}} \int_{\hat{A}_c} p(t,z,y)d\mu(y)d\mu(z) \\
= & \int_{\hat{A}_c} p(t,\mathbf{s},y)d\mu(y).
\end{aligned}$$

□

Corollary 15 *Under the same assumptions as in Theorem 17 one has for all divergence-free vector fields b on $\mathbf{M}$ and all $f \in \mathcal{B}_b(\mathbf{S}_a^d)$*

$$\sup_{x\in\mathbf{S}_a^d} K_{b,t}f(x) \leq T_t\hat{f}(\mathbf{n}),$$

$$\inf_{x\in\mathbf{S}_a^d} K_{b,t}f(x) \geq T_t\hat{f}(\mathbf{s}).$$

Proof: For an arbitrary $x \in \mathbf{S}_a^d$ one has by theorem 17

$$\begin{aligned}
& K_{b,t}f(x) \\
= & \int_{\mathbf{S}_a^d} p_b(t,x,y)f(y)d\mu(y) \\
= & \int_0^\infty \int_{\{x; f+\inf f\geq s\}} p_b(t,x,y)d\mu(y)ds - \inf f \\
\leq & \int_0^\infty \int_{\{x; \hat{f}+\inf \hat{f}\geq s\}} p(t,\mathbf{n},y)d\mu(y)ds - \inf \hat{f} \\
= & \int_{\mathbf{S}_a^d} p(t,\mathbf{n},y)\hat{f}(y)d\mu(y) \\
= & T_t\hat{f}(\mathbf{n})
\end{aligned}$$

and

$$
\begin{aligned}
& K_{b,t} f(x) \\
= & \int_{\mathbf{S}_a^d} p_b(t,x,y) f(y) d\mu(y) \\
= & \int_0^\infty \int_{\{x; f + \inf f \geq s\}} p_b(t,x,y) d\mu(y) ds - \inf f \\
\geq & \int_0^\infty \int_{\{x; \hat{f} + \inf \hat{f} \geq s\}} p(t,\mathbf{s},y) d\mu(y) ds - \inf \hat{f} \\
= & \int_{\mathbf{S}_a^d} p(t,\mathbf{s},y) \hat{f}(y) d\mu(y) \\
= & T_t \hat{f}(\mathbf{s})
\end{aligned}
$$

□

Theorem 18 *For all smooth vector fields b on $\mathbf{S}_a^d$ with $\operatorname{div}(b) = 0$ and all $t > 0$ one has*

$$
\sup_{x,y \in \mathbf{S}_a^d} p_b(t,x,y) \leq p(t,\mathbf{n},\mathbf{n}),
$$
$$
\inf_{x,y \in \mathbf{S}_a^d} p_b(t,x,y) \geq p(t,\mathbf{n},\mathbf{s}).
$$

Proof: Let $B_r(y)$ be the ball of radius $r \geq 0$ around $y \in \mathbf{S}_a^d$. Put $c(r,y) := \mu(B_r(y))$. One has for all $x, y \in \mathbf{S}_a^d$ by theorem 17

$$
\begin{aligned}
& p_b(t,x,y) \\
= & \lim_{r \longrightarrow 0} \frac{1}{c(r,y)} \int_{B_r(y)} p_b(t,x,z) p(s,x,z) d\mu(z) \\
\leq & \lim_{r \longrightarrow 0} \frac{1}{c(r,y)} \int_{\hat{A}_{c(r,y)}} p(t,\mathbf{n},z) d\mu(z) \\
= & p(t,\mathbf{n},\mathbf{n})
\end{aligned}
$$

and

$$
\begin{aligned}
& p_b(t,x,y) \\
= & \lim_{r \longrightarrow 0} \frac{1}{c(r,y)} \int_{B_r(y)} p_b(t,x,z) p(s,x,z) d\mu(z) \\
\geq & \lim_{r \longrightarrow 0} \frac{1}{c(r,y)} \int_{\hat{A}_{c(r,y)}} p(t,\mathbf{s},z) d\mu(z) \\
= & p(t,\mathbf{s},\mathbf{n})
\end{aligned}
$$

This proves the theorem. $\square$

The following three results follow directly from the previous theorems.

Corollary 16 *For all smooth vector fields b on $\mathbf{S}_a^d$ with $\operatorname{div}(b) = 0$ and all $t > 0$ one has for all $f \in \mathcal{B}_b(\mathbf{S}_a^d)$*

$$\|K_{b,t} f\|_\infty \leq \|T_t \hat{f}\|_\infty.$$

Proof: One has for all $x \in \mathbf{S}_a^d$ by Corollary 15

$$T_t \hat{f}(\mathbf{s}) \leq \inf_{x \in \mathbf{S}_a^d} K_{b,t} f(x) \leq \sup_{x \in \mathbf{S}_a^d} K_{b,t} f(x) \leq T_t \hat{f}(\mathbf{n}).$$

$\square$

Corollary 17 *There exists a $C > 0$ and a $\lambda > 0$ such that for all smooth vector fields b on $\mathbf{S}_a^d$ with $\operatorname{div}(b) = 0$ and for all $f \in \mathcal{B}_b(\mathbf{S}_a^d)$ with $\int_{\mathbf{S}_a^d} f d\mu = 0$ and for all $t \geq 0$*

$$\|K_{b,t} f\|_\infty \leq C \|f\|_\infty \exp(-\lambda t).$$

Proof: By a classical result from ergodic theory (see [BenLioPap] p.365) there exist a $C > 0$ and a $\lambda > 0$ such that for all $f \in \mathcal{B}_b(\mathbf{S}^d)$ with $\int_{\mathbf{S}_a^d} f d\mu = 0$ and for all $t \geq 0$ one has

$$\|T_t f\|_\infty \leq C \|f\|_\infty \exp(-\lambda t).$$

Further, one has $\int_{\mathbf{S}_a^d} f d\mu = \int_{\mathbf{S}_a^d} \hat{f} d\mu$. Therefore, one gets by Corollary 16

$$\|K_{b,t} f\|_\infty \leq \|T_t \hat{f}\|_\infty \leq C \|\hat{f}\|_\infty \exp(-\lambda t) = C \|f\|_\infty \exp(-\lambda t).$$

$\square$

Corollary 18 *There exist a $\tilde{C} > 0$ such that for all vector fields b on $\mathbf{S}_a^d$ with $\operatorname{div}(b) = 0$ and for all $f \in \mathcal{B}_b(\mathbf{S}_a^d)$ with $\int_{\mathbf{S}_a^d} f d\mu = 0$ and for all solutions $u \in \mathcal{C}^2(\mathbf{S}_a^d)$ of the Poisson equation $A_b u = f$ one has*

$$\|u\|_\infty \leq \tilde{C} \|f\|_\infty.$$

Proof: By a classical result from functional analysis the solution of the Poisson equation $A_b u = f$ is given by $u(x) = \int_0^\infty K_{b,t} f(x) dt$ if $\int_{\mathbf{S}_a^d} f d\mu = 0$. Therefore, by Corollary 17 one gets

$$\begin{aligned}
\|u\|_\infty &= \| \int_0^\infty K_{b,t} f(.) dt \|_\infty \\
&\leq \int_0^\infty \|K_{b,t} f(.)\|_\infty dt \\
&\leq \int_0^\infty C \exp(-\lambda t) \|f\|_\infty dt \\
&= \frac{C}{\lambda} \|f\|_\infty .
\end{aligned}$$

□

9 Appendix

Let $[a, b]$ be an open Interval and $F : [a, b] \longrightarrow I\!R$ a real increasing, left continuous function. Let $F^- : [F(a), F(b)] \longrightarrow [a, b]$ be the following function:

$$F^-(y) := \sup(F^{-1}([F(a), y])).$$

F^- is well defined since $F^{-1}(([F(a), y]))$ is never empty and always included in $[a, b]$. Furthermore, F^- is increasing and right continuous.

Lemma 33 *If F is strictly increasing then $F^- \circ F = id$.*

Proof: $z \in F^{-1}([F(a), F(x)])$ implies $F(z) \in [F(a), F(x)]$. Now $z > x$ would imply $F(z) > F(x)$ since F is strictly increasing in x. Therefore, $F^{-1}([F(a), F(x)]) \subset [a, x]$. On the other hand $z \in [a, x]$ implies $F(z) \in [F(a), F(x)]$ and from this follows $z \in F^{-1}([F(a), F(x)])$. Therefore, one gets $F^{-1}([F(a), F(x)]) = [a, x]$ for all $x \in [a, b]$. From this we obtain

$$F^-(F(x)) = \sup(F^{-1}([F(a), F(x)])) = \sup([a, x]) = x.$$

□

Lemma 34 *If $F^-(x) \leq z$ then one has $x \leq F(z)$.*

Proof: Assume that $x > F(z)$.
By the left continuity of F one has $F(z) = \inf_{y>z} F(y)$.
Therefore there would exist a $y > z$ such that $F(z) \leq F(y) < x$.
For the same y one would have $z < y \in F^{-1}([a, x])$.
From this would follow $z < y \leq \sup(F^{-1}([a, x])) = F^-(x)$.
This would be a contradiction to the assumption of the lemma.
Therefore, one has $x \leq F(z)$. □

Lemma 35 *If $x < F(z)$ then one has $F^-(x) \leq z$.*

Proof: Assume $F^-(x) > z$.
From this one can conclude that there exists an $\epsilon > 0$ such that
$\sup(F^{-1}([a, x])) > z + \epsilon$.
From this follows $z \in F^{-1}([a, x])$.
And this implies $F(z) \leq x$.
This is a contradiction to the assumption of the lemma.
Therefore, one has $F^-(x) < z$. □

Lemma 36 *If F is strictly increasing then $F^- \in \mathcal{C}([F(a), F(b)])$.*

Proof: Since F is strictly increasing, for all $\epsilon > 0$ and for all $x \in [a, b]$ there exist $x_{1,\epsilon} \in]x - \epsilon, x[$ and $x_{,\epsilon} \in]x, x + \epsilon[$ with $F(x_{1,\epsilon}) < F(x) < F(x_{2,\epsilon})$.
Therefore, for all $y \in [F(a), F(b)]$ there exist $x_{1,\epsilon} \in]F^-(y) - \epsilon, F^-(y)[$ and $x_{,\epsilon} \in]F^-(y), F^-(y) + \epsilon[$ with $F(x_{1,\epsilon}) < F(F^-(y)) < F(x_{2,\epsilon})$.
Now it is easy to see that

$$y \in]F(x_{1,\epsilon}), F(x_{2,\epsilon})[\neq \emptyset.$$

Furthermore, by Lemma 33

$$F^-(]F(x_{1,\epsilon}), F(x_{2,\epsilon})[) \subset [x_{1,\epsilon}, x_{2,\epsilon}] \subset B_{2\epsilon}(F^-(y)).$$

This proves the continuity of F^- in $y \in [F(a), F(b)]$. $\square$

The following lemma holds for distribution functions.

Lemma 37 *Let $f, f_n \in \mathcal{B}_b(\mathbf{M})$ and $n \in I\!N$. If f_n converges to f in measure, then $F^-_{f_n}$ converges to F^-_f λ-almost everywhere.*

Proof: It is well know that distribution functions are left continuous. Further it is well know, that if f_n converges to f in measure, then F_{f_n} converges to F_f in all points where F_f is continuous (see [Bre] p.159). Furthermore, one knows that the points where F_f is continuous form a dense set in $[a, b]$. Furthermore, one knows that the set K of all $x \in [0, \mu_{\mathbf{M}}(\mathbf{M})]$ with the property that there exists a small neighbourhood U of $F^-_f(x)$ with $F_f(x') = F_f(F^-_f(x))$ for all $x' \in U$ is a zero set in $[0, \mu_{\mathbf{M}}(\mathbf{M})]$. For a point $x \in [0, \mu_{\mathbf{M}}(\mathbf{M})] \cap K^c$ one can determine the smallest interval $[x_1, x_2]$ such that $F^{-1}_f([x_1, x_2])$ is non empty. x_1 may be equal to x_2. By the left continuity of F_f the set $F^{-1}_f([x_1, x_2])$ consists of a single point $F^{-1}_f(\{x_1\})$. Therefore, one has for all $x' \in [x_1, x_2]$

$$F^-_f(x') = \sup(F^{-1}_f([0, y]) = \sup(F^{-1}_f([0, x_1]) = F^-_f(x_2) = F^-_f(x_1).$$

Since the set of all points, where F_f is continuous is a dense set, one can find for any $\epsilon > 0$ points $y_{\epsilon,1} \in]F^-_f(x_1) - \epsilon, F^-_f(x_1)[$ and $y_{\epsilon,2} \in]F^-_f(x_2), F^-_f(x_2) + \epsilon[$ where F_f is continuous. Since $x \notin K$, one knows that

$$F_f(y_{\epsilon,1}) < x < F_f(y_{\epsilon,2}).$$

Now one can choose a $\delta > 0$ such that the interval $[F_f(y_{\epsilon,1}) + \delta, F_f(y_{\epsilon,2}) - \delta]$ is non empty and contains x. Since F_{f_n} converges to F_f in all points where F_f is continuous, there exists a $n_0 \in I\!N$ such that for all $n \geq n_0$

$$F_{f_n}(y_{\epsilon,2}) \in B_\delta(F_f(y_{\epsilon,2}))$$

and

$$F_{f_n}(y_{\epsilon,1}) \in B_\delta(F_f(y_{\epsilon,1})).$$

For the same $n \in I\!N$ one has

$$y_{\epsilon,2} \in F_{f_n}^{-1}(B_\delta(F_f(y_{\epsilon,2})))$$

and

$$y_{\epsilon,1} \in F_{f_n}^{-1}(B_\delta(F_f(y_{\epsilon,1}))).$$

Since $F_{f_n}^-$ is increasing, one has for all $x' \in [F_f(y_{\epsilon,1}) + \delta, F_f(y_{\epsilon,2}) - \delta]$

$$F_{f_n}^-(x') \in [F_{f_n}^-(F_f(y_{\epsilon,1}) + \delta), F_{f_n}^-(F_f(y_{\epsilon,2}) - \delta)] \subset [y_{\epsilon,1}, y_{\epsilon,2}].$$

Therefore, one has for all $n \geq n_0$:

$$F_{f_n}^-([F_f(y_{\epsilon,1}) + \delta, F_f(y_{\epsilon,2}) - \delta]) \subset [y_{\epsilon,1}, y_{\epsilon,2}].$$

Therefore, one gets for all $n \geq n_0$

$$F_{f_n}^-(x) \in [y_{\epsilon,1}, y_{\epsilon,2}] \subset B_\epsilon(F_f^-(x)).$$

Since $\epsilon > 0$ can be chosen arbitrary small one gets the convergence of $F_{f_n}^-(x)$ to $F_f^-(x)$ for all $x \notin K$. Since K was a zero set, this implies the almost everywhere convergence of $F_{f_n}^-$ to F_f^-. $\square$

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