PENTAKUN, THE MOD 5 MARKOV CHAIN AND A MARTIN BOUNDARY

ATSUSHI IMAI

Dedicated to Professor Hiroshi Sato on the occasion of his 60th birthday

ABSTRACT. Let $\mathcal{P}$ denote the p.c.f. self-similar set defined by mapping the regular pentagon into itself by five self-similarities each leaving one vertex fixed. We define the canonical Markov chain for $\mathcal{P}$ and denote its Markov operator by P. We show that its Martin boundary $\mathcal{M}$ is homeomorphic to $\mathcal{P}$. The associated Dirichlet problem $(P - I)f = 0$ and $f = g$ on $\mathcal{P}$ has a unique solution such that $f(\xi) = \mathcal{P}_\xi$ for $\xi \in \mathcal{P}$. We obtain an integral representation for kernel functions on $\mathcal{P}$ (Poisson integral type).

1. PRELIMINARIES

1.1 Introduction.

Classical potential theory has its origins in Coulomb's law. It states that two charges attract each other with a force in the direction of their connecting line whose magnitude is proportional to the quotient of the product of the two charges and the square of their distance.

Another observation made in the 19th century plays an important role in the understanding of potential theory. In 1826 the botanist Brown observed that microscopic particles, when left alone in a liquid, are seen to move constantly in the fluid along erratic paths. Much later Einstein investigated this movement as a statistical law which describes how a large number of particles spread over a period of time. His predictions were verified in experiment.

The above two ideas are linked by the Laplace operator. The deep connection between the two theories was first revealed in the papers of Doob [7], Kac [19], Kakutani [20] and Knapp [23]. This can be expressed by the fact that the harmonic measures which occur in the solution of the Dirichlet problem are hitting distributions for Brownian motion or, equivalently, that the positive hyperharmonic functions for the Laplace equation are the excessive functions of the Brownian semigroup. This equivalence allows potential theoretic results and notions to be given a probabilistic meaning.

Therefore, harmonic functions play a central role in the analysis to understand the above-mentioned phenomena from a probabilistic viewpoint. These functions are characterised in different ways.

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Let U be a bounded open domain in $\mathbb{R}^d$ with boundary ∂U. The Laplace operator $\Delta := \sum_{i=1}^d \partial^2/\partial x_i^2$ acts on twice differentiable functions on U; by definition, its kernel consists of the harmonic functions . Apart from this description of harmonic functions f, that is by $\Delta f \equiv 0$, it is a well known fact that harmonic functions can be characterised by geometrically defined averaging properties.

Let $a \in U$ and denote by $B(a,r)(\subset U)$ the open ball centred at a with radius $r > 0$, the boundary $\partial B(a,r)$ is the sphere $S(a,r)$ of radius $r > 0$ centred at a.

Let $\mu_{a,r}$ denote the uniform probability measure on $S(a,r)$. With this notation we are able to define the averaging operator $H_{a,r}$ on $\mu_{a,r}$-integrable functions defined on $\overline{B(a,r)}$ by

$$H_{a,r}f(x) = \int_{S(a,r)} \frac{r^2 - |x-a|^2}{|x-y|^d} r^{d-2} f(y) d\mu_{a,r}(y).$$

Then it is known that a function $f \in C^2(U)$ is harmonic if and only if for all $a \in U$ and $r > 0$, $f|_{\overline{B(a,r)}}$ is a fixed point of $H_{a,r}$.

The Dirichlet problem can be stated as follows. One has to find a continuous function on $\overline{U}$ which is harmonic in U and takes prescribed boundary values on ∂U, i.e. we have a boundary condition $f \equiv g$ on ∂U where $g \in C(\partial U)$.

Let $f : S(a,r) \to \overline{\mathbb{R}}$ be a Borel measurable function which is bounded from below.
(1) If f is $\mu_{a,r}$-integrable, then $H_{a,r}f$ is harmonic on $B(a,r)$.
(2) If f is continuous at $z \in S(a,r)$, then

$$\lim_{x \to z} H_{a,r}f(x) = f(z).$$

We briefly discuss the connections with stochastic processes.

Let $\mathbf{X} = (X_t)_{t>0}$ denote the Brownian motion on $\mathbb{R}^d$, i.e. it is given by the transition density semigroup

$$p_t(x,y) = \left(\frac{1}{\sqrt{2\pi t}}\right)^d \exp\left(-\frac{|x-y|^2}{2t}\right)$$

which is considered as a Markov process with respect to the distributions P^x when starting in $x \in \mathbb{R}^d$.

Since $\mathbf{X}$ has continuous paths, it leaves a bounded open domain U within a finite time τ_u, once it started in U. We set

$$\tau_U(y) = \inf\{t > 0 | X_t(y) \in U^c\}.$$

Let f be a bounded measurable function. Then the theory of Brownian motion asserts that

$$H'_U f(x) := \int_{\{y \in \Omega | \tau_U(y) < \infty\}} f \circ X(\tau_U(y)) dP^x(y)$$

is harmonic in U (in particular, we also have that $f \in C^2(U)$).

Also, if f is continuous at $z \in \{s \in \partial U | P^s(\tau_U = 0) = 1\}$, then it follows that

$$\lim_{\substack{x \to z \\ x \in \overline{U}}} H'_U f(x) = f(z).$$

The geometric averaging property of harmonic functions is not restricted to the structure given by the Brownian motion or averaging on spheres, as well as the abstract properties of the Laplace operator are not restricted to its specific definition. In fact, this concept has a direct extension to Markov processes. Let P denote a Markov operator, and call a function f defined on the state space *harmonic* if $Pf = f$, so that the Laplacian is defined by $\Delta f = (P - I)f$. In order to turn this concept into a Dirichlet problem one needs to assign a boundary ∂E to E. Denker/Sato (see [6]) solved this problem in the special case of the word space (cf. Section 2 of Chapter 3), that is,

Let E be a countable set, called word space, and ∂E denote the Martin boundary (associated in Dynkin [9]) of E. For a real valued continuous function g on ∂E,

$$\begin{cases} ([P - I]f)(x) = 0, & x \in E \\ \lim_{x \to \xi} f(x) = g(\xi), & \xi \in \partial E \end{cases}$$

has a unique solution f in the space of harmonic uniformly continuous functions on E, given by

$$f(x) = \int_{\partial E} k(x, y) g(y) d\mu(y)$$

where μ denotes some finite measure and k kernel function.

Lately there have been made attempts to define the concept of the Laplace operator and the Dirichlet problem for p.c.f. self-similar sets.

A p.c.f. self-similar set K is defined by a family $f_i : \mathbb{R}^d \to \mathbb{R}^d$ of contractions which satisfy

$$K = \bigcup_{i=1}^{m} f_i(K).$$

It is known (see [15, 17]) that a family f_i always defines a unique self-similar set. P.c.f. self-similar means that $f_i(K) \cap f_j(K)$ is finite for all $1 \leq i \neq j \leq m$. Kigami [21, 22] has defined a method of geometric averaging in corresponding fractals of this type and also described the Laplace operator. He showed that the Dirichlet problem for the Poisson equation on K, which consists in, for given real valued continuous function h on K, finding a real valued continuous function f on K such that

$$\begin{cases} \Delta f = 0 \\ f = h \text{ on } \partial K, \end{cases}$$

has a unique solution f. Details will be given below.

On the other hand, a few years ago, Denker/Sato [4] have initiated the study of the relation between one of the best known examples for a fractal set, the Sierpiński gasket, and a Martin exit boundary; that is to say, the Sierpiński gasket is represented as the Martin boundary of a Markov chain and harmonic functions have an integral representation using the Martin kernel of a certain canonical random walk and Dynkin's theorem [9]. In addition, Denker and Koch [3] proved a Poisson formula for bounded harmonic functions on the Sierpiński gasket as an application of [4]. These results may be considered as a new approach in harmonic analysis and Martin boundary theory.

In this paper we connect the extension to the mod 5 Markov chain and Denker-Sato's approach. We show that the Pentakun (the self-similar Pentagon) suggested by Kumagai agrees with the Martin boundary of an appropriately chosen Markov chain.

1.2 Martin boundaries.

Here we recall some facts about Martin boundaries which are needed and explain the background of our investigation.

It is one of the main goals to identify Martin boundaries (which always exist [9]). Clearly, the Martin boundary may be trivial (i.e. consisting of one point) which occurs for a recurrent Markov Chain. One of the first examples of a nontrivial Martin boundary is due to Ney and Spitzer [27] (see Woess [33]). We begin by describing the necessary notation and definitions.

The Martin boundary for the state space of a discrete Markov chain was introduced by Doob [8], Dynkin [9], Feller [11] and Hunt [16], among others. In this chapter, we use the notation of Dynkin [9] who uses Hunt's probabilistic approach.

We consider a sub-Markovian kernel on a countable set as starting point.

Definition 1.1. Let E be a countable set. We call $p : E \times E \to [0,1]$ a *sub-Markovian Kernel*, if

$$\sum_{y\in E} p(x,y) \le 1 \tag{1.1}$$

for every $x \in E$.

Definition 1.2. Let f be a non-negative function on E. The associated *Markov operator P* is defined by

$$(Pf)(x) = \sum_{y\in E} p(x,y)f(y) \quad (x \in E) \tag{1.2}$$

and f is called P*-excessive* if $Pf \le f$ and P*-harmonic* if $Pf = f$ where we admit the value $+\infty$.

Let μ be a measure on E. The associated *dual Markov operator* P^* is defined by

$$(\mu P^*)(y) = \sum_{x \in E} \mu(x) p(x, y) \quad (y \in E) \tag{1.3}$$

and μ is called P^*-*excessive* if $\mu P^* \leq \mu$ and P^*-*harmonic* if $\mu P^* = P^*$.

Iterating the procedure in (1.2) and (1.3) we obtain

$$\begin{cases} (\underbrace{P \circ P \circ \cdots \circ P}_{n\text{-times}} f)(x) = \sum_{y \in E} p(n; x, y) f(y) & x \in E \\ (\mu \underbrace{P^* \circ P^* \circ \cdots \circ P^*}_{n\text{-times}})(y) = \sum_{x \in E} \mu(x) p(n; x, y) & y \in E, \end{cases} \tag{1.4}$$

where $p(n; x, y)$ is defined recursively by

$$\begin{cases} p(0, x, y) = \delta_{x,y} \\ p(n; x, y) = \sum_{z \in E} p(n-1; x, z) p(z, y), \end{cases}$$

where $\delta_{x,y}$ is the Kronecker delta.

Introducing the absorbing state ∞ and setting

$$p(x, \infty) = 1 - \sum_{y \in E} p(x, y) \text{ and } p(\infty, \infty) = 1$$

we may define a Markov chain $(X_n)_{n \in \mathbb{N}_0}$ with state space $E \cup \{\infty\}$ where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Denote

$$T = \min\{n \in \mathbb{N}_0 | X_n = \infty\}.$$

Then $\{T = \infty\}$ means that X_n is never absorbed by ∞.

Definition 1.3. The *Green function* g on E is defined by

$$g(x, y) = \sum_{n=0}^{\infty} p(n, x, y) \tag{1.5}$$

whenever the series converges for all $x, y \in E$.

In addition, we suppose that a *finite standard measure* γ on E (in the sense that $\sum_{y \in E} \gamma(y) < \infty$ and $\gamma(z) > 0$ for any $z \in E$) exists. Then we define

$$\eta(x) := \sum_{y \in E} \gamma(y) g(y, x) > 0$$

for each $x \in E$.

Definition 1.4. The *Martin kernel* (for g and γ) is defined by

$$k(x, y) = \frac{g(x, y)}{\eta(y)} \quad (x, y \in E) \tag{1.6}$$

whenever $g(x, y)$ and $\eta(y)$ exist.

We call a function $l : E \to \mathbb{N}$ such that $l(x_n) \to \infty$ as $n \to \infty$, an *index* or a *terminal moment*. In this chapter, we suppose that

$$a(x) := \sup_{y \in E} k(x, y) < \infty. \tag{1.7}$$

We define a metric ρ on E by

$$\rho(x, y) = |2^{-l(x)} - 2^{-l(y)}| + \sum_{z \in E} \frac{|k(z, x) - k(z, y)|}{a(z)\{1 + |k(z, x) - k(z, y)|\}}. \tag{1.8}$$

Using (1.8), we can assume that E is a metric space (E, ρ) and can construct the ρ-completion of E, denoted by $\overline{E}$. Note that E is an open set. The next Lemmas 1.5 and 1.6 will give a deeper insight into the topological structure of $\overline{E}$.

Lemma 1.5. The map $k(z, \cdot)$ is uniformly continuous in the metric ρ.

Proof. It is obvious from the fact that a sequence $\{x_n\}_{n \in \mathbb{N}} \subset E$ is Cauchy if and only if the sequence of real numbers $\{k(z, x_n)\}_{n \in \mathbb{N}}$ is Cauchy. □

By the above lemma, the map $k(z, \cdot)$ and the metric (1.8) extend to $\overline{E}$, respectively. Therefore, the extension is also denoted by k and ρ, respectively.

Lemma 1.6. $\overline{E}$ is a compact metric space.

Proof. Let $\{x_n\}_{n \in \mathbb{N}} \subset \overline{E}$. First note that the sequence $\{k(z, x_n)\}_{n \in \mathbb{N}}$ is bounded for fixed $z \in E$. Hence we can choose subsequence $\{x_{N(n)}\}_{n \in \mathbb{N}} \subset \overline{E}$ such that $\{k(z, x_{N(n)})\}_{n \in \mathbb{N}}$ is Cauchy. Then by (1.8), it follows that $\{x_{N(n)}\}_{n \in \mathbb{N}}$ is Cauchy in $\overline{E}$ and thus $\overline{E}$ is sequentially compact. □

Definition 1.7. $\overline{E}$ is called the *Martin space* associated to p. The boundary of $\overline{E}$, that is, $\partial \overline{E} = \overline{E} \setminus E^{\circ} = \overline{E} \setminus E$ is called *Martin boundary* and is denoted by M. Note that M is a compact metric space.

The main theorem in this section (proved in Dynkin [9]) is the following.

Theorem 1.8. There exists a Borel set $B \subset M$, called the *space of exits*, such that the following holds:

(1) The function $k(\cdot, z)$ is P-harmonic on E for every $z \in B$.

(2) For every γ-integrable P-excessive function $h \geq 0$ there exists a unique finite measure μ_h on $\overline{E}$ such that

$$h(x) = \int_{E \cup B} k(x, z) d\mu_h(z) \text{ and } \mu_h(M \setminus B) = 0.$$

(3) $z \in B$ if and only if $\mu_{k(\cdot, z)}(x) = \delta_{x,z}$.

(4) For every γ-integrable P-harmonic function $h \geq 0$ there exists a unique finite measure μ_h on $\overline{E}$ such that

$$h(x) = \int_B k(x, y) d\mu_h(y).$$

(5) For every bounded P-harmonic function $h \geq 0$, μ_h is absolutely continuous with respect to μ_1 with bounded Radon-Nikodym derivative $\frac{d\mu_h}{d\mu_1}$ such that

$$
\begin{aligned}
&h(x) = \int_B k(x,y)\frac{d\mu_h}{d\mu_1}(y)d\mu_1(y) \\
&\lim_{n\to\infty} h(X_n) = \frac{d\mu_h}{d\mu_1}(X_\infty) \quad \text{Pr}_x\text{-a.e. on } \{T=\infty\} \quad \forall x \in E \\
&\exists X_\infty \ni h(x) = \mathbf{E}^x\left[\frac{d\mu_h}{d\mu_1}(X_\infty)\right] \quad \forall x \in E.
\end{aligned}
$$

Here $\{X_n | n \in \mathbb{N}_0\}$ denotes the associated Markov chain and Pr_x is the probability measure concentrated on the paths starting from x given by

$$\text{Pr}_x[X_0 = x_0, X_1 = x_1, ..., X_n = x_n] = \delta_{x,x_0} p(x_0,x_1)p(x_1,x_2)\cdots p(x_{n-1},x_n).$$

(6) Statement (5) holds for every bounded P-harmonic function if P is *conservative*, i.e. $P1 = 1$ (Recall that p is a Markovian kernel.).

(7) If f is a non-negative μ_1-integrable function on M then

$$h_f(x) := \int_B k(x,y)f(y)d\mu_1(y) \tag{1.9}$$

is P-harmonic on E and

$$f(X_\infty) = \lim_{n\to\infty} h_f(X_n) \quad \text{Pr}_x\text{-a.e. on } \{T=\infty\} \quad \forall x \in E.$$

The identification problem is investigated by many authors (see [28, 33] etc.). We mention some aspects of the theory which are connected with our research.

Random walks on infinite graphs and (as a special case) groups are among the most interesting topics. The definition of a random walk adopted here is that of a time-homogeneous Markov chain whose transition probabilities are adapted in some way (which has to be specified more precisely) to a graph structure of the underlying discrete state space. It goes without saying that a graph can be associated with any time-homogeneous Markov chain on a countable state space, so that one could say that this notion of random walks coincides with that of arbitrary Markov chains.

We suppose that E is an *infinite graph*; we consider the non-oriented edge set as a symmetric subset of $E \times E$ and write $x \approx y$ if x and y are neighbours. The *degree* of $x \in E$, denoted by $\deg(x)$, is the number of neighbours of x. A *path of length* n from x to y is a sequence $x = x_0, x_1, ..., x_n = y$ of distinct vertices such that $x_{i-1} \approx x_i$. We now assume that E is infinite and *locally finite,* i.e. for every $x, y \in E$ there exists a finite path from x to y. The *distance* $c(x,y)$ between two vertices x and y is the minimal length of a path connecting the two.

Definition 1.9. A *random walk* on E is defined by a transition matrix

$$\mathbf{P} = (p(x,y))_{x,y \in E}$$

which describes the one-step transition of a Markov chain $(X_n)_{n \in \mathbb{N}_0}$ with state space E.

To model X_n, we consider the *trajectory space* $E^{\mathbb{N}_0}$, equipped with the usual product sigma-algebra arising from n-th projection $E^{\mathbb{N}_0} \to E$. This describes the random walk starting at $x \in E$, if $E^{\mathbb{N}_0}$ is equipped with the probability measure Pr_x given in Theorem 1.8.

It is clear that $p(n; x, y) = \mathrm{Pr}_x[X_n = y]$. This is the (x, y)-entry of $\mathbf{P}^n$, with $\mathbf{P}^0 = \mathbf{I}$, the identity matrix over E.

We also assume that $(X_n)_{n \in \mathbb{N}_0}$ is *irreducible*, that is, for every $x, y \in E$ there exists some $n \in \mathbb{N}_0$ such that $p(n; x, y) > 0$.

Irreducibility alone is, of course, not enough to say that the random walk is well adapted to the underlying graph structure. Thus we now present some conditions which will serve to meet this requirement in some form.

The random walk has *finite range* if $\{y | p(x, y) > 0\}$ is a finite set for every $x \in E$. In particular, the random walk has *bounded range* if $\sup\{c(x, y) | p(x, y) > 0\} < \infty$.

1.3 Self-similar fractals.

In this section, we review the theory of self-similar sets and analysis in p.c.f self-similar sets.

We begin with a description of a general construction for fractals (see [15, 17]). Examples are the Cantor set, the von Koch curve and the von Koch island.

Let D be a closed subset of $\mathbb{R}^d$. A mapping $S : D \to D$ is called a contraction on D if there exists some c with $0 < c < 1$ such that $|S(x) - S(y)| \leq c|x - y|$ for all $x, y \in D$. If equality holds, then S maps sets to geometrically similar ones, and we call S a similarity and c its similitude ratio. According to [10, 15, 17], families of contractions, or iterated function schemes as they are sometimes called, define unique non-empty compact invariant sets. That is,

Theorem 1.10. [10, 15, 17] Let $\{f_i\}_{1 \leq i \leq m}$ be contractions on $D \subset \mathbb{R}^d$. Then there exists a unique non-empty compact set K that satisfies

$$K = \bigcup_{i=1}^{m} f_i(K).$$

Moreover, if we define a transformation of f on the class $\mathfrak{T}$ of non-empty compact sets by

$$f(K_0) = \bigcup_{i=1}^{m} f_i(K_0),$$

then

$$K = \bigcap_{n=1}^{\infty} \underbrace{f \circ f \circ \cdots \circ f}_{n\text{-times}}(K_0) \tag{1.10}$$

for any $K_0 \in \mathfrak{T}$ such that $f_i(K_0) \subset K_0$ for each i.

We call K a fractal set. In (1.10), taking K_0 as a simplex in $\mathbb{R}^d$ and f_i the three appropriate similarities with similitude ratio $1/2$, we have that K is the Sierpiński gasket in $\mathbb{R}^d$ originated from Sierpiński's work [30]. The term was later introduced by Mandelbrot [25].

Moreover, by the following definition we can understand that a self-similar structure is an abstraction of topological features from the concepts of the self-similar sets studied in [15, 17].

Let K be a compact metric space, $\mathcal{A}$ a finite set, $\mathcal{A}^\infty$ the space of one-sided infinite sequences, $\mathcal{W}$ the word space generated by $\mathcal{A}$. For each $a \in \mathcal{A}$, let $F_a : K \to K$ be a continuous injection and $\omega^a : \mathcal{A}^\infty \to \mathcal{A}^\infty$ the map defined by $\omega^a(\mathbf{w}) = a\mathbf{w}$ where $a \in \mathcal{A}$. Then $(K, \mathcal{A}, \{F_a\}_{a\in\mathcal{A}})$ is said to be a *self-similar structure* on K (or simply, K is a *self-similar set*) if there exists a continuous surjection $\pi : \mathcal{A}^\infty \to K$ satisfying

$$\pi \circ \omega^{w_1} \circ \omega^{w_2} \circ \cdots \circ \omega^{w_n} = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_n} \circ \pi$$

for any $w_1 w_2 \cdots w_n \in \mathcal{W}$. In particular, $F_\emptyset$ is the identity map of K.

For fractal sets K the geometry has been investigated for quite some time. More recently, one has investigated the connection between multifractal spectrum and harmonic analysis [2].

Here, we are interested in the notion of Brownian motion in harmonic analysis.

Goldstein [14] and Kusuoka [24] independently constructed a Brownian motion on the Sierpiński gasket. Barlow/Pelkins [1] obtained a remarkable estimate on its transition probability density with respect to an appropriate Hausdorff measure.

This investigation may be viewed as the part corresponding to the point of view originating from Brownian motion. In fact, Fukushima/Shima [13] and Shima [29] determined the eigenvalues of the Laplacian on Sierpiński gasket.

As mentioned before, Kigami [21] studied the problem from the averaging viewpoint. He found the direct and natural definition of the Laplace operator on the Sierpiński gasket as the limit of difference operators and then established a theory which solved the associated Dirichlet problem for the Poisson equation, Gauss-Green's formula and so on. He later expanded the theory to a class of self-similar sets called p.c.f. self-similar sets using the theory of Dirichlet forms [22]

We would like to mention that in [26] a suitable class of "harmonic functions" on the Sierpiński gasket is constructed such that these functions satisfy a minimum principle and Harnack's inequality. Furthermore, in [31, 32], a dynamical approach is taken using modified Cayley graphs and dynamical zeta functions. There is also an approach using fractal differentiation [12]. In addition, as an application of [4], a Poisson formula for P-harmonic functions is established by application of the fibre dynamical property [3].

We consider the discrete approximation of a p.c.f. self-similar set by averages of its "boundary". The corresponding Markov chains are recurrent so that the Martin boundaries are trivial. In order to get a non trivial Martin boundary it is necessary to define Markov chains more suitable to represent harmonic functions on the p.c.f. self-similar sets. This means that there exists a Markov chain with discrete state space such that K is homeomorphic to the Martin boundary of the Markov chain. Note that the coding $\pi : \mathcal{A}^\infty \to K$ is a coding by the space of *ends*, i.e. an equivalence class of infinite paths. In the Pentakun case, two paths are equivalent if they differ by only finitely many vertices, and in general there exists an analogous definition for the equivalence classes. However, in general, it is not necessary to consider the space of ends.

We introduce the Pentakun $\mathcal{P}$.

Let $p_i \in \mathbb{R}^2$ for $i = -2, -1, 0, 1, 2$ in a Euclidean space and

$$|\overrightarrow{p_{-2}p_{-1}}| = |\overrightarrow{p_{-1}p_0}| = |\overrightarrow{p_0p_1}| = |\overrightarrow{p_1p_2}| = |\overrightarrow{p_2p_{-2}}| = 1$$

then

$$\Delta(p_{-2}, ..., p_2) := \bigcup_{j=-2}^{0} \{x | \overrightarrow{p_{-2}x} = s_j \overrightarrow{p_{-2}p_j} + t_j \overrightarrow{p_{-2}p_{j+2}} : s_j, t_j \geq 0, 0 \leq s_j + t_j \leq 1\}$$

is called a *regular simplex* if the vectors $\overrightarrow{p_{-2}p_{-1}}$ and $\overrightarrow{p_{-2}p_2}$ are linearly independent.

For $-2 \leq i, j \leq 2$, we define the points

$$p_{ij} = \frac{3 - \sqrt{5}}{2} p_i + \frac{\sqrt{5} - 1}{2} p_j$$

and for $-2 \leq \ell \leq 2$ we let

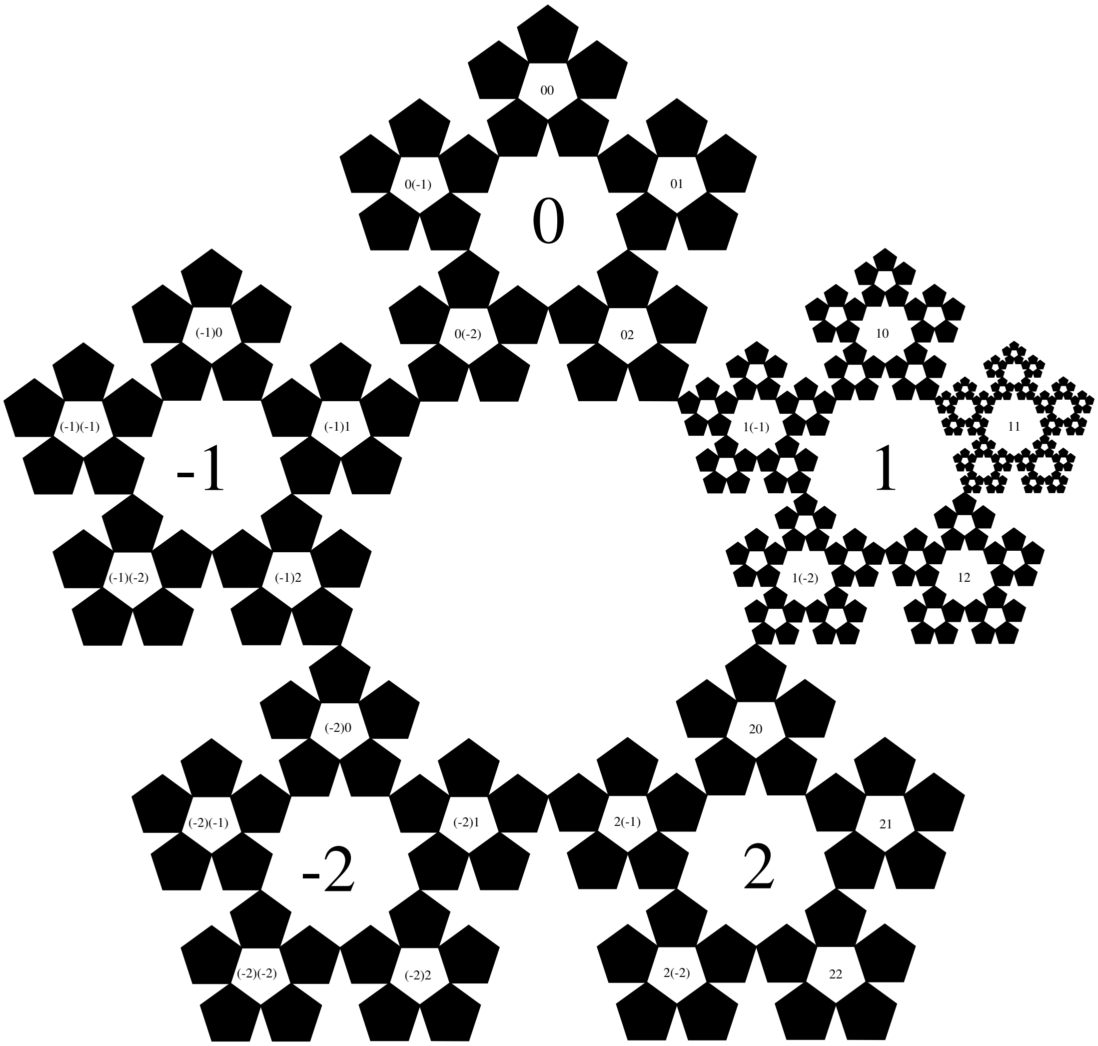
$$F_\ell : \Delta(p_{-2}, ..., p_2) \to \Delta(p_{-2}, ..., p_2)$$

denote the affine mappings onto the simplex generated by $p_{j\ell}$ and satisfying $F_\ell(p_j) = p_{j\ell}$. It is clear that p_ℓ is a fixed point of F_ℓ.

Let $\mathcal{A} = \{-2, -1, 0, 1, 2\}$ be the alphabet of five letters equipped with a module structure with the additive operation $\oplus$ modulo 5. Let $\mathcal{A}^n$ denote the collection of words consisting of n symbols and $\mathcal{A}^\infty$ the space of one-sided infinite sequences. In particular, $\mathcal{A}^0 = \{\emptyset\}$ where $\emptyset$ denotes the empty word. Then, for $\mathbf{w} \in \mathcal{W} \cup \mathcal{A}^\infty$, we define the conjugate $\mathbf{w}^{\#}$ of $\mathbf{w}$ by

$$\begin{cases} \mathbf{w}_0(a \oplus d/2)(a \oplus -d/2) & \text{if } \mathbf{w} = \mathbf{w}_0 a(a \oplus d)^k, d \in \{-2, 2\} \\ \mathbf{w} & \text{otherwise} \end{cases} \tag{1.11}$$

and an equivalence relation $\sim$ on $\mathcal{W} \cup \mathcal{A}^\infty$ by $\mathbf{x} = \mathbf{y}$ or $\mathbf{x}^{\#} = \mathbf{y}$ where $\mathcal{W} = \bigcup_{n=0}^{\infty} \mathcal{A}^n$, $k \in \mathbb{N} \cup \{\infty\}$, $a \in \mathcal{A}$ and $\mathbf{w}_0 \in \mathcal{W}$.



For $\mathbf{x} \in \mathcal{A}^n$ we define

$$F_{\mathbf{x}} = \begin{cases} F_{x_1} \circ F_{x_2} \circ \cdots \circ F_{x_n} & \text{if } \mathbf{x} = x_1 x_2 \cdots x_n \\ \text{identity} & \text{if } \mathbf{x} = \emptyset \end{cases}$$

and

$$\Delta(\mathbf{x}) = F_{\mathbf{x}}(\Delta(p_{-2}, ..., p_2)).$$

The *Pentakun* $\mathcal{P}$ is defined as

$$\mathcal{P} = \bigcap_{m=0}^{\infty} \bigcup_{\mathbf{x} \in \mathcal{A}^m} \Delta(\mathbf{x}).$$

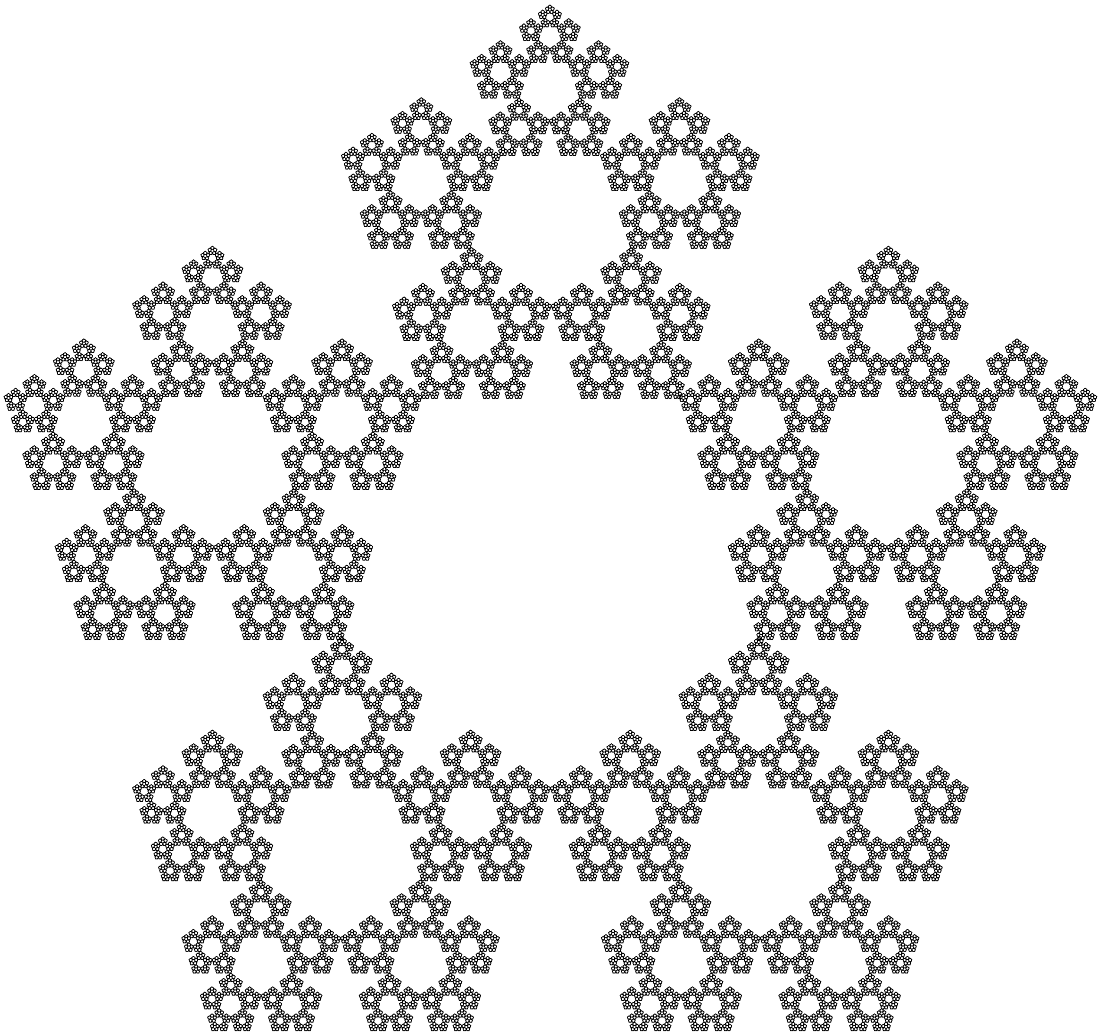


FIGURE 1. $\mathcal{P}$:Pentakun

It is clear that $\mathcal{P}$ is a compact metric space with the Euclidean distance in $\mathbb{R}^2$ restricted to $\mathcal{P}$.

It is known that for the Sierpiński gasket [4] and the Pentakun $\mathcal{P}$ (private communication of M. Denker, see the below theorem) the fractal K in (1.10) is homeomorphic to a quotient space $\mathcal{A}^\infty/\sim$.

Theorem 1.11. The space $\mathcal{A}^\infty/\sim$ and the Pentakun $\mathcal{P}$ are bi-Lipschitz equivalent.

Idea of proof. We introduce the metric

$$d(x,y) = \sum_{j=1}^{\infty} \left(\frac{3-\sqrt{5}}{2} \right)^j (1-\delta_{x_j,y_j})$$

on $\mathcal{A}^\infty$.

For $x = x_1x_2\cdots \in \mathcal{A}^\infty$ define $\mathbf{x}_n = x_1x_2\cdots x_n$ and $\Pi : \mathcal{A}^\infty \to \mathcal{P}$ by

$$\Pi(x) = \lim_{n\to\infty} \Delta(\mathbf{x}_n).$$

Since

$$|\Pi(\mathbf{x}) - \Pi(\mathbf{y})| \leq (2+\sqrt{5})d(\mathbf{x},\mathbf{y})$$

we have that Π is Lipschitz continuous and onto. Moreover, $\Pi(x) = \Pi(y)$ if and only if $x \sim y$.

Define $\pi : (\mathcal{A}^\infty/{\sim}) \to \mathcal{P}$ by

$$\pi(\tilde{x}) = \Pi(x) \quad \text{for } x \in \tilde{x} \in \mathcal{A}^\infty/{\sim}\ .$$

This map is well defined by the Lipschitz continuity of Π. It is also a bijection by general topology theory.

Define for $x, y \in \mathcal{A}^\infty$

$$d_0(x,y) = \begin{cases} d(x,y) & \text{if } x \not\sim y \\ 0 & \text{if } x \sim y, \end{cases}$$

and for $\tilde{x}, \tilde{y} \in \mathcal{A}^\infty/{\sim}$

$$\tilde{d}(\tilde{x},\tilde{y})$$
$$= \inf \left\{ \sum_{j=1}^{m} d_0(w^{j-1}, w^j) \,\middle|\, m \in \mathbb{N} : w^0 \in \tilde{x}, w^1 \in \mathcal{A}^\infty, ..., w^{m-1} \in \mathcal{A}^\infty, w^m \in \tilde{y} \right\}.$$

By some discussions, we obtain the following properties:

(1) $\tilde{d}$ is a metric, that is, $\mathcal{A}^\infty/{\sim}$ is metrizable with respect to the metric $\tilde{d}$.

(2) $|\pi(\tilde{x}) - \pi(\tilde{y})| < (3+\sqrt{5})\tilde{d}(\tilde{x},\tilde{y})$ whenever $\tilde{x} \neq \tilde{y}$.

(3) $\tilde{d}(\pi^{-1}(\xi), \pi^{-1}(\xi')) < (5-\sqrt{5})|\xi - \xi'|$ whenever $\xi \neq \xi'$.

1.4 Outline of the method.

The main goal of this paper is to prove that $\mathcal{P}$ can be represented as the Martin boundary of a canonical Markov chain which is not irreducible and does not have bounded range structure. This can be accomplished by defining the Martin kernel, and in fact the n-step transition probabilities.

Consider a finite alphabet $\mathcal{A}$ and the space $\mathcal{W}$ of finite words. Suppose we have already defined an equivalence relation $\sim$ on $\mathcal{A}^\infty$, so that $\mathcal{A}^\infty/{\sim}$ is homeomorphic to the fractal. This extends to $\mathcal{W}$ by defining $\mathbf{w}a \sim \mathbf{v}b$ if $\mathbf{w}a^\infty \sim \mathbf{v}b^\infty$ (see (1.11)).

A natural Markov chain for the fractal is given by a Markov chain with state space $\mathcal{W}$ and positive transition probabilities. Furthermore, it has a following properties:

- $\mathbf{w}$ is successor of $\mathbf{v}$ or its equivalent (dual) word.
- Transition probabilities are uniformly.

Consider a fractal which is totally disconnected. With $N = \#\mathcal{A}$ we choose the transition probabilities $p(\mathbf{w}, \mathbf{w}a) = 1/N$. In this case $p(n, \mathbf{w}, \mathbf{v}) = N^{-n}$ because there exists only one ancestor for each word $\mathbf{u}$. In case $(\mathcal{A}^\infty/\sim) \cong$ Sierpiński gasket there are exactly two ancestors (except "boundary" word $\mathbf{u}$, see [4]). Thus there exists a simple formula for $p(n, \mathbf{w}, \mathbf{v})$. In the Pentakun case we have a mixture of both phenomena. We now give a description of the estimate for the Green function of the natural modulo 5 fractal Markov chains.

We denote by $(X_n)_{n\in\mathbb{N}_0}$ the Markov chain with state space $\mathcal{W}$ and stationary transition probabilities

$$p(\mathbf{w}, \mathbf{w}a) = p(\mathbf{w}, \mathbf{w}^\# a) = \begin{cases} 1/10 & \text{if } \mathbf{w} \neq \mathbf{w}^\#, a \in \mathcal{A} \\ 1/5 & \text{if } \mathbf{w} = \mathbf{w}^\#, a \in \mathcal{A}. \end{cases} \tag{1.12}$$

The *Green function* $g(\mathbf{v}, \mathbf{w})$ on $\mathcal{W}$ is given by

$$g(\mathbf{v}, \mathbf{w}) = p(l(\mathbf{w}) - l(\mathbf{v}); \mathbf{v}, \mathbf{w}) := \sum_{\mathbf{u}\in\mathcal{W}} p(l(\mathbf{w}) - l(\mathbf{v}) - 1; \mathbf{v}, \mathbf{u}) p(\mathbf{u}, \mathbf{w}) \tag{1.13}$$

where $l(\mathbf{w})$ denotes the length of $\mathbf{w}$ and where $g(\mathbf{v}, \mathbf{w}) = \delta_{\mathbf{v},\mathbf{w}}$ whenever $l(\mathbf{v}) = l(\mathbf{w})$.

As a consequence of the above definitions (1.12) and (1.13), we became aware that the key to the estimation of the Green function is the behaviour of the difference of the last two different letters in a word, in particular, whether they are 2 (or -2) or not. The investigation is based upon this interesting discovery.

Superficially, the Pentakun is similar to the Sierpiński gasket as a geometric structure in $\mathbb{R}^2$, but in reality they are radically different in the behaviour of their critical sets as p.c.f. self-similar sets.

The structure of the Martin kernel $k(\mathbf{v}, \xi)$ (i.e. the normalised Green function in rough sense, see (1.6)) is such that it resembles the word space metric. If $\mathbf{v} \to \eta$, then $k(\mathbf{v}, \xi)$ vanishes outside a neighbourhood $U(\mathbf{v})$ also contracting to η. This immediately gives the final result that the Martin boundary is equal to $\mathcal{A}^\infty/\sim$.

The organisation of this paper is as follows. In section II we first give basic definitions and define the Martin function, called the Martin kernel. Next, we define the conjugate area and the non-conjugate area for one fixed word and derive the first result which is the estimation of the Martin kernels of the natural mod 5 Markov chain by considering the difference between two letters and the determination of a p-partial for the ancestor of a word. In section III we show that $\mathcal{P}$ is homeomorphic to the Martin boundary and also a space of exits using our result in section II. Finally, as a corollary to [5, 6], we describe the associated Dirichlet problem for $P-I$ on $\mathcal{W}$ using Dynkin's Theorem (Theorem 1.8).

2. The natural mod 5 Markov chain

2.1 The Pentakun graph.

In this section we study the symbolic representation of the Pentakun. We define the state space for the natural Markov chain and derive basic properties.

Definition 2.1. Let $\mathcal{A} := \{-2,-1,0,1,2\}$ be an *alphabet* of five letters and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We consider $\mathcal{A}$ as a *module* with addition mod 5, denoted by $\oplus$.

The following definitions give some basic concepts and notation.

(1) Let $\mathcal{A}^\infty$ be the space of one-sided infinite sequences.

(2) For $n \in \mathbb{N}_0$, let $\mathcal{A}^n$ be the collection of words consisting of n symbols. In particular, $\mathcal{A}^0 = \{\emptyset\}$ where $\emptyset$ denotes the *empty word.* Then the *word space* is defined by

$$\mathcal{W} = \bigcup_{n \in \mathbb{N}_0} \mathcal{A}^n.$$

We consider $(\mathcal{W}, \mathcal{O}_1)$ as a topological space where $\mathcal{O}_1$ is a discrete topology.

(3) For $n \in \mathbb{N}_0 \cup \{\infty\}$, the *length* of a word is defined by $l(\mathbf{x}) = n$ where $\mathbf{x} \in \mathcal{A}^n$.

(4) The *product* of two words is defined by $\mathbf{x}\mathbf{y} = x_1x_2\cdots x_ny_1y_2\cdots$ where $\mathbf{x} = x_1x_2\cdots x_n \in \mathcal{W}$ and $\mathbf{y} = y_1y_2\cdots \in \mathcal{W} \cup \mathcal{A}^\infty$.

(5) Let $\mathbf{w} = w_1w_2\cdots w_n \in \mathcal{W} \setminus \{\emptyset\}$. Then we define

$$\mathbf{w}^- = \begin{cases} w_1w_2\cdots w_n & \text{if } n \geq 2 \\ \emptyset & \text{if } n = 1. \end{cases}$$

(6) For fixed $p \in \mathcal{A}$, we define the function $\sigma_p : \mathcal{A} \to \mathcal{A}$ by $\sigma_p(a) = a \oplus p$.

Next, we define the *conjugate* of a word using a kind of involution.

Definition 2.2. Let $\mathcal{A} = \mathfrak{G}^\circ \cup \mathfrak{G}^\#$ where $\mathfrak{G}^\circ = \{-1,0,1\}$ and $\mathfrak{G}^\# = \{-2,2\}$.

(1) Let $\mathbf{w} \in \mathcal{W} \cup \mathcal{A}^\infty$. Then the *conjugate* $\mathbf{w}^\#$ of $\mathbf{w}$ is defined by

$$\mathbf{w}^\# = \begin{cases} \mathbf{w}_0\sigma_{d/2}(a)\sigma_{-d/2}(a)^k & \text{if } \mathbf{w} = \mathbf{w}_0a\sigma_d(a)^k, d \in \mathfrak{G}^\# \\ \mathbf{w} & \text{otherwise,} \end{cases} \tag{2.1}$$

where $k \in \mathbb{N} \cup \{\infty\}$, $a \in \mathcal{A}$ and $\mathbf{w}_0 \in \mathcal{W}$.

For example, since $-2 = \sigma_2(1)$, we have

$$(\mathbf{w}_01(-2)^k)^\# = (\mathbf{w}_01\sigma_2(1)^k)^\# = \mathbf{w}_0\sigma_1(1)\sigma_{-1}(1)^k = \mathbf{w}_020^k$$

which implies

$$(\mathbf{w}_0 1(-2)^k)^{\#\#} = (\mathbf{w}_0 20^k)^{\#} = (\mathbf{w}_0 2\sigma_{-2}(2)^k)^{\#} = \mathbf{w}_0\sigma_{-1}(2)\sigma_1(2)^k = \mathbf{w}_0 1(-2)^k.$$

(2) Define the function θ of $\mathbf{x} \in \mathcal{W}$ by

$$\theta(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x}^{\#} \neq \mathbf{x} \\ 2 & \text{if } \mathbf{x}^{\#} = \mathbf{x}. \end{cases}$$

We define the relation $\sim$ on $\mathcal{W} \cup \mathcal{A}^{\infty}$ by

$$\mathbf{x} \sim \mathbf{y} \iff \mathbf{x} = \mathbf{y} \text{ or } \mathbf{x} = \mathbf{y}^{\#}.$$

The next lemma is an immediate consequence.

Lemma 2.3. The relation $\sim$ is an equivalence relation on $\mathcal{W} \cup \mathcal{A}^{\infty}$.

Hence we can define the *Modulo 5 fractal* by $\mathcal{A}^{\infty}/\!\sim$.

For a fixed alphabet $\mathcal{A} = \{-2, -1, 0, 1, 2\}$ the Markov chain will be defined by the following transition probabilities $p(\cdot, \cdot)$ on $\mathcal{W} \times \mathcal{W}$, so it has state space $\mathcal{W}$.

Definition 2.4.

(1) We denote by $(X_n)_{n\in\mathbb{N}_0}$ the *Markov chain* state space $\mathcal{W}$ and define the *transition probabilities*

$$p(\mathbf{w}, \mathbf{w}a) = p(\mathbf{w}, \mathbf{w}^{\#}a) = \frac{\theta(\mathbf{w})}{10} \tag{2.2}$$

where $\mathbf{w} \in \mathcal{W}$ and $a \in \mathcal{A}$.

We call $\mathfrak{X} = (X_n)_{n\in\mathbb{N}_0}$ the natural Markov chain for the Pentakun.

(2) The *n-th step transition probabilities* on $\mathcal{W}$ are defined recursively by

$$\begin{cases} p(0; \mathbf{v}, \mathbf{w}) = \delta_{\mathbf{v},\mathbf{w}} \\ p(n; \mathbf{v}, \mathbf{w}) = \displaystyle\sum_{\mathbf{u}\in\mathcal{W}} p(n-1; \mathbf{v}, \mathbf{u}) p(\mathbf{u}, \mathbf{w}) \quad \mathbf{v}, \mathbf{w} \in \mathcal{W}, n \in \mathbb{N}. \end{cases} \tag{2.3}$$

Lemma 2.5. Let $\mathbf{v}, \mathbf{w} \in \mathcal{W}$. Then we have $p(n; \mathbf{v}, \mathbf{w}) > 0$ only if $n = l(\mathbf{w}) - l(\mathbf{v})$.

Proof. We shall prove this lemma by induction over n. If $p(1; \mathbf{v}, \mathbf{w}) > 0$, then by (2.3) we have $\sum_{\mathbf{u}\in\mathcal{W}} p(0; \mathbf{v}, \mathbf{u}) p(\mathbf{u}, \mathbf{w}) > 0$ which occurs only if

$$\begin{cases} \mathbf{v} = \mathbf{u} \\ \mathbf{w} = \mathbf{u}a \text{ or } \mathbf{u}^{\#}a \text{ for some } a \in \mathcal{A} \end{cases}$$

and hence $l(\mathbf{w}) - l(\mathbf{v}) = l(\mathbf{u}) + 1 - l(\mathbf{u}) = 1$. This argument also gives the induction step. $\square$

Definition 2.6. The non-oriented edge set as a symmetric subset of $\mathcal{W} \times \mathcal{W}$, that is,

$$\big\{\text{edges } [\mathbf{x}, \mathbf{y}] | \mathbf{x}, \mathbf{y} \in \mathcal{W} : \mathbf{y} = \mathbf{x}a \text{ or } \mathbf{y} = \mathbf{x}^{\#}a, a \in \mathcal{A}\big\}$$

is called the *Pentakun graph*.

Let $\mathbf{x}, \mathbf{y} \in \mathcal{W}$. A path from $\mathbf{x}$ to $\mathbf{y}$ is a collection $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_s\} \subset \mathcal{W}$ such that $l(\mathbf{u}_i) = l(\mathbf{u}_{i-1}) + 1, \mathbf{u}_1 = \mathbf{x}, \mathbf{u}_s = \mathbf{y}$ and $p(\mathbf{u}_i, \mathbf{u}_{i+1}) > 0$ for all $1 \le i < s$. Define

$$\mathbf{n}(\mathbf{x}, \mathbf{y}) = \begin{cases} \text{the number of paths from } \mathbf{x} \text{ to } \mathbf{y} & \text{if } p(l(\mathbf{y}) - l(\mathbf{x}); \mathbf{x}, \mathbf{y}) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.7. The Pentakun graph is *connected*, i.e. for every $\mathbf{x}, \mathbf{y} \in \mathcal{W}$ there exists a finite path from $\mathbf{x}$ to $\mathbf{y}$.

Proof. Obvious. □

2.2 The Green function and the Martin kernel.

The object of this section is to estimate the Martin kernel defined in (2.5). The key to the estimation is the difference between two letters and the determination of a p-partial for the ancestor of a word.

We first define the Green function and the Martin kernel.

By Lemma 2.5, we have

$$\sum_{n=0}^{\infty} p(n; \mathbf{v}, \mathbf{w}) = p(l(\mathbf{w}) - l(\mathbf{v}); \mathbf{v}, \mathbf{w})$$

and hence by (1.4), the Green function on $\mathcal{W}$ is defined as follows.

Definition 2.8. The *Green function* on $\mathcal{W}$ is given by

$$g(\mathbf{v}, \mathbf{w}) = p(l(\mathbf{w}) - l(\mathbf{v}); \mathbf{v}, \mathbf{w}) \tag{2.4}$$

and $\mathbf{v}$ is called an *ancestor* of $\mathbf{w}$ if $g(\mathbf{v}, \mathbf{w}) > 0$. In particular, if $g(\mathbf{v}, \mathbf{w}) > 0$ and $l(\mathbf{w}) - l(\mathbf{v}) = k$, then $\mathbf{w}$ is called k*-ancestor* of $\mathbf{w}$. We denote the collection of k-*ancestors* of $\mathbf{w}$ by $\mathbf{Anc}_k[\mathbf{w}]$.

Lemma 2.9. Let $\mathbf{x}, \mathbf{y} \in \mathcal{W}$ such that $l(\mathbf{x}) \le l(\mathbf{y})$ and let $a \in \mathcal{A}$. Then we have

$$g(\mathbf{x}, \mathbf{y}a) = \frac{1}{10}\{g(\mathbf{x}, \mathbf{y}) + g(\mathbf{x}, \mathbf{y}^{\#})\}.$$

Proof. Note that $l(\mathbf{y}) = l(\mathbf{y}^{\#})$ for all $\mathbf{y} \in \mathcal{W}$. Since $g(\mathbf{x}, \mathbf{y}a) = g(\mathbf{x}, \mathbf{y}) = g(\mathbf{x}, \mathbf{y}^{\#}) = 0$ whenever $l(\mathbf{x}) > l(\mathbf{y})$, we may assume that $l(\mathbf{x}) \le l(\mathbf{y})$. By (2.2) and (2.4), we have

$$\begin{aligned} &g(\mathbf{x}, \mathbf{y}a) \\ =&p(l(\mathbf{y}) + 1 - l(\mathbf{x}); \mathbf{x}, \mathbf{y}a) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\mathbf{u}\in\mathcal{W}} p(l(\mathbf{y})-l(\mathbf{x});\mathbf{x},\mathbf{u})p(\mathbf{u},\mathbf{y}a)\\
&= \begin{cases} \dfrac{1}{5}\times p(l(\mathbf{y})-l(\mathbf{x});\mathbf{x},\mathbf{y}) & \text{if } \mathbf{y}=\mathbf{y}^{\#}\\ \dfrac{1}{10}\times p(l(\mathbf{y})-l(\mathbf{x});\mathbf{x},\mathbf{y})+\dfrac{1}{10}\times p(l(\mathbf{y}^{\#})-l(\mathbf{x});\mathbf{x},\mathbf{y}^{\#}) & \text{if } \mathbf{y}\neq\mathbf{y}^{\#}\end{cases}\\
&= \begin{cases} \dfrac{1}{5}\times g(\mathbf{x},\mathbf{y}) & \text{if } \mathbf{y}=\mathbf{y}^{\#}\\ \dfrac{1}{10}\times g(\mathbf{x},\mathbf{y})+\dfrac{1}{10}\times g(\mathbf{x},\mathbf{y}^{\#}) & \text{if } \mathbf{y}\neq\mathbf{y}^{\#},\end{cases}
\end{aligned}
$$

and hence the lemma follows. □

Lemma 2.10. Let $\mathbf{x}\in\mathcal{W}$. Then we have $g(\emptyset,\mathbf{x})=1/5^{l(\mathbf{x})}$.

Proof. Using induction over $l(\mathbf{x})$, the assertion follows immediately. In the case of $l(\mathbf{x})\leq 1$, the lemma is an immediate consequence of (2.4). We now assume $g(\emptyset,\mathbf{x})=1/5^{l(\mathbf{x})}$. Let $\mathbf{y}\in\mathcal{W}$ such that $l(\mathbf{y})=l(\mathbf{x})$ and let $a\in\mathcal{A}$. Then by Lemma 2.9 we have

$$
\begin{aligned}
g(\emptyset,\mathbf{y}a) &= \frac{1}{10}\times g(\emptyset,\mathbf{y})+\frac{1}{10}\times g(\emptyset,\mathbf{y}^{\#})\\
&= \frac{1}{10}\times\frac{1}{5^{l(\mathbf{y})}}+\frac{1}{10}\times\frac{1}{5^{l(\mathbf{y}^{\#})}}\\
&= \frac{1}{10}\times\frac{1}{5^{l(\mathbf{y})}}+\frac{1}{10}\times\frac{1}{5^{l(\mathbf{y})}}\\
&= \frac{1}{5^{l(\mathbf{y})+1}}.
\end{aligned}
$$

The lemma is proved. □

Referring [4], (1.6) and the above lemma, the Martin kernel on $\mathcal{W}$ is defined as follows.

Definition 2.11. The *Martin kernel* (for g) on $\mathcal{W}$ is defined by

$$
k(\mathbf{v},\mathbf{w})=\frac{g(\mathbf{v},\mathbf{w})}{g(\emptyset,\mathbf{w})}=5^{l(\mathbf{w})}g(\mathbf{v},\mathbf{w}). \tag{2.5}
$$

Note that if $l(\mathbf{w})>l(\mathbf{v})$, then we have

$$
k(\mathbf{v},\mathbf{w})=5^{l(\mathbf{w})-l(\mathbf{v})}\psi(\mathbf{v})g(\mathbf{v},\mathbf{w})
$$

where $\psi(\mathbf{v}):=5^{l(\mathbf{v})}\theta(\mathbf{v})$.

The following proposition will play an important role throughout this paper. It states that the ancestor of an arbitrary finite word lies in the neighbourhood of its first letter.

Proposition 2.12. Let $\mathbf{x}, \mathbf{y}_\ell \in \mathcal{W}$ such that $\mathbf{y}_\ell = \mathbf{x}a_1a_2\cdots a_\ell a_{\ell+1}$ where $a_j \in \mathcal{A}, j = 1, 2, ..., \ell, \ell+1$ and $\ell \in \mathbb{N}$. Then we have

$$\mathbf{Anc}_\ell[\mathbf{y}_\ell] \cup \mathbf{Anc}_\ell[\mathbf{y}_\ell^\#] \subset \bigcup_{p\in\mathfrak{G}^\circ} \{\mathbf{x}\sigma_p(a_1), (\mathbf{x}\sigma_p(a_1))^\#\}. \tag{2.6}$$

Proof. We shall prove the proposition by induction over ℓ. We set $a_2 = a_1 \oplus d_1$. In the case $\ell = 1$, by (2.3) and (2.4),

$$\begin{aligned} 0 <& g(\mathbf{v}, \mathbf{x}a_1a_2) \\ =& p(1; \mathbf{v}, \mathbf{x}a_1a_2) \\ =& p(0; \mathbf{v}, \mathbf{x}a_1)p(\mathbf{x}a_1, \mathbf{x}a_1a_2) + p(0; \mathbf{v}, (\mathbf{x}a_1)^\#)p((\mathbf{x}a_1)^\#, \mathbf{x}a_1a_2) \end{aligned} \tag{2.7}$$

which implies $\mathbf{Anc}_1[\mathbf{x}a_1a_2] = \{\mathbf{x}a_1, (\mathbf{x}a_1)^\#\}$. On the other hand, by (2.1)

$$(\mathbf{x}a_1a_2)^\# = \begin{cases} \mathbf{x}a_1a_2 & \text{if } d_1 \in \mathfrak{G}^\circ \\ \mathbf{x}\sigma_{d_1/2}(a_1)\sigma_{-d_1/2}(a_1) & \text{if } d_1 \in \mathfrak{G}^\# \end{cases}$$

and by the same argument as above, we have

$$\mathbf{Anc}_1[(\mathbf{x}a_1a_2)^\#] = \begin{cases} \{\mathbf{x}a_1, (\mathbf{x}a_1)^\#\} & \text{if } d_1 \in \mathfrak{G}^\circ \\ \{\mathbf{x}\sigma_{d_1/2}(a_1), (\mathbf{x}\sigma_{d_1/2}(a_1))^\#\} & \text{if } d_1 \in \mathfrak{G}^\#. \end{cases}$$

Since $d_1/2 \in \mathfrak{G}^\circ$ whenever $d_1 \in \mathfrak{G}^\#$, we obtain that (2.6) is true for $\ell = 1$. Assume that (2.6) is true for ℓ. Then for $a_{\ell+2} \in \mathcal{A}$, by the induction hypothesis and (2.3),

$$\begin{aligned} \mathbf{Anc}_{\ell+1}[\mathbf{y}_{\ell+1}] &= \mathbf{Anc}_\ell[\mathbf{y}_\ell] \cup \mathbf{Anc}_\ell[\mathbf{y}_\ell^\#] \\ &\subset \bigcup_{p\in\mathfrak{G}^\circ} \{\mathbf{x}\sigma_p(a_1), (\mathbf{x}\sigma_p(a_1))^\#\} \end{aligned}$$

and

$$\mathbf{Anc}_\ell[((\mathbf{x}a_1)a_2a_3\cdots a_{\ell+1}a_{\ell+2})^\#] \subset \bigcup_{p\in\mathfrak{G}^\circ} \{(\mathbf{x}a_1)\sigma_p(a_2), ((\mathbf{x}a_1)\sigma_p(a_2))^\#\}. \tag{2.8}$$

Hence by (2.8), we have for $a_2 = \sigma_q(a_1)$

$$\begin{aligned} \mathbf{Anc}_{\ell+1}[\mathbf{y}_{\ell+1}^\#] &= \bigcup_{\mathbf{z}\in\mathbf{Anc}_\ell[\mathbf{y}_{\ell+1}^\#]} \mathbf{Anc}_1[\mathbf{z}] \\ &\subset \bigcup_{p\in\mathfrak{G}^\circ} \{\mathbf{Anc}_1[\mathbf{x}a_1\sigma_{p\oplus q}(a_1)] \cup \mathbf{Anc}_1[(\mathbf{x}a_1\sigma_{p\oplus q}(a_1))^\#]\} \\ &= \begin{cases} \bigcup_{p\in\{0,q\}} \{\mathbf{x}\sigma_p(a_1), (\mathbf{x}\sigma_p(a_1))^\#\} & \text{if } q \in \mathfrak{G}^\circ \\ \bigcup_{p\in\mathfrak{G}^\circ} \{\mathbf{x}\sigma_p(a_1), (\mathbf{x}\sigma_p(a_1))^\#\} & \text{if } q \in \mathfrak{G}^\#, \end{cases} \end{aligned}$$

and thus the proposition follows. $\square$

Consequently, we have

$$
\begin{aligned}
&g(\mathbf{x}a, \mathbf{u}) \\
&= \sum_{\mathbf{z} \in \mathbf{Anc}_k[\mathbf{u}]} p(\ell - k; \mathbf{x}a, \mathbf{z}) p(k; \mathbf{z}, \mathbf{u}) \\
&= \sum_{p \in \mathfrak{G}^\circ} \{ g(\mathbf{x}a, \mathbf{x}a_1 a_2 \cdots a_{\ell-k} \sigma_p(a_{\ell-k+1})) g(\mathbf{x}a_1 a_2 \cdots a_{\ell-k} \sigma_p(a_{\ell-k+1}), \mathbf{u}) \\
&\qquad + g(\mathbf{x}a, (\mathbf{x}a_1 a_2 \cdots a_{\ell-k} \sigma_p(a_{\ell-k+1}))^\#) g((\mathbf{x}a_1 a_2 \cdots a_{\ell-k} \sigma_p(a_{\ell-k+1}))^\#, \mathbf{u}) \}
\end{aligned}
$$

for any $\mathbf{u} \in \bigcup_{q \in \mathfrak{G}^\circ} \{ \mathbf{y}_\ell^- \sigma_q(a_{\ell+1}), (\mathbf{y}_\ell^- \sigma_q(a_{\ell+1}))^\# \}$ and $1 \le k \le \ell$.

This leads us to the definition of a p-partial and the neighbourhood of a word.

Definition 2.13. Let $\mathbf{x} \in \mathcal{W}$ and $a \in \mathcal{A}$. Then a *p-partial* of $\mathbf{x}a$ is defined by

$$U_p(\mathbf{x}a) := \{ \mathbf{x}\sigma_p(a), (\mathbf{x}\sigma_p(a))^\# \}.$$

Furthermore, the *neighbourhood* of $\mathbf{x}a$ is defined by

$$U(\mathbf{x}a) := \bigcup_{p \in \mathfrak{G}^\circ} \{ \mathbf{x}\sigma_p(a), (\mathbf{x}\sigma_p(a))^\# \} = \bigcup_{p \in \mathfrak{G}^\circ} U_p(\mathbf{x}a).$$

For $a_k \in \mathcal{A}, k \in \mathbb{N}$ we define $d_k = a_{k+1} - a_k$. d_k is called the *difference* between letters a_k and a_{k+1}. Note that d_k is also $\mathcal{A}$-valued.

Definition 2.14. Let $\mathbf{x} \in \mathcal{W}$ such that $\mathbf{x} = a_1 a_2 \cdots a_\ell a_{\ell+1}$ where $a_j \in \mathcal{A}, j = 1, 2, ..., \ell, \ell+1$ and $\ell \in \mathbb{N}$. The part of $\mathbf{x}$ from a_{ℓ_1} to $a_{\ell_1+\ell_2}$ is called *conjugate area* (resp. *non-conjugate area*) if $d_k \in \mathfrak{G}^\#$ (resp. $d_k \in \mathfrak{G}^\circ$) for $\ell_1 \le \forall k \le \ell_1 + \ell_2 - 1$ where $\ell_1, \ell_2 \in \{1, 2, ..., \ell\}$ with $\ell_1 + \ell_2 \le \ell + 1$.

$\mathbf{x}$ is always divided into two parts of area whenever $\mathbf{x} \in \bigcup_{n=2}^\infty \mathcal{A}^n$. For example, let $\mathbf{x} \in \mathcal{W}$ such that

$$\mathbf{x} = 1(-2)(-1)2021 =: a_1 a_2 a_3 a_4 a_5 a_6 a_7.$$

Since $d_1, d_3, d_4, d_5 \in \mathfrak{G}^\#$ and $d_2, d_6 \in \mathfrak{G}^\circ$, we have that the conjugate area of $\mathbf{x}$ is from a_1 to a_2 and a_3 to a_6 and that the non-conjugate area of $\mathbf{x}$ is from a_2 to a_3 and a_6 to a_7.

The following Lemma 2.15 is the cornerstone of our discussion in the conjugate area. This lemma states that one fundamental difference between two letters is the sequence in which 2 and -2 appear alternately.

We set $d^* = d/2$ for $d \in \mathfrak{G}^\#$.

Lemma 2.15. Let $\mathbf{x}, \mathbf{y}_\ell \in \mathcal{W}$ such that $\mathbf{y}_\ell = \mathbf{x}a_1 a_2 \cdots a_\ell a_{\ell+1}$ where $a_j \in \mathcal{A}, j = 1, 2, ..., \ell, \ell+1, d_k \in \mathfrak{G}^\#, k = 1, 2, ..., \ell$ and $\ell \in \mathbb{N}$.

(1) If $d_k = d_1(-1)^{k-1}$, then we have

$$\mathbf{Anc}_\ell[\mathbf{y}_\ell] = \bigcup_{p\in\{0,d_1^*(1-\delta_{1,\ell})\}} U_p(\mathbf{x}a_1)$$

and

$$\mathbf{Anc}_\ell[\mathbf{y}_\ell^\#] = \bigcup_{p\in\{d_1^*\delta_{1,\ell},d_1^*(1-\delta_{2,\ell})\}} U_p(\mathbf{x}a_1).$$

Consequently we have

(2)

$$\mathbf{Anc}_\ell[\mathbf{y}_\ell] = \begin{cases} \displaystyle\bigcup_{p\in\{0,d_1^*(1-\delta_{1,\ell})\}} U_p(\mathbf{x}a_1) & \text{if } \ell = 1,2 \\ \displaystyle\bigcup_{k=1}^{\ell-2} \bigcup_{p\in\{0,d_1^*,-d_1^*\delta_{d_k,d_{k+1}}\}} U_p(\mathbf{x}a_1) & \text{if } \ell \geq 3 \end{cases}$$

and

$$\mathbf{Anc}_\ell[\mathbf{y}_\ell^\#] = \begin{cases} U_{d_1^*}(\mathbf{x}a_1) & \text{if } \ell = 1 \\ \displaystyle\bigcup_{p\in\{0,-d_1^*\delta_{d_1,d_2}\}} U_p(\mathbf{x}a_1) & \text{if } \ell = 2 \\ \displaystyle\bigcup_{p\in\{0,d_1^*,-d_1^*(\delta_{d_2,d_3}-\delta_{d_1,d_2})\}} U_p(\mathbf{x}a_1) & \text{if } \ell = 3 \\ \displaystyle\bigcup_{k=1}^{\ell-3} \bigcup_{p\in\{0,d_1^*,-d_1^*\delta_{d_k,d_{k+1}},-d_1^*(\delta_{d_{\ell-1},d_\ell}-\delta_{d_{\ell-2},d_{\ell-1}})\}} U_p(\mathbf{x}a_1) & \text{if } \ell \geq 4. \end{cases}$$

Proof. We begin with the proof of (1). We only show the case where $\ell \geq 3$. Note that $\mathbf{y}_3 = \mathbf{x}a_1\sigma_{d_1}(a_1)a_1\sigma_{d_1}(a_1)$ whence $\mathbf{y}_3^\# = \mathbf{x}a_1\sigma_{d_1}(a_1)\sigma_{d_1^*}(a_1)\sigma_{-d_1^*}(a_1)$. In the case $\ell = 3$, by the same argument as that in (2.7), we can construct the following diagram:

$$\begin{array}{ccccc}
\mathbf{y}_3 & & & & \mathbf{y}_3^\# \\
\uparrow & \nwarrow & & & \uparrow \\
\begin{array}{c}(\mathbf{x}a_1\sigma_{d_1}(a_1)a_1)^\# \\ = \ \mathbf{x}a_1\sigma_{d_1^*}(a_1)\sigma_{-d_1}(a_1)\end{array} & & \mathbf{x}a_1\sigma_{d_1}(a_1)a_1 & & \mathbf{x}a_1\sigma_{d_1}(a_1)\sigma_{d_1^*}(a_1) \\
\uparrow & & \uparrow & \nwarrow\!\!\nearrow & \uparrow \\
\mathbf{x}a_1\sigma_{d_1^*}(a_1) & & \mathbf{x}a_1\sigma_{d_1}(a_1) & & \begin{array}{c}(\mathbf{x}a_1\sigma_{d_1}(a_1))^\# \\ = \ \mathbf{x}\sigma_{d_1^*}(a_1)\sigma_{-d_1^*}(a_1)\end{array} \\
\uparrow & \nwarrow\!\!\nearrow & \uparrow & & \nearrow\nwarrow \\
\mathbf{x}a_1 & & (\mathbf{x}a_1)^\# & & \mathbf{x}\sigma_{d_1^*}(a_1) \quad (\mathbf{x}\sigma_{d_1^*}(a_1))^\#
\end{array}$$

(2.9) transition diagram for $\ell = 3, d_1 = \pm 2$ and $d_2 = \mp 2$, which corresponds to (1) for $\ell = 3$. Assume that (1) is true for $\ell(\geq 3)$. Then by the induction hypothesis,

$$\mathbf{Anc}_\ell[\mathbf{x}a_2a_3\cdots a_{\ell+1}a_{\ell+2}] = \mathbf{Anc}_\ell[(\mathbf{x}a_2a_3\cdots a_{\ell+1}a_{\ell+2})^{\#}] = \bigcup_{p\in\{0,d_2^*\}} U_p(\mathbf{x}a_2),$$

so that we have

$$\mathbf{Anc}_\ell[\mathbf{y}_{\ell+1}] = \mathbf{Anc}_\ell[\mathbf{y}_{\ell+1}^{\#}] = \bigcup_{p\in\{0,d_2^*\}} U_p(\mathbf{x}a_1a_2),$$

and since $d_1 = -d_2$ and

$$\mathbf{Anc}_{\ell+1}[\mathbf{y}_{\ell+1}] = \mathbf{Anc}_1[\mathbf{Anc}_\ell[\mathbf{y}_{\ell+1}]],$$

we have

$$\begin{aligned}
\mathbf{Anc}_{\ell+1}[\mathbf{y}_{\ell+1}] &= \mathbf{Anc}_{\ell+1}[\mathbf{y}_{\ell+1}^{\#}] \\
&= \bigcup_{\mathbf{z}\in\{\mathbf{y}_1,\mathbf{y}_1^{\#},\mathbf{y}_1^{-}\sigma_{d_2^*}(a_2),(\mathbf{y}_1^{-}\sigma_{d_2^*}(a_2))^{\#}\}} \mathbf{Anc}_1[\mathbf{z}] \\
&= \bigcup_{\mathbf{z}\in\{\mathbf{y}_1^{-}\sigma_{d_1}(a_1),\mathbf{y}_1^{-}\sigma_{d_1^*}(a_1),\mathbf{x}\sigma_{d_1^*}(a_1)\sigma_{-d_1^*}(a_1)\}} \mathbf{Anc}_1[\mathbf{z}] \\
&= \bigcup_{p\in\{0,d_1^*\}} U_p(\mathbf{x}a_1)
\end{aligned}$$

which implies (1). By (1), for the remaining parts of the proof it suffices to show that if there exists $p \in \{1, 2, ..., \ell-2\}$ such that $d_p = d_{p+1}$, then we have

$$\mathbf{Anc}_\ell[\mathbf{y}_\ell] = U(\mathbf{x}a_1) \tag{2.10}$$

We assume $\ell \geq 3$. In the case $\ell = 3$ and $d_1 = d_2$, by the discussion following (2.9), we obtain the following diagram:

$$\begin{array}{cccc}
 & \mathbf{y}_3 = \mathbf{x}a_1\sigma_{d_1}(a_1)\sigma_{-d_1^*}(a_1)\sigma_{d_1^*}(a_1) & & \\
\nearrow & & & \nwarrow \\
\mathbf{x}a_1\sigma_{d_1}(a_1)\sigma_{-d_1^*}(a_1) & & & \begin{array}{c}(\mathbf{x}a_1\sigma_{d_1}(a_1)\sigma_{-d_1^*}(a_1))^{\#} \\ = \mathbf{x}a_1\sigma_{-d_1}(a_1)\sigma_{d_1^*}(a_1)\end{array} \\
\uparrow & \nwarrow & \nearrow & \uparrow \\
\begin{array}{c}(\mathbf{x}a_1\sigma_{d_1}(a_1))^{\#} \\ = \mathbf{x}\sigma_{d_1^*}(a_1)\sigma_{-d_1^*}(a_1)\end{array} & \mathbf{x}a_1\sigma_{d_1}(a_1) & \mathbf{x}a_1\sigma_{-d_1}(a_1) & \begin{array}{c}(\mathbf{x}a_1\sigma_{-d_1}(a_1))^{\#} \\ = \mathbf{x}\sigma_{-d_1^*}(a_1)\sigma_{d_1^*}(a_1)\end{array} \\
\nearrow\nwarrow & \uparrow \quad \nwarrow\!\!\!\!\nearrow & \uparrow & \nearrow\nwarrow \\
\mathbf{x}\sigma_{d_1^*}(a_1) \quad (\mathbf{x}\sigma_{d_1^*}(a_1))^{\#} & \mathbf{x}a_1 & (\mathbf{x}a_1)^{\#} & \mathbf{x}\sigma_{-d_1^*}(a_1) \quad (\mathbf{x}\sigma_{-d_1^*}(a_1))^{\#}
\end{array}$$

transition diagram for $\ell = 3, d_1 = d_2 = \pm 2$.

Suppose that (2.10) is true for ℓ (≥ 3). Then, by the induction hypothesis and (1), we have that if $d_1 = d_2, d_k = d_2(-1)^{k-2}$ $(k \geq 2)$, then

$$\mathbf{Anc}_\ell[\mathbf{y}_{\ell+1}] = \bigcup_{p \in \{0, d_2^*\}} U_p(\mathbf{x}a_1a_2)$$

and if $d_{k_0} = d_{k_0+1}$ for some $k_0 \in \{2, 3, ..., \ell - 1\}$, then

$$\mathbf{Anc}_\ell[\mathbf{y}_{\ell+1}] = U(\mathbf{x}a_1a_2).$$

In either case, we obtain $\mathbf{Anc}_{\ell+1}[\mathbf{y}_{\ell+1}] = U(\mathbf{x}a_1)$ and thus the lemma follows. $\square$

The next lemma gives an ancestor of the non-conjugate area.

Lemma 2.16. Let $\mathbf{x}, \mathbf{y}_\ell \in \mathcal{W}$ such that $\mathbf{y}_\ell = \mathbf{x}a_1a_2 \cdots a_\ell a_{\ell+1}$ where $a_j \in \mathcal{A}, j = 1, 2, ..., \ell, \ell+1, d_k \in \mathfrak{G}^\circ, k = 1, 2, ..., \ell$ and $\ell \in \mathbb{N}$. Then we have

(1) $\mathbf{Anc}_\ell[\mathbf{y}_\ell] = \mathbf{Anc}_\ell[\mathbf{y}_\ell^\#] = U_0(\mathbf{x}a_1)$.

(2) $\mathbf{Anc}_\ell[(\mathbf{y}_\ell^- \sigma_{d_\ell}(a_{\ell+1}))^\#] = \begin{cases} \displaystyle\bigcup_{p \in \{d_1\delta_{1,\ell}, d_1\}} U_p(\mathbf{x}a_1) & \text{if } d_k \equiv d_1 (\neq 0) \\ U_0(\mathbf{x}a_1) & \text{otherwise.} \end{cases}$

Proof. We only have to show (2). By the discussion following (2.9), we know that the lemma holds for $\ell = 1, 2$.

For $\ell \geq 2$ and $d_k \equiv d_1$ $(\neq 0)$, we assume

$$\mathbf{Anc}_\ell[(\mathbf{y}_\ell^- \sigma_{d_\ell}(a_{\ell+1}))^\#] = \bigcup_{p \in \{0, d_1\}} U_p(\mathbf{x}a_1)$$

which implies for $a_{\ell+2} = \sigma_{d_{\ell+1}}(a_{\ell+1})$

$$\mathbf{Anc}_\ell[(\mathbf{x}a_2a_3 \cdots a_\ell a_{\ell+1}\sigma_{d_{\ell+1}}(a_{\ell+2}))^\#] = \bigcup_{p \in \{0, d_2\}} U_p(\mathbf{x}a_2)$$

and therefore we have

$$\begin{aligned} \mathbf{Anc}_\ell[(\mathbf{y}_{\ell+1}\sigma_{d_{\ell+1}}(a_{\ell+2}))^\#] &= \{\mathbf{x}a_1a_2, \mathbf{x}a_1\sigma_{d_2}(a_2), (\mathbf{x}a_1\sigma_{d_2}(a_2))^\#\} \\ &= \{\mathbf{x}a_1a_2, \mathbf{x}a_1\sigma_{2d_1}(a_1), \mathbf{x}\sigma_{d_1}(a_1)\sigma_{-d_1}(a_1)\}. \end{aligned}$$

In all other cases, we may assume that there exists some $p \in \{1, 2, ..., \ell\}$ such that

$$\begin{cases} d_p = \{-d_{p+1}, 0\} \\ d_k \equiv d_{p+1} \quad (p+1 \leq k \leq \ell) \end{cases}$$

where $d_{p+1} \in \{-1, 1\}$. Then we obtain

$$\begin{aligned} &\mathbf{Anc}_{\ell-p}[(\mathbf{y}_\ell^- \sigma_{d_\ell}(a_{\ell+1}))^\#] \\ &= \bigcup_{q \in \{0, d_{p+1}\}} U_q(\mathbf{x}a_1a_2 \cdots a_pa_{p+1}) \end{aligned}$$

$$= \begin{cases} \{\mathbf{x}a_1a_2\cdots a_pa_p, (\mathbf{x}a_1a_2\cdots a_pa_p)^{\#}, \mathbf{x}a_1a_2\cdots a_p\sigma_{d_{p+1}}(a_p)\} & \text{if } d_p = 0 \\ \{\mathbf{x}a_1a_2\cdots a_pa_p, (\mathbf{x}a_1a_2\cdots a_pa_p)^{\#}, \mathbf{x}a_1a_2\cdots a_p\sigma_{-d_{p+1}}(a_p)\} & \text{if } d_p = -d_{p+1}. \end{cases}$$

and hence the lemma follows. □

Next, we shall calculate the Martin kernel on $\mathcal{W}$. In the last two lemmas, we have already constructed ancestors. Hence the following discussion assumes their existence.

Henceforth we will take $\mathfrak{G}^\circ$ as the representative element of $\mathbb{Z}/3\mathbb{Z}$ when no confusion can arise.

Define the map $\varphi : \mathfrak{G}^\circ \to \mathfrak{G}^\circ$ by $\varphi(\ell) := -\ell$. Since the projection $\mathbb{Z}$ onto $\mathfrak{G}^\circ$ is well-defined, we may assume that the domain of φ is $\mathbb{Z}$. For example, $\varphi(2) = \varphi(-1) = 1, \varphi(-5) = \varphi(1) = -1$, etc.

Lemma 2.17. Let $\mathbf{x}, \mathbf{y}_\ell \in \mathcal{W}$ such that $\mathbf{y}_\ell = \mathbf{x}a_1a_2\cdots a_\ell a_{\ell+1}$ where $a_j \in \mathcal{A}, j = 1, 2, ..., \ell, \ell+1, d_k = d_1(-1)^{k-1}, d_1 \in \mathfrak{G}^{\#}, k = 1, 2, ..., \ell$ and $\ell \in \mathbb{N}$. Then we have for $q \in \mathfrak{G}^\circ$ and $\mathbf{v} \in U_q(\mathbf{x}a_1)$

$$(1) \quad k(\mathbf{v}, \mathbf{y}_\ell) = \frac{\psi(\mathbf{v})}{6}\left(1 + \varphi(d_1^*q - 1) + \frac{2\varphi(d_1^*q+1)(-1)^\ell}{2^\ell}\right).$$

$$(2) \quad k(\mathbf{v}, \mathbf{y}_\ell^{\#}) = \frac{\psi(\mathbf{v})}{6}\left(1 + \varphi(d_1^*q - 1) - \frac{4\varphi(d_1^*q+1)(-1)^\ell}{2^\ell}\right).$$

$$(3) \quad k(\mathbf{v}, (\mathbf{y}_\ell^- \sigma_{d_\ell^*}(a_{\ell+1}))^{\#})$$
$$= \frac{\psi(\mathbf{v})}{6}\left(1 + \varphi(d_1^*q - 1) + \frac{\{3-(-1)^\ell\}\varphi(d_1^*q+1) + 6\varphi(d_1^*q)}{2^\ell}\right).$$

Proof. By (1) of Lemma 2.15 we have

$$\mathbf{Anc}_\ell[\mathbf{y}_\ell] \cup \mathbf{Anc}_\ell[\mathbf{y}_\ell^{\#}] = \bigcup_{p\in\{0,d_1^*\}} U_p(\mathbf{x}a_1).$$

It is easy to check that the following recursion formula holds

$$\begin{cases} k(\mathbf{v}, \mathbf{y}_1) = \dfrac{\psi(\mathbf{v})}{6}\{1 + \varphi(d_1^*q-1) - \varphi(d_1^*q+1)\} \\ k(\mathbf{v}, \mathbf{y}_1^{\#}) = \dfrac{\psi(\mathbf{v})}{6}\{1 + \varphi(d_1^*q-1) + 2\varphi(d_1^*q+1)\} \\ k(\mathbf{v}, \mathbf{y}_{\ell+1}) = \dfrac{1}{2}\{k(\mathbf{v}, \mathbf{y}_\ell) + k(\mathbf{v}, \mathbf{y}_\ell^{\#})\} \\ k(\mathbf{v}, \mathbf{y}_{\ell+1}^{\#}) = k(\mathbf{v}, \mathbf{y}_\ell) \end{cases}$$

and we have proved (1) and (2). Since $d_{\ell-1} = d_\ell \oplus d_\ell^*$, we have by (2) of Lemma 2.15

$$\mathbf{Anc}_\ell[(\mathbf{y}_\ell^- \sigma_{d_\ell^*}(a_{\ell+1}))^{\#}] = \begin{cases} \bigcup_{p\in -d_1^*, -d_1^*(1-\delta_{2,\ell})\}} U_p(\mathbf{x}a_1) & \text{if } \ell = 1, 2 \\ U(\mathbf{x}a_1) & \text{if } \ell \geq 3. \end{cases}$$

A straightforward computation using (1) and (2) yields

$$\begin{cases} k(\mathbf{v}, (\mathbf{y}_1^- \sigma_{d_1^*}(a_2))^\#) = \dfrac{\psi(\mathbf{v})}{6}\{1 + \varphi(d_1^* q + 1) + 2\varphi(d_1^* q)\} \\ k(\mathbf{v}, (\mathbf{y}_{\ell+2}^- \sigma_{d_{\ell+2}^*}(a_{\ell+3}))^\#) = \dfrac{k(\mathbf{v}, (\mathbf{y}_\ell^- \sigma_{d_\ell^*}(a_{\ell+1}))^\#)}{4} + \dfrac{\psi(\mathbf{v})}{8}(1 + \varphi(d_1^* q - 1)) \end{cases}$$

and hence the lemma is proved. □

We now define the sequences for words of fixed length.

Let $k_j^{(i)}, \ell_j^{(i)} \in \mathbb{N}_0, j = 1, 2, ..., p, p \in \mathbb{N}, m_i \in \mathbb{N}_0$ and $i \in \mathbb{N}_0$. Confining ourselves to Martin kernels, we associate the quantities $K^i, L^i, \mathbf{L}_i, \mathbf{L}, \alpha_{m_i}, \beta_{m_i}^+, \beta_{m_i}^-, \beta_{m_i}$ and $\zeta_{i,j,r}$. They are defined as follows:

$$\begin{aligned}
&\cdot K^i(p) = \sum_{j=1}^{p} k_j^{(i)}, \\
&\cdot L^i(p) = \sum_{j=1}^{p} \ell_j^{(i)} \text{ with the convention } L^i(0) = 0, \\
&\cdot \mathbf{L}_i = K^i(L^i(m_i) + 1), \\
&\cdot \mathbf{L}(p) = \sum_{i=1}^{p} \mathbf{L}_i \text{ with the convention } \mathbf{L}(0) = 0, \\
&\cdot \alpha_{m_i}(q) = (-1)^{k^{(i)}_{L^i(m_i)+1}} \Psi_{m_i}^{(i)}(\varphi(-q-1)(-1)^{k_1^{(i)}})), \\
&\cdot \beta_{m_i}^{\pm}(q) = \Psi_{m_i}^{(i)}(\varphi(-q-1)(-1)^{k_1^{(i)}} \pm (-1)^{m_i}), \\
&\cdot \beta_{m_i}(q) = \beta_{m_i}^+(q) - \beta_{m_i}^-(q), \\
&\cdot \zeta_{i,j,r}(q) = \varphi(j + \Psi_{r-1}^{(i)}(\varphi((-1)^r + \varphi(q+1)(-1)^{k_1^{(i)}})))
\end{aligned} \tag{2.11}$$

where $q \in \mathfrak{G}^\circ$ and where

$$\Psi_{m_i}^{(i)}(x) = \begin{cases} x & \text{if } m_i = 0 \\ \varphi(\ell_{m_i}^{(i)} + \varphi(\ell_{m_i-1}^{(i)} + \cdots + \varphi(\ell_2^{(i)} + \varphi(\ell_1^{(i)} + x)) \cdots)) & \text{if } m_i \in \mathbb{N}. \end{cases}$$

Let $a_j \in \mathcal{A}, j = 1, 2, ..., \mathbf{L}(p), \mathbf{L}(p) + 1$ where $\mathbf{L}(p)$ is defined in (2.11). Then we define $\mathbf{H}_{p-1}^p = \{q | \mathbf{L}(p-1) + 1 \leq q \leq \mathbf{L}(p) - 1 : d_q = d_{q+1}\}$.

Lemma 2.18. Let $k_j^{(0)}, \ell_j^{(0)}, K^0, L^0$ and $\mathbf{L}_0$ be defined as in (2.11). Let $\mathbf{x}, \mathbf{y} \in \mathcal{W}$ satisfying $\mathbf{y} = \mathbf{x} a_1 a_2 \cdots a_{\mathbf{L}_0} a_{\mathbf{L}_0+1}$ where $a_k \in \mathcal{A}, k = 1, 2, ..., \mathbf{L}_0, \mathbf{L}_0 + 1, d_k \in \mathfrak{G}^\#, k = 1, 2, ..., \mathbf{L}_0$ and $\ell_1^{(0)} \neq 0$. Moreover, we suppose that

$$\mathbf{H}_0^1 = \{K^0(1), K^0(2), ..., K^0(L^0(m_0))\}$$

and if $m_0 \geq 2$, then $d_{K^0(L^0(t-1)+1)} \neq d_{K^0(L^0(t)+1)}$ whenever $t = 1, 2, ..., m_0 - 1$. Then we have for $q \in \mathfrak{G}^\circ$ and $\mathbf{v} \in U_q(\mathbf{x}a_1)$

$$(1) \quad k(\mathbf{v}, \mathbf{y}) = \frac{\psi(\mathbf{v})}{6}\left(1 + \varphi(d_1^* q - 1) + \sum_{r=1}^{m_0}\sum_{j=1}^{\ell_r^{(0)}} \frac{\zeta_{0,j,r}(d_1^* q)}{2^{K^0(L^0(r-1)+j)}} - \frac{2\alpha_{m_0}(d_1^* q)}{2^{\mathbf{L}_0}}\right).$$

$$(2) \quad k(\mathbf{v}, \mathbf{y}^{\#}) = \frac{\psi(\mathbf{v})}{6}\left(1 + \varphi(d_1^* q - 1) + \sum_{r=1}^{m_0}\sum_{j=1}^{\ell_r^{(0)}} \frac{\zeta_{0,j,r}(d_1^* q)}{2^{K^0(L^0(r-1)+j)}} + \frac{4\alpha_{m_0}(d_1^* q)}{2^{\mathbf{L}_0}}\right).$$

$$\begin{aligned}(3) \quad & k(\mathbf{v}, (\mathbf{y}^{-}\sigma_{d^*_{\mathbf{L}_0}}(a_{\mathbf{L}_0+1}))^{\#}) \\ =& \frac{\psi(\mathbf{v})}{6}\left(1 + \varphi(d_1^* q - 1) + \sum_{r=1}^{m_0}\sum_{j=1}^{\ell_r^{(0)}} \frac{\zeta_{0,j,r}(d_1^* q)}{2^{K^0(L^0(r-1)+j)}} + \frac{\alpha_{m_0}(d_1^* q) + 3\beta_{m_0}(d_1^* q)}{2^{\mathbf{L}_0}}\right).\end{aligned}$$

Proof. Define

$$\begin{aligned}\mathfrak{K}_j^{r\#} =& \frac{1}{\psi(\mathbf{v})} k(\mathbf{v}, (\mathbf{x}a_1 a_2 \cdots a_{K^0(j)}\sigma_r(a_{K^0(j)+1}))^{\#}) \\ \mathfrak{K}_j^{r} =& \frac{1}{\psi(\mathbf{v})} k(\mathbf{v}, \mathbf{x}a_1 a_2 \cdots a_{K^0(j)}\sigma_r(a_{K^0(j)+1})) \\ \mathfrak{K}_j =& {}^t[\mathfrak{K}_j^{d^*_{K^0(j)}\#}, \mathfrak{K}_j^0, \mathfrak{K}_j^{0\#}]\end{aligned}$$

where ${}^tX$ is the transposed matrix of X.

Note that (2.1) and $d_{K^0(j)} \oplus (-d^*_{K^0(j)}) \in \mathfrak{G}^\circ$ whence $\mathfrak{K}_j^{-d^*_{K^0(j)}\#} = \mathfrak{K}_j^0$. Hence by Lemma 2.17, we have for $1 \leq j \leq L^0(m_0)$

$$\begin{cases} \mathfrak{K}_{j+1}^{d^*_{K^0(j+1)}\#} &= \frac{1}{6}\left(1 - \frac{(-1)^{k_{j+1}^{(0)}}+3}{2^{k_{j+1}^{(0)}}}\right)\mathfrak{K}^{d^*_{K^0(j)}\#} + \left(\frac{1}{2} + \frac{1}{2^{k_{j+1}^{(0)}}}\right)\mathfrak{K}_j^0 \\ & \quad + \frac{1}{6}\left(2 + \frac{(-1)^{k_{j+1}^{(0)}}-3}{2^{k_{j+1}^{(0)}}}\right)\mathfrak{K}_j^{0\#} \\ \mathfrak{K}_{j+1}^0 &= \frac{1}{6}\left(1 + \frac{2(-1)^{k_{j+1}}}{2^{k_{j+1}^{(0)}}}\right)\mathfrak{K}_j^{d^*_{K^0(j)}\#} + \frac{1}{2}\mathfrak{K}_j^0 + \frac{1}{3}\left(1 - \frac{(-1)^{k_{j+1}^{(0)}}}{2^{k_{j+1}^{(0)}}}\right)\mathfrak{K}_j^{0\#} \\ \mathfrak{K}_{j+1}^{0\#} &= \frac{1}{6}\left(1 - \frac{4(-1)^{k_{j+1}}}{2^{k_{j+1}^{(0)}}}\right)\mathfrak{K}_j^{d^*_{K^0(j)}\#} + \frac{1}{2}\mathfrak{K}_j^0 + \frac{1}{3}\left(1 + \frac{2(-1)^{k_{j+1}^{(0)}}}{2^{k_{j+1}^{(0)}}}\right)\mathfrak{K}_j^{0\#}, \end{cases}$$

it follows

$${}^t[\mathfrak{K}_{j+1}^{d^*_{K^0(j+1)}\#}, \mathfrak{K}_{j+1}^0, \mathfrak{K}_{j+1}^{0\#}] = \left(\frac{A_{j+1}}{2^{k_{j+1}^{(0)}}} + B\right)\ {}^t[\mathfrak{K}_j^{d^*_{K^0(j)}\#}, \mathfrak{K}_j^0, \mathfrak{K}_j^{0\#}]$$

where

$$A_{j+1} = \begin{bmatrix} -\frac{(-1)^{k_{j+1}^{(0)}}+3}{6} & 1 & \frac{(-1)^{k_{j+1}^{(0)}}-3}{6} \\ \frac{(-1)^{k_{j+1}^{(0)}}}{3} & 0 & -\frac{(-1)^{k_{j+1}^{(0)}}}{3} \\ -\frac{2(-1)^{k_{j+1}^{(0)}}}{3} & 0 & \frac{2(-1)^{k_{j+1}^{(0)}}}{3} \end{bmatrix} \text{ and } B = \frac{1}{6}\begin{bmatrix} 1 & 3 & 2 \\ 1 & 3 & 2 \\ 1 & 3 & 2 \end{bmatrix}, \tag{2.12}$$

so that we obtain

$$\mathfrak{K}_{L^0(m_0+1)} = \prod_{j=1}^{L^0(m_0)} \left(\frac{A_{L^0(m_0)+2-j}}{2^{k^{(0)}_{L^0(m_0)+2-j}}} + B \right) \mathfrak{K}_1.$$

Now define the matrices M_1 and M_2 for the case where $k_{j+1}^{(0)}$ is odd and even in (2.12) respectively, that is,

$$M_1 = \begin{bmatrix} -1/3 & 1 & -2/3 \\ -1/3 & 0 & 1/3 \\ 2/3 & 0 & -2/3 \end{bmatrix} \text{ and } M_2 = \begin{bmatrix} -2/3 & 1 & -1/3 \\ 1/3 & 0 & -1/3 \\ -2/3 & 0 & 2/3 \end{bmatrix}.$$

Note that by some linear algebra one can show

$$M_1^{n-1} = \frac{1}{3}\begin{bmatrix} 2\varphi(n+1)-\varphi(n) & 2\varphi(n)-\varphi(n-1) & 2\varphi(n-1)-\varphi(n+1) \\ \varphi(n-1) & \varphi(n+1) & \varphi(n) \\ -2\varphi(n-1) & -2\varphi(n+1) & -2\varphi(n) \end{bmatrix}$$

and for $j \in \{1,2,3\}$

$$\begin{cases} (M_1^{\ell_s^{(0)}-1} M_2 M_1^{\ell_{s-1}^{(0)}-1} M_2 \cdots M_2 M_1^{\ell_1^{(0)}-1})_{1j} &= \frac{2}{3}\Psi_s^{(0)}(\varphi(j-1)-(-1)^s) \\ & \quad -\frac{1}{3}\Psi_s^{(0)}(\varphi(j-1)) \\ (M_1^{\ell_s^{(0)}-1} M_2 M_1^{\ell_{s-1}^{(0)}-1} M_2 \cdots M_2 M_1^{\ell_1^{(0)}-1})_{2j} &= \frac{1}{3}\Psi_s^{(0)}(\varphi(j-1)+(-1)^s) \\ (M_1^{\ell_s^{(0)}-1} M_2 M_1^{\ell_{s-1}^{(0)}-1} M_2 \cdots M_2 M_1^{\ell_1^{(0)}-1})_{3j} &= -\frac{2}{3}\Psi_s^{(0)}(\varphi(j-1)+(-1)^s) \end{cases}$$

where X_{ij} is the (i,j)-entry of X.

Suppose that $m_0 \geq 2$ and $\ell_1^{(0)} \geq 2$. Since $A_{j+1}B = 0$, $B^2 = B$, $M_1^{1-1}M_2 = M_2$ and for $j \geq 2$

$$k_j^{(0)} = \begin{cases} \text{even} & \text{if } j = L^0(1)+1, L^0(2)+1, L^0(3)+1, ..., L^0(m_0-1)+1 \\ \text{odd} & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned}
&\prod_{j=1}^{L^0(m_0)}\left(\frac{A_{L^0(m_0)+2-j}}{2^{k^{(0)}_{L^0(m_0)+2-j}}}+B\right)\\
&=B+\sum_{j=2}^{\ell_1^{(0)}}\frac{BM_1^{j-1}}{2^{k_2^{(0)}+\cdots+k_j^{(0)}}}+\sum_{s=1}^{m_0-1}\sum_{j=1}^{\ell_{s+1}^{(0)}}\frac{BM_1^{j-1}M_2M_1^{\ell_s^{(0)}-1}M_2\cdots M_2M_1^{\ell_1^{(0)}-1}}{2^{k_2^{(0)}+\cdots+k^{(0)}_{L^0(s)+j}}}\\
&\quad+\frac{A_{L^0(m_0)+1}M_1^{\ell_{m_0}^{(0)}-1}M_2\cdots M_2M_1^{\ell_1^{(0)}-1}}{2^{k_2^{(0)}+\cdots+k^{(0)}_{L^0(m_0)+1}}}.
\end{aligned}\tag{2.13}$$

We have already constructed $\mathfrak{K}_1$ in Lemma 2.17 and thus the result follows. Similar calculations show the remaining part of the lemma. □

Here is an example of Lemma 2.18 with

$$\mathbf{y}=\mathbf{x}1(-2)1(-2)1(-1)1(-1)1(-2)0(-2)1(-2)0(-2)1(-1).$$

Since $d_4=d_5=-2$, we have by (2) of Lemma 2.15

$$\mathbf{Anc}_{17}[\mathbf{y}]=U(\mathbf{x}1)=\{\mathbf{x}1,(\mathbf{x}1)^\#,\mathbf{x}0,(\mathbf{x}0)^\#,\mathbf{x}2,(\mathbf{x}2)^\#\},$$

so that by (2.11) we can set

$$\begin{cases}k_1^{(0)}=4,k_2^{(0)}=4,k_3^{(0)}=1,k_4^{(0)}=2,k_5^{(0)}=2,k_6^{(0)}=2,k_7^{(0)}=1,k_8^{(0)}=1,\\ \ell_1^{(0)}=1,\ell_2^{(0)}=2,\ell_3^{(0)}=1,\ell_4^{(0)}=1,\ell_5^{(0)}=2,\\ \mathbf{H}_0^1=\{K^0(1),K^0(2),...,K^0(L^0(5))\}=\{4,8,9,11,13,15,16\}\end{cases}$$

and obtain

$$\begin{aligned}
&k(\mathbf{x}1,\mathbf{y})=k((\mathbf{x}1)^\#,\mathbf{y})\\
&=\frac{\psi(\mathbf{x}1)}{6}\left(2+\sum_{r=1}^{5}\sum_{j=1}^{\ell_r^{(0)}}\frac{\zeta_{0,j,r}(0)}{2^{K^0(L^0(r-1)+j)}}-\frac{2\alpha_5(0)}{2^{17}}\right)\\
&=\frac{\psi(\mathbf{x}1)}{6}\left(2+\sum_{j=1}^{1}\frac{\varphi(j-1)}{2^{K^0(j)}}+\sum_{j=1}^{2}\frac{\varphi(j-1)}{2^{K^0(1+j)}}+\sum_{j=1}^{1}\frac{\varphi(j+1)}{2^{K^0(3+j)}}+\sum_{j=1}^{1}\frac{\varphi(j)}{2^{K^0(4+j)}}\right.\\
&\qquad\left.+\sum_{j=1}^{2}\frac{\varphi(j+1)}{2^{K^0(5+j)}}+\frac{2}{2^{K^0(7+1)}}\right)\\
&=\frac{\psi(\mathbf{x}1)}{6}\left(2-\frac{1}{2^{K(3)}}+\frac{1}{2^{K(4)}}-\frac{1}{2^{K(5)}}+\frac{1}{2^{K(6)}}+\frac{2}{2^{K(8)}}\right)\\
&=\frac{\psi(\mathbf{x}1)}{6}\left(2-\frac{1}{2^{9}}+\frac{1}{2^{11}}-\frac{1}{2^{13}}+\frac{1}{2^{15}}+\frac{2}{2^{17}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\psi(\mathbf{x}1)}{6} \times \frac{261942}{2^{17}} \\
&= \frac{\psi(\mathbf{x}1) \times 43657}{2^{17}}.
\end{aligned}
$$

A similarly calculation yields

$$
\begin{aligned}
k(\mathbf{x}2, \mathbf{y}) = k((\mathbf{x}2)^{\#}, \mathbf{y}) &= \tfrac{\psi(\mathbf{x}2) \times 20565}{2^{17}} \\
k(\mathbf{x}0, \mathbf{y}) = k((\mathbf{x}0)^{\#}, \mathbf{y}) &= \tfrac{\psi(\mathbf{x}0) \times 1314}{2^{17}}.
\end{aligned}
$$

Lemma 2.19. Let $\mathbf{x}, \mathbf{y}_\ell \in \mathcal{W}$ such that $\mathbf{y}_\ell = \mathbf{x}a_1a_2\cdots a_\ell a_{\ell+1}$ where $a_j \in \mathcal{A}, j = 1, 2, ..., \ell, \ell+1, d_k \in \mathfrak{G}^\circ, k = 1, 2, ..., \ell$ and $\ell \in \mathbb{N}$. Then we have for $q \in \mathfrak{G}^\circ$ and $\mathbf{v} \in U_q(\mathbf{x}a_1)$

$$
(1) \quad k(\mathbf{v}, \mathbf{y}_\ell) = k(\mathbf{v}, \mathbf{y}_\ell^{\#}) = \frac{\delta_{q,0}\psi(\mathbf{v})}{2}.
$$

$$
(2) \quad k(\mathbf{v}, (\mathbf{y}_\ell^- \sigma_{d_\ell}(a_{\ell+1}))^{\#}) = \begin{cases} \psi(\mathbf{v}) \left(\dfrac{\delta_{0,q}}{2} + \dfrac{\phi(q)}{2^\ell} \right) & \text{if } d_k \equiv d_1 (\neq 0) \\ \dfrac{\delta_{0,q}\psi(\mathbf{v})}{2} & \text{otherwise} \end{cases}
$$

where $\phi(q) = \varphi(d_1 q + 1)$.

Proof. We only have to show (2) in the case where $d_k \equiv d_1 (\neq 0)$ for $1 \leq k \leq \ell$. Note that

$$
\mathbf{Anc}_\ell[(\mathbf{y}_\ell^- \sigma_{d_\ell}(a_{\ell+1}))^{\#}] = \bigcup_{p \in \{d_1\delta_{1,\ell}, d_1\}} U_p(\mathbf{x}a_1)
$$

by Lemma 2.16. Let $a_{\ell+2} \in \mathcal{A}$ such that $a_{\ell+2} = a_{\ell+1} \oplus d_1$. It is easy to check that the following recursion formula holds

$$
\begin{cases} k(\mathbf{v}, (\mathbf{y}_1^- \sigma_{d_1}(a_2))^{\#}) = \dfrac{\psi(\mathbf{v})}{2}(\delta_{0,q} + \phi(q)) \\ 2 \cdot k(\mathbf{v}, (\mathbf{y}_{\ell+1}^- \sigma_{d_{\ell+1}}(a_{\ell+2}))^{\#}) = k(\mathbf{v}, (\mathbf{y}_\ell^- \sigma_{d_\ell}(a_{\ell+1}))^{\#}) + k(\mathbf{v}, \mathbf{y}_\ell). \end{cases}
$$

This together with the fact that $k(\mathbf{v}, \mathbf{y}_\ell) = \delta_{0,q}\psi(\mathbf{v})/2$ implies the result. □

2.3 The estimation of the Martin kernel.

In this section we shall prove the following theorem.

Theorem A. Let $k_j^{(i)}, \ell_j^{(i)}, K^i, L^i, \mathbf{L}_i$ and $\mathbf{L}$ be as in (2.11). Let $\mathbf{x}, \mathbf{y} \in \mathcal{W}$ satisfying

$$
\mathbf{y} = \mathbf{x}a_1a_2\cdots a_{\mathbf{L}(2N+1)}a_{\mathbf{L}(2N+1)+1}
$$

where $a_k \in \mathcal{A}, k = 1, 2, ..., \mathbf{L}(2N+1), \mathbf{L}(2N+1)+1$ and $N \in \mathbb{N}$.

(1) If $\mathbf{v} \notin U(\mathbf{x}a_1)$ (i.e. $\mathbf{v} \in U_p(\mathbf{x}a_1), p \in \mathfrak{G}^{\#}$), then

$$
k(\mathbf{v}, \mathbf{y}) = k(\mathbf{v}, \mathbf{y}^{\#}) = 0.
$$

(2) If $\mathbf{v} \in U(\mathbf{x}a_1)$, then the Martin kernel k has the form:

• If $d_1 \in \mathfrak{G}^{\#}$, then there exist sequences

$\cdot\ s_j = s_j(\mathbf{v}, \mathbf{y}) \in \mathfrak{G}^{\circ}$ with $\sum_{\mathbf{v}\in U(\mathbf{x}a_1)} s_j = 0$

$$\cdot\ p_j = \begin{cases} p_1(\mathbf{v}, \mathbf{y}) \text{ with } 0 \le p_1 \le 3 \text{ and } \sum_{\mathbf{v}\in U(\mathbf{x}a_1)} p_1 = 6 & \text{if } j = 1 \\ p_j(\mathbf{y}) \text{ with } 0 \le p_j \le 1 & \text{if } 2 \le j \le N+1 \end{cases}$$

and $\alpha \in \mathfrak{G}^{\circ}$ such that

$$\begin{aligned} k(\mathbf{v}, \mathbf{y}) &= \frac{\psi(\mathbf{v})}{6}\left(p_1 + \sum_{j=2}^{N+1} \frac{s_1 s_2 \cdots s_{j-1}}{2^{\mathbf{L}(2j-2)}} p_j + \frac{6 s_1 s_2 \cdots s_N \alpha}{2^{\mathbf{L}(2N+1)+1}}\right) \\ k(\mathbf{v}, \mathbf{y}^{\#}) &= \frac{\psi(\mathbf{v})}{6}\left(p_1 + \sum_{j=2}^{N+1} \frac{s_1 s_2 \cdots s_{j-1}}{2^{\mathbf{L}(2j-2)}} p_j - \frac{6 s_1 s_2 \cdots s_N \alpha}{2^{\mathbf{L}(2N+1)+1}}\right). \end{aligned}$$

• If $d_1 \in \mathfrak{G}^{\circ}$, then there exist sequences

$\cdot\ s_j \in \mathfrak{G}^{\circ}$
$\cdot\ p_j = p_j(\mathbf{y})$ with $0 \le p_j \le 1$
$\cdot\ t = t(\mathbf{v}) \in \mathfrak{G}^{\circ}$ with $\sum_{\mathbf{v}\in U(\mathbf{x}a_1)} t = 0$

and $\alpha \in \mathfrak{G}^{\circ}$ such that

$$\begin{aligned} k(\mathbf{v}, \mathbf{y}) &= \frac{\psi(\mathbf{v})}{6}\left\{3\delta_{0,q} + t(\mathbf{v})\left(\sum_{j=1}^{N+1} \frac{s_1 s_2 \cdots s_{j-1}}{2^{\mathbf{L}(2j-2)}} p_j + \frac{6 s_1 s_2 \cdots s_N \alpha}{2^{\mathbf{L}(2N+1)+1}}\right)\right\} \\ k(\mathbf{v}, \mathbf{y}^{\#}) &= \frac{\psi(\mathbf{v})}{6}\left\{3\delta_{0,q} + t(\mathbf{v})\left(\sum_{j=1}^{N+1} \frac{s_1 s_2 \cdots s_{j-1}}{2^{\mathbf{L}(2j-2)}} p_j - \frac{6 s_1 s_2 \cdots s_N \alpha}{2^{\mathbf{L}(2N+1)+1}}\right)\right\}. \end{aligned}$$

We give the exact formula in the following proposition.

Proposition 2.20. Let $k_j^{(i)}, \ell_j^{(i)}, K^i, L^i, \mathbf{L}_i$ and $\mathbf{L}$ be as in (2.11). Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{W}$ satisfying

$$\mathbf{y} = \mathbf{x}a_1 a_2 \cdots a_{\mathbf{L}(2N)} a_{\mathbf{L}(2N)+1} \text{ and } \mathbf{z} = \mathbf{x}a_1 a_2 \cdots a_{\mathbf{L}(2N+1)} a_{\mathbf{L}(2N+1)+1}$$

where $a_k \in \mathcal{A}, k = 1, 2, ..., \mathbf{L}(2N+1), \mathbf{L}(2N+1)+1$ and $N \in \mathbb{N}$.

Moreover, we suppose that if $d_k \in \mathfrak{G}^{\#}$ for $\mathbf{L}(i-1)+1 \le \forall k \le \mathbf{L}(i)$, then

$$\mathbf{H}_{i-1}^i = \{K^i(1) + \mathbf{L}(i-1), K^i(2) + \mathbf{L}(i-1), ..., K^i(L^i(m_i)) + \mathbf{L}(i-1)\}$$

with the convention $\mathbf{H}_{i-1}^i = \emptyset$ whenever $m_i = 0$.

(1) If $d_k \in \begin{cases} \mathfrak{G}^\circ & \text{for } \mathbf{L}(2t-1)+1 \leq k \leq \mathbf{L}(2t), t=1,2,...,N \\ \mathfrak{G}^\# & \text{for } \mathbf{L}(2t-2)+1 \leq k \leq \mathbf{L}(2t-1), t=1,2,...,N,N+1, \end{cases}$

then there exists a sequence $\{s_j^{(q)}\}_j \subset \mathfrak{G}^\circ$ satisfying $\sum_{q\in\mathfrak{G}^\circ} s_1^{(q)}s_2^{(q)}\cdots s_j^{(q)} = 0$ for any j such that

$$
\begin{aligned}
&k(\mathbf{v},\mathbf{y}) = k(\mathbf{v},\mathbf{y}^\#) \\
&\quad= \begin{cases} \dfrac{\psi(\mathbf{v})}{6} p_1(d_1^* q) & \text{if } N=1 \\ \dfrac{\psi(\mathbf{v})}{6}\left(p_1(d_1^*q) + \displaystyle\sum_{j=2}^{N} \frac{s_1^{(q)}s_2^{(q)}\cdots s_{j-1}^{(q)} p_{2j-1}(1)}{2^{\mathbf{L}(2j-2)}}\right) & \text{if } N \geq 2, \end{cases} \\
&k(\mathbf{v},\mathbf{z}) \\
&\quad= \frac{\psi(\mathbf{v})}{6}\left(p_1(d_1^*q) + \sum_{j=2}^{N+1} \frac{s_1^{(q)}s_2^{(q)}\cdots s_{j-1}^{(q)} p_{2j-1}(1)}{2^{\mathbf{L}(2j-2)}} + \frac{6 s_1^{(q)}s_2^{(q)}\cdots s_N^{(q)} \alpha_{m_{2N+1}}(1)}{2^{\mathbf{L}(2N+1)+1}}\right), \\
&k(\mathbf{v},\mathbf{z}^\#) \\
&\quad= \frac{\psi(\mathbf{v})}{6}\left(p_1(d_1^*q) + \sum_{j=2}^{N+1} \frac{s_1^{(q)}s_2^{(q)}\cdots s_{j-1}^{(q)} p_{2j-1}(1)}{2^{\mathbf{L}(2j-2)}} - \frac{6 s_1^{(q)}s_2^{(q)}\cdots s_N^{(q)} \alpha_{m_{2N+1}}(1)}{2^{\mathbf{L}(2N+1)+1}}\right)
\end{aligned}
$$

for $q \in \mathfrak{G}^\circ$ and $\mathbf{v} \in U_q(\mathbf{x}a_1)$.

(2) If $d_k \in \begin{cases} \mathfrak{G}^\# & \text{for } \mathbf{L}(2t-1)+1 \leq k \leq \mathbf{L}(2t), t=1,2,...,N \\ \mathfrak{G}^\circ & \text{for } \mathbf{L}(2t-2)+1 \leq k \leq \mathbf{L}(2t-1), t=1,2,...,N,N+1, \end{cases}$

then there exists a sequence $\{s_j\}_j \subset \mathfrak{G}^\circ$ such that

$$
\begin{aligned}
k(\mathbf{v},\mathbf{y}) &= \frac{\psi(\mathbf{v})}{6}\left\{3\delta_{0,q} + \phi(q)\left(\sum_{j=1}^{N} \frac{s_1 s_2\cdots s_j p_{2j}(1)}{2^{\mathbf{L}(2j-1)}} + \frac{6 s_1 s_2 \cdots s_N \alpha_{m_{2N}}(1)}{2^{\mathbf{L}(2N)+1}}\right)\right\}, \\
k(\mathbf{v},\mathbf{y}^\#) &= \frac{\psi(\mathbf{v})}{6}\left\{3\delta_{0,q} + \phi(q)\left(\sum_{j=1}^{N} \frac{s_1 s_2\cdots s_j p_{2j}(1)}{2^{\mathbf{L}(2j-1)}} - \frac{6 s_1 s_2 \cdots s_N \alpha_{m_{2N}}(1)}{2^{\mathbf{L}(2N)+1}}\right)\right\}, \\
k(\mathbf{v},\mathbf{z}) &= k(\mathbf{v},\mathbf{z}^\#) = \frac{\psi(\mathbf{v})}{6}\left(3\delta_{0,q} + \phi(q)\sum_{j=1}^{N} \frac{s_1 s_2\cdots s_j p_{2j}(1)}{2^{\mathbf{L}(2j-1)}}\right)
\end{aligned}
$$

for $q \in \mathfrak{G}^\circ$ and $\mathbf{v} \in U_q(\mathbf{x}a_1)$, where ϕ is defined in Lemma 2.19 and p_j is defined as follows:

$$p_j(q) = \begin{cases} 1+\varphi(q-1)-\varphi(q+1)(-1/2)^{\mathbf{L}_j} & \text{if } \mathbf{H}^j_{j-1} = \emptyset \\ 1+\varphi(q-1)+\displaystyle\sum_{v=1}^{m_j}\sum_{w=1}^{\ell_v^{(j)}} \frac{\zeta_{j,w,v}(q)}{2^{K^j(L^j(v-1)+w)}} + \frac{\alpha_{m_j}(q)}{2^{\mathbf{L}_j}} & \text{if } \mathbf{H}^j_{j-1} \neq \emptyset. \end{cases}$$

Proof. We show (1) in several steps. For $1 \le \ell \le \mathbf{L}(2N+1)$, we set $\mathbf{x}^r_\ell = \mathbf{x}a_1a_2\cdots a_\ell\sigma_r(a_{\ell+1})$.

(First step) For any $p \in \{1,2,...,N\}$ and $\mathbf{w}_{\mathbf{L}(2p+1)} \in U(\mathbf{x}^0_{\mathbf{L}(2p+1)})$, we have

$$\begin{aligned} k(\mathbf{v},\mathbf{w}_{\mathbf{L}(2p+1)}) =& \frac{1}{2}\{k(\mathbf{v},\mathbf{x}^0_{\mathbf{L}(2p-1)}) + k(\mathbf{v},\mathbf{x}^{0\#}_{\mathbf{L}(2p-1)})\} \\ &+ \frac{5^{\mathbf{L}_{2p+1}}\chi_p^{(q)}\psi(\mathbf{v})}{2^{\mathbf{L}(2p)}} g(\mathbf{x}^{d_{\mathbf{L}(2p)}\#}_{\mathbf{L}(2p)}, \mathbf{w}_{\mathbf{L}(2p+1)}) \end{aligned} \tag{2.14}$$

where

$$\chi_p^{(q)} = \begin{cases} \dfrac{2^{\mathbf{L}(2p-1)}}{\psi(\mathbf{v})}\{k(\mathbf{v},\mathbf{x}^{d^*_{\mathbf{L}(2p-1)}\#}_{\mathbf{L}(2p-1)}) - k(\mathbf{v},\mathbf{x}^{0\#}_{\mathbf{L}(2p-1)})\} \\ \qquad \text{if } \begin{cases} \mathbf{x}^{d_{\mathbf{L}(2p)}}_{\mathbf{L}(2p)} \in \mathbf{Anc}_{\mathbf{L}(2N+1)-\mathbf{L}(2p)}[\mathbf{z}], \\ d_j \equiv d^*_{\mathbf{L}(2p-1)} \quad \text{for } \mathbf{L}(2p-1)+1 \le j \le \mathbf{L}(2p) \end{cases} \\ \dfrac{2^{\mathbf{L}(2p-1)}}{\psi(\mathbf{v})}\{k(\mathbf{v},\mathbf{x}^0_{\mathbf{L}(2p-1)}) - k(\mathbf{v},\mathbf{x}^{0\#}_{\mathbf{L}(2p-1)})\} \\ \qquad \text{if } \begin{cases} \mathbf{x}^{d_{\mathbf{L}(2p)}}_{\mathbf{L}(2p)} \in \mathbf{Anc}_{\mathbf{L}(2N+1)-\mathbf{L}(2p)}[\mathbf{z}], \\ d_j \equiv -d^*_{\mathbf{L}(2p-1)} \quad \text{for } \mathbf{L}(2p-1)+1 \le j \le \mathbf{L}(2p) \end{cases} \\ 0 \quad \text{otherwise.} \end{cases} \tag{2.15}$$

Indeed, we can prove (2.14) by the following way.

Suppose that

$$\begin{cases} \mathbf{x}^{d_{\mathbf{L}(2p)}\#}_{\mathbf{L}(2p)} \in \mathbf{Anc}_{\mathbf{L}(2N+1)-\mathbf{L}(2p)}[\mathbf{z}], \\ d_j \equiv d^*_{\mathbf{L}(2p-1)} \quad \text{for } \mathbf{L}(2p-1)+1 \le j \le \mathbf{L}(2p). \end{cases} \tag{2.16}$$

By the definition of the transition probability

$$k(\mathbf{v},\mathbf{w}_{\mathbf{L}(2p+1)}) = \sum_{\mathbf{t}\in U(\mathbf{x}^0_{\mathbf{L}(2p)})} k(\mathbf{v},\mathbf{t})g(\mathbf{t},\mathbf{w}_{\mathbf{L}(2p+1)}).$$

Hence by (2.16) and Lemma 2.19, we have

$$k(\mathbf{v},\mathbf{w}_{\mathbf{L}(2p+1)})$$

$$
\begin{aligned}
= & \sum_{\mathbf{t}\in U_0(\mathbf{x}^0_{\mathbf{L}(2p-1)})} k(\mathbf{v},\mathbf{t})\left(\frac{1}{2}-\frac{1}{2^{\mathbf{L}_{2p}}}\right)5^{\mathbf{L}_{2p+1}} g(\mathbf{x}^{d_{\mathbf{L}(2p)}\#}_{\mathbf{L}(2p)},\mathbf{w}_{\mathbf{L}(2p+1)}) \\
& + \sum_{\mathbf{t}\in U_{d^*_{\mathbf{L}(2p-1)}}(\mathbf{x}^0_{\mathbf{L}(2p-1)})} k(\mathbf{v},\mathbf{t})\frac{5^{\mathbf{L}_{2p+1}} g(\mathbf{x}^{d_{\mathbf{L}(2p)}\#}_{\mathbf{L}(2p)},\mathbf{w}_{\mathbf{L}(2p+1)})}{2^{\mathbf{L}_{2p}}} \\
& + \sum_{\mathbf{t}\in U_0(\mathbf{x}^0_{\mathbf{L}(2p-1)})} k(\mathbf{v},\mathbf{t})\frac{1-5^{\mathbf{L}_{2p+1}} g(\mathbf{x}^{d_{\mathbf{L}(2p)}\#}_{\mathbf{L}(2p)},\mathbf{w}_{\mathbf{L}(2p+1)})}{2} \\
= & \frac{k(\mathbf{v},\mathbf{x}^0_{\mathbf{L}(2p-1)})+k(\mathbf{v},\mathbf{x}^{0\#}_{\mathbf{L}(2p-1)})}{2} \\
& - \frac{k(\mathbf{v},\mathbf{x}^0_{\mathbf{L}(2p-1)})+k(\mathbf{v},\mathbf{x}^{0\#}_{\mathbf{L}(2p-1)})}{2^{\mathbf{L}_{2p}}}5^{\mathbf{L}_{2p+1}} g(\mathbf{x}^{d_{\mathbf{L}(2p)}\#}_{\mathbf{L}(2p)},\mathbf{w}_{\mathbf{L}(2p+1)}) \\
& + \frac{k(\mathbf{v},\mathbf{x}^{d^*_{\mathbf{L}(2p-1)}\#}_{\mathbf{L}(2p-1)})+k(\mathbf{v},\mathbf{x}^{d^*_{\mathbf{L}(2p-1)}}_{\mathbf{L}(2p-1)})}{2^{\mathbf{L}_{2p}}}5^{\mathbf{L}_{2p+1}} g(\mathbf{x}^{d_{\mathbf{L}(2p)}\#}_{\mathbf{L}(2p)},\mathbf{w}_{\mathbf{L}(2p+1)}) \\
= & \frac{k(\mathbf{v},\mathbf{x}^0_{\mathbf{L}(2p-1)})+k(\mathbf{v},\mathbf{x}^{0\#}_{\mathbf{L}(2p-1)})}{2} \\
& + \frac{k(\mathbf{v},\mathbf{x}^{d^*_{\mathbf{L}(2p-1)}\#}_{\mathbf{L}(2p-1)})-k(\mathbf{v},\mathbf{x}^{0\#}_{\mathbf{L}(2p-1)})}{2^{\mathbf{L}_{2p}}}5^{\mathbf{L}_{2p+1}} g(\mathbf{x}^{d_{\mathbf{L}(2p)}\#}_{\mathbf{L}(2p)},\mathbf{w}_{\mathbf{L}(2p+1)})
\end{aligned}
$$

which is (2.14).

(Second step) We shall prove

$$
\chi_p^{(q)} \in \mathfrak{G}^\circ \text{ and } \sum_{q\in\mathfrak{G}^\circ}\chi_p^{(q)} = 0 \text{ for any } p\in\{1,2,...,N\}. \tag{2.17}
$$

Since $\sum_{q\in\mathfrak{G}^\circ}\chi_p^{(q)}=0$ follows from (2.16), we only have to show that $\chi_p^{(q)}\in\mathfrak{G}^\circ$. Using induction over p, the assertion follows immediately. If $p=1$, then by Lemmas 2.17 and 2.18 we have

$$
\begin{aligned}
\chi_1^{(q)} \in \Bigg\{ & \varphi(d_1^*q+1)(-1)^{\mathbf{L}_1}, \frac{(d_1^*q+1)(1+(-1)^{\mathbf{L}_1})+2(-1)^{\mathbf{L}_1}\varphi(d_1^*q)}{2}, \\
& -\alpha_{m_1}(d_1^*q), \frac{\beta_{m_1}(d_1^*q)-\alpha_{m_1}(d_1^*q)}{2}\Bigg\}.
\end{aligned}
$$

Noting that

$$
\beta_{m_i}(d_1^*q)-\alpha_{m_i}(d_1^*q) = 2(-1)^{k^{(i)}_{L^i(m_i)+1}}\Psi^{(i)}_{m_i}((-1)^{k^{(i)}_{L^i(m_i)+1}+m_i}-\varphi(d_1^*q+1)(-1)^{k_1^{(i)}}),
$$

this implies that (2.17) is true for $p = 1$. Suppose that (2.17) is true for p. By (2.14), we obtain

$$\begin{aligned}&\frac{2^{\mathbf{L}(2p+1)}}{\psi(\mathbf{v})}\{k(\mathbf{v},\mathbf{x}_{\mathbf{L}(2p+1)}^{d^*_{\mathbf{L}(2p+1)}\#})-k(\mathbf{v},\mathbf{x}_{\mathbf{L}(2p+1)}^{0\#})\}\\ =\ &\frac{10^{\mathbf{L}_{2p+1}}\chi_p^{(q)}}{2^{\mathbf{L}(2p)}}\{g(\mathbf{x}_{\mathbf{L}(2p)}^{d_{\mathbf{L}(2p)}\#},\mathbf{x}_{\mathbf{L}(2p+1)}^{d^*_{\mathbf{L}(2p+1)}\#})-g(\mathbf{x}_{\mathbf{L}(2p)}^{d_{\mathbf{L}(2p)}\#},\mathbf{x}_{\mathbf{L}(2p+1)}^{0\#})\}.\end{aligned}$$

On the other hand, by Lemmas 2.17 and 2.18 again

$$\begin{aligned}&5^{\mathbf{L}_{2p+1}}\{g(\mathbf{x}_{\mathbf{L}(2p)}^{d_{\mathbf{L}(2p)}\#},\mathbf{x}_{\mathbf{L}(2p+1)}^{d^*_{\mathbf{L}(2p+1)}\#})-g(\mathbf{x}_{\mathbf{L}(2p)}^{d_{\mathbf{L}(2p)}\#},\mathbf{x}_{\mathbf{L}(2p+1)}^{0\#})\}\\ &\in\left\{\frac{(-1)^{\mathbf{L}_{2p+1}}-1}{2^{\mathbf{L}_{2p+1}+1}},\frac{\beta_{m_{2p+1}}(1)-\alpha_{m_{2p+1}}(1)}{2^{\mathbf{L}_{2p+1}+1}}\right\},\end{aligned}$$

so that

$$\begin{aligned}\chi_{p+1}^{(q)}\in\Bigg\{&(-1)^{\mathbf{L}_{2p+1}}\chi_p^{(q)},\frac{(-1)^{\mathbf{L}_{2p+1}}-1}{2}\chi_p^{(q)},\\ &-\alpha_{m_{2p+1}}(1)\chi_p^{(q)},\frac{\beta_{m_{2p+1}}(1)-\alpha_{m_{2p+1}}(1)}{2}\chi_p^{(q)}\Bigg\}\end{aligned}\tag{2.18}$$

and we have proved (2.17).

(Third step) Now by (2.14)

$$k(\mathbf{v},\mathbf{z})=\sum_{\mathbf{t}\in U_0(\mathbf{x}^0_{\mathbf{L}(2N-1)})}\frac{k(\mathbf{v},\mathbf{t})}{2}+\frac{\psi(\mathbf{v})}{6}\left(\frac{\chi_N^{(q)}p_{2N+1}(1)}{2^{\mathbf{L}(2N)}}+\frac{6\chi_N^{(q)}\alpha_{2N+1}(1)}{2^{\mathbf{L}(2N+1)+1}}\right).$$

Using (2.14) again, we have

$$\begin{aligned}&\sum_{\mathbf{t}\in U_0(\mathbf{x}^0_{\mathbf{L}(2N-1)})}\frac{k(\mathbf{v},\mathbf{t})}{2}\\ =\ &\sum_{\mathbf{t}\in U_0(\mathbf{x}^0_{\mathbf{L}(2N-3)})}\frac{k(\mathbf{v},\mathbf{t})}{2}+\sum_{\mathbf{t}\in U_0(\mathbf{x}^0_{\mathbf{L}(2N-1)})}\frac{5^{\mathbf{L}_{2N-1}}\chi_{N-1}^{(q)}}{2^{\mathbf{L}(2N-2)+1}}p(\mathbf{L}_{2N-1};\mathbf{x}_{\mathbf{L}(2N-2)}^{d^*_{\mathbf{L}(2N-2)}\#},\mathbf{t}).\end{aligned}$$

This together with the fact that

$$\sum_{\mathbf{t}\in U_0(\mathbf{x}^0_{\mathbf{L}(2N-1)})}5^{\mathbf{L}_{2N-1}}p(\mathbf{L}_{2N-1};\mathbf{x}_{\mathbf{L}(2N-2)}^{d^*_{\mathbf{L}(2N-2)}\#},\mathbf{t})=2p_{2N-1}(1)$$

implies

$$k(\mathbf{v},\mathbf{z})=\sum_{\mathbf{t}\in U_0(\mathbf{x}^0_{\mathbf{L}(2N-3)})}\frac{k(\mathbf{v},\mathbf{t})}{2}+\frac{\psi(\mathbf{v})}{6}\left(\sum_{j=N}^{N+1}\frac{\chi_{j-1}^{(q)}p_{2j-1}(1)}{2^{\mathbf{L}(2j-2)}}+\frac{6\chi_N^{(q)}\alpha_{2N+1}(1)}{2^{\mathbf{L}(2N+1)+1}}\right).$$

Repeating this process, we have

$$k(\mathbf{v},\mathbf{z}) = \frac{\psi(\mathbf{v})}{6}\left(p_1(d_1^* q) + \sum_{j=2}^{N+1}\frac{\chi_{j-1}^{(q)} p_{2j-1}(1)}{2^{\mathbf{L}(2j-2)}} + \frac{6\chi_N^{(q)}\alpha_{2N+1}(1)}{2^{\mathbf{L}(2N+1)+1}}\right).$$

By construction of $\chi_N^{(q)}$ in (2.18), there exists a sequence $\{s_j^{(q)}\}_j \subset \mathfrak{G}^\circ$ such that $\chi_N^{(q)} = s_1^{(q)} s_2^{(q)} \cdots s_N^{(q)}$, and hence the theorem follows. □

Remark 2.21. This is an additional remark to Proposition 2.20.

(1) If $d_k \in \mathfrak{G}^\#$ or $d_k \in \mathfrak{G}^\circ$ for any k, then we come to the conclusions that Lemmas 2.18 (or 2.17) or 2.19 hold, respectively.

(2) Let M_1, M_2 and B be defined as in Lemma 2.18. It is easy to see that

$$\mathbf{M} = \left\{ M \,\middle|\, M = [m_1, m_2, m_3] : m_1, m_2, m_3 \in \left\{ \begin{array}{l} {}^t[-1/3, -1/3, 2/3] \\ {}^t[-2/3, 1/3, -2/3] \\ {}^t[1, 0, 0] \end{array} \right\} \right\}$$

where $\mathbf{M}$ is the collection of consisting of finite arbitrary product M_1 and M_2. Furthermore, we define $\mathbf{M}_0 = \{M | M = BM' : M' \in \mathbf{M}\}$. By the same argument as above, we have

$$\mathbf{M}_0 = \left\{ M \,\middle|\, M = [m_1, m_2, m_3] : m_1, m_2, m_3 \in \left\{ \begin{array}{l} {}^t[-1/6, -1/6, -1/6] \\ {}^t[1/6, 1/6, 1/6] \\ {}^t[0, 0, 0] \end{array} \right\} \right\}.$$

Then by (2.13), it follows that there exists a sequence $\{\gamma_k^{(j,q)}\}_k \subset \mathfrak{G}^\circ$ and constant $\gamma^{(j,q)} \in \mathfrak{G}^\circ$ satisfying the condition $\sum_{q\in\mathfrak{G}^\circ} \gamma_k^{(j,q)} = \sum_{q\in\mathfrak{G}^\circ} \gamma^{(j,q)} = 0$ such that

$$p_j(q) = \begin{cases} 1 + \varphi(q-1) - \varphi(q+1)(-1/2)^{\mathbf{L}_j} & \text{if } \mathbf{H}_{j-1}^j = \emptyset \\ 1 + \varphi(q-1) + \displaystyle\sum_{k\in\mathbf{H}_{j-1}^j} \frac{\gamma_k^{(j,q)}}{2^{k-\mathbf{L}(j-1)}} + \frac{\gamma^{(j,q)}}{2^{\mathbf{L}_j}} & \text{if } \mathbf{H}_{j-1}^j \neq \emptyset. \end{cases} \tag{2.19}$$

We also have that

$$\begin{aligned} \chi_{p+1}^{(q)} \in \quad & \{\varphi(q+1)(-1)^{\mathbf{L}_{2p+1}}\chi_p^{(q)}, -\varphi(q+1)(-1)^{\mathbf{L}_{2p+1}}\chi_p^{(q)}, \\ & \tfrac{\varphi(q+1)\{1-(-1)^{\mathbf{L}_{2p+1}}\}+2\varphi(q)}{2}\chi_p^{(q)}, \tfrac{\varphi(q+1)\{-1+(-1)^{\mathbf{L}_{2p+1}}\}-2\varphi(q)}{2}\chi_p^{(q)}, \\ & \tfrac{\varphi(q+1)\{1+(-1)^{\mathbf{L}_{2p+1}}\}+2\varphi(q)}{2}\chi_p^{(q)}, \tfrac{\varphi(q+1)\{-1-(-1)^{\mathbf{L}_{2p+1}}\}-2\varphi(q)}{2}\chi_p^{(q)}\}. \end{aligned}$$

Compare this result with (2.18).

(3) It is clear that

$$k(\mathbf{v},\mathbf{y}), k(\mathbf{v},\mathbf{y}^\#), k(\mathbf{v},\mathbf{z}), k(\mathbf{v},\mathbf{z}^\#) \le \frac{\psi(\mathbf{v})}{2}$$

for any $\mathbf{v} \in U(\mathbf{x}a_1)$. In particular, by elementary computations, we have for any $\mathbf{v} \in U_0(\mathbf{x}a_1)$

$$\frac{\psi(\mathbf{v})}{4} \leq k(\mathbf{v},\mathbf{y}), k(\mathbf{v},\mathbf{y}^{\#}), k(\mathbf{v},\mathbf{z}), k(\mathbf{v},\mathbf{z}^{\#}) \leq \frac{\psi(\mathbf{v})}{2}. \tag{2.20}$$

3. Pentakun as a Martin boundary

3.1 The Martin boundary of $\mathfrak{X}$.

In this section we identify the Pentakun $\mathcal{P}$ with the Martin boundary $\mathcal{M}$ of the transition probability function defined in (2.2). Since by Theorem 1.11 the Modulo 5 fractal $\mathcal{A}^{\infty}/{\sim}$ and the Pentakun $\mathcal{P}$ are bi-Lipschitz equivalent, it suffices to show the existence of a homeomorphism

$$T : (\mathcal{A}^{\infty}/{\sim}) \to \mathcal{M}.$$

We define the map $\rho : \mathcal{W} \times \mathcal{W} \to \mathbb{R}_+$

$$\rho(\mathbf{x},\mathbf{y}) = \left| r^{l(\mathbf{x})} - r^{l(\mathbf{y})} \right| + \sum_{\mathbf{u}\in\mathcal{W}} a(\mathbf{u}) \frac{|k(\mathbf{u},\mathbf{x}) - k(\mathbf{u},\mathbf{y})|}{1 + |k(\mathbf{u},\mathbf{x}) - k(\mathbf{u},\mathbf{y})|} \tag{3.1}$$

where $r = \frac{3-\sqrt{5}}{2}$ and $\{a(\mathbf{u}); \mathbf{u} \in \mathcal{W}\}$ is some fixed sequence of strictly positive numbers such that $\sum_{\mathbf{u}\in\mathcal{W}} a(\mathbf{u}) = 1$.

Lemma 3.1. The map ρ is a metric on $\mathcal{W}$.

Proof. We assume that $\rho(\mathbf{x},\mathbf{y}) = 0$. Then by (3.1) we have

$$l(\mathbf{x}) = l(\mathbf{y}) \text{ and } k(\mathbf{u},\mathbf{x}) = k(\mathbf{u},\mathbf{y}) \text{ for all } \mathbf{u} \in \mathcal{W}.$$

Taking $\mathbf{u} = \mathbf{x}$ and by (2.5), we obtain $\mathbf{x} = \mathbf{y}$. The remaining parts are obvious. □

This metric has its source in (1.8) and is called *Martin metric* [5, 9]. By (3) of Remark 2.21, the Martin kernel defined in (2.5) satisfies (1.7). Hence by the same discussion as in Section 2 of Chapter 1, we can consider $(\mathcal{W}, \mathcal{O}_2)$ as a topological space where $\mathcal{O}_2$ is a natural topology.

The ρ-completion of $\mathcal{W}$ is called the *Martin space* associated to p where p is defined in (2.2), denoted by $\overline{\mathcal{W}}$. The boundary of $\overline{\mathcal{W}}$, that is, $\partial\overline{\mathcal{W}} = \overline{\mathcal{W}} \setminus \mathcal{W}^{\circ} = \overline{\mathcal{W}} \setminus \mathcal{W}$ is called the *Martin boundary* and is denoted by $\mathcal{M}$. $\mathcal{M}$ is a compact metric space (see Lemma 1.6). Recall that $\mathcal{O}_1$ is defined in (2) of Definition 2.1. The next lemma is an immediate consequence.

Lemma 3.2. $\mathcal{O}_1 = \mathcal{O}_2$.

Since p is a Markovian kernel, we can also define the Markov operator on $\mathcal{W}$.

Let f be a non-negative function on $\mathcal{W}$. The associated *Markov operator* P is defined by

$$(Pf)(\mathbf{w}) = \sum_{\mathbf{v}\in\mathcal{W}} p(\mathbf{w},\mathbf{v})f(\mathbf{v})$$

and f is called P*-excessive* if $Pf \leq f$ and P*-harmonic* if $Pf = f$.

Let μ be a measure on $\mathcal{W}$. The associated *dual Markov operator* P^* is defined by

$$(\mu P^*)(\mathbf{v}) = \sum_{\mathbf{w}\in\mathcal{W}} \mu(\mathbf{w})p(\mathbf{w},\mathbf{v})$$

and μ is called P^**-excessive* if $\mu P^* \leq \mu$ and P^**-harmonic* if $\mu P^* = \mu$.

Theorem B. $\mathcal{A}^\infty/\!\sim$ is the Martin boundary $\mathcal{M}$ of a certain, naturally defined the mod 5 Markov chain $\mathfrak{X}$ (in Definition 2.4).

In order to prove the theorem, it is enough to establish a map $T : (\mathcal{A}^\infty/\!\sim) \to \mathcal{M}$ with the following Lemmas 3.3 through 3.5.

Lemma 3.3. Let $x = \{x_k\}$ be an infinite sequence of letters and define $\mathbf{x}_\ell = x_1x_2\cdots x_\ell x_{\ell+1}$ and $\ell \in \mathbb{N}$. Then $T_0(x) = \{\mathbf{x}_\ell\}$ is a Cauchy sequence in $(\mathcal{W}, \rho)$.

Proof. Let $\mathbf{v} \in \mathcal{W}$. If $\mathbf{v} \notin U(\mathbf{x}_{l(\mathbf{v})})$, then by Proposition 2.12 we have $k(\mathbf{v}, \mathbf{x}_\ell) = 0$ for all ℓ, so that we may assume without loss of generality that $\ell \geq l(\mathbf{v})$ and $\mathbf{v} \in U(\mathbf{x}_{l(\mathbf{v})})$.

Recall that $K^i, L^i, \mathbf{L}_i$ and $\mathbf{L}$ are defined in (2.11). We may set

$$x_{\ell+1}x_{\ell+2}\cdots x_{\ell'}x_{\ell'+1} = x_{\ell+1}x_{\ell+2}\cdots x_{\ell+\mathbf{L}(2N+1)}x_{\ell+\mathbf{L}(2N+1)+1},$$

$$d_k \in \begin{cases} \mathfrak{G}^\circ & \text{if } \ell+\mathbf{L}(2t-1)+1 \leq k \leq \ell+\mathbf{L}(2t) \\ \mathfrak{G}^\# & \text{if } \ell+\mathbf{L}(2t-2)+1 \leq k \leq \ell+\mathbf{L}(2t-1), \end{cases}$$

$t = 1, 2, ..., N, N+1$ and $N \in \mathbb{N}$.

If $d_\ell \in \mathfrak{G}^\circ$, then by Proposition 2.20 we have

$$|k(\mathbf{v}, \mathbf{x}_{\ell'}) - k(\mathbf{v}, \mathbf{x}_\ell)| \leq \frac{\psi(\mathbf{v})}{6}\left|\sum_{j=1}^{N+1}\frac{p_{2j-1}(1)}{2^{\ell+\mathbf{L}(2j-2)}} + \frac{6\alpha_{m_{2N+1}}(1)}{2^{\ell+\mathbf{L}(2N+1)+1}}\right|.$$

If $d_\ell \in \mathfrak{G}^\#$, then by replacing $\ell - n_0$ by $K^0(L^0(m_0)+1)$ in Lemma 2.18 and Proposition 2.20 again, we have

$$\begin{aligned} &|k(\mathbf{v}, \mathbf{x}_{\ell'}) - k(\mathbf{v}, \mathbf{x}_\ell)| \\ \leq\ & \frac{\psi(\mathbf{v})}{6}\left|\sum_{j=2}^{N+1}\frac{p_{2j-1}(1)}{2^{\ell+\mathbf{L}(2j-2)}} + \frac{6\alpha_{m_{2N+1}}(1)}{2^{\ell+\mathbf{L}(2N+1)+1}}\right. \\ &\left. + \frac{1}{2^{n_0}}\left(\frac{6}{2^{K^0(L^0(m_0)+1)+1}} + \sum_{q=1}^{m_1}\sum_{j=1}^{\ell_q^{(1)}}\frac{1}{2^{K^1(L^1(q-1)+j)+\ell-n_0}} + \frac{1}{2^{\ell+\mathbf{L}(1)-n_0}}\right)\right| \end{aligned}$$

$$\leq \quad \frac{\psi(\mathbf{v})}{6}\left|\sum_{j=2}^{N+1}\frac{1}{2^{\ell+\mathbf{L}(2j-2)}}+\frac{6}{2^{\ell+\mathbf{L}(2N+1)+1}}\right.$$

$$\left.+\frac{6}{2^{\ell+1}}+\sum_{q=1}^{m_1}\sum_{j=1}^{\ell_q^{(1)}}\frac{1}{2^{K^1(L^1(q-1)+j)+\ell}}+\frac{1}{2^{\ell+\mathbf{L}(1)}}\right|$$

where $n_0 = \max\{n | l(\mathbf{v}) \leq n < \ell, d_n \in \mathfrak{G}^\circ\}$.

In either case, we obtain

$$|k(\mathbf{v},\mathbf{x}_{\ell'}) - k(\mathbf{v},\mathbf{x}_\ell)| \leq \frac{\psi(\mathbf{v})}{2^\ell}\sum_{n=0}^{\infty}\frac{1}{2^n} \quad \to 0 \text{ as } \ell \to \infty.$$

The remaining parts can be shown analogously. □

Lemma 3.4. If $\{\mathbf{x}_\ell\}$ is a Cauchy sequence in $\mathcal{W}$, then $\{\mathbf{x}_\ell^\#\}$ is also a Cauchy sequence. $\{\mathbf{x}_\ell^\#\}$ is equivalent to $\{\mathbf{x}_\ell\}$ if $l(\mathbf{x}_\ell) \to \infty$.

Proof. Let $\mathbf{v} \in U(\mathbf{x}_{l(\mathbf{v})})$. By the triangle inequality:

$$\rho(\mathbf{x}_\ell, \mathbf{x}_{\ell'}) \leq \rho(\mathbf{x}_\ell, \mathbf{x}_\ell^\#) + \rho(\mathbf{x}_\ell^\#, \mathbf{x}_{\ell'}^\#) + \rho(\mathbf{x}_{\ell'}^\#, \mathbf{x}_{\ell'})$$

and by (3.1), it suffices to show that

$$\lim_{\ell\to\infty} |k(\mathbf{v},\mathbf{x}_\ell) - k(\mathbf{v},\mathbf{x}_\ell^\#)| = 0.$$

If $\lim_{\ell\to\infty} l(\mathbf{x}_\ell) < \infty$, then the sequence $\mathbf{x}_\ell$ is eventually constant, so is $\mathbf{x}_\ell^\#$, hence it is also a Cauchy sequence. Thus, by Proposition 2.20 we have

$$|k(\mathbf{v},\mathbf{x}_\ell) - k(\mathbf{v},\mathbf{x}_\ell^\#)| \leq \frac{\psi(\mathbf{v})}{2^\ell} \to 0 \text{ as } \ell \to \infty.$$

□

Lemma 3.5. If $T_0(x) = T_0(y)$ for $x = x_1x_2\cdots$ and $y = y_1y_2\cdots \in \mathcal{A}^\infty$, then we have $x \sim y$.

Proof. Let $x, y \in \mathcal{A}^\infty$ satisfying $x \neq y$. Then there exists $\mathbf{w} \in \mathcal{W}$ such that

$$x = \mathbf{w}x_1x_2x_3\cdots, \quad y = \mathbf{w}y_1y_2y_3\cdots \text{ and } x_1 \neq y_1.$$

Assume that $T_0(x) = T_0(y)$ and define $\mathbf{x}_m = \mathbf{w}x_1x_2\cdots x_m$ and $\mathbf{y}_m = \mathbf{w}y_1y_2\cdots y_m$. Note that by the definition of ρ and T_0, we have for any $\mathbf{t} \in \mathcal{W}$

$$\lim_{m\to\infty} k(\mathbf{t},\mathbf{x}_m) = k(\mathbf{t},x) = k(\mathbf{t},y) = \lim_{m\to\infty} k(\mathbf{t},\mathbf{y}_m). \tag{3.2}$$

Taking $\mathbf{t} = \mathbf{x}_1$ in (3.2), we have by (2.20)

$$0 < \lim_{m\to\infty} k(\mathbf{x}_1,\mathbf{x}_m) = \lim_{m\to\infty} k(\mathbf{x}_1,\mathbf{y}_m)$$

and hence by Proposition 2.12 we obtain $\{\mathbf{x}_1\} \cap U(\mathbf{y}_1) \neq \emptyset$ which implies $y_1 = \sigma_d(x_1)$ where $d \in \{-1, 1\}$. Taking $\mathbf{t} = \mathbf{x}_2$ in (3.2) and the same argument as above, we have $\{\mathbf{x}_2\} \cap U(\mathbf{y}_2) \neq \emptyset$ and hence $y_2 = \sigma_{-d}(x_1)$ and $x_2 = \sigma_{2d}(x_1)$. We define $\mathcal{X} = \{k \geq 3 | x_k \neq \sigma_{2d}(x_1)\}, \mathcal{Y} = \{k \geq 3 | y_k \neq \sigma_{-d}(x_1)\}$ and denote ℓ and ℓ' by $\ell = \min \mathcal{X}$ and $\ell' = \min \mathcal{Y}$ if $\mathcal{X} \neq \emptyset$ and $\mathcal{Y} \neq \emptyset$, respectively. We shall prove

$$\mathcal{X} = \mathcal{Y} = \emptyset. \tag{3.3}$$

If $\mathcal{X} \neq \emptyset$ and $\mathcal{Y} = \emptyset$, then by taking $\mathbf{t} = \mathbf{x}_\ell$ in (3.2), it follows that $\{\mathbf{x}_\ell\} \cap U(\mathbf{y}_\ell) = \emptyset$ which is a contradiction. Interchanging x and y, the remaining parts of the proof for (3.3) suffice in considering the case where $\mathcal{X} \neq \emptyset$ and $\mathcal{Y} \neq \emptyset$. Notice that we may automatically assume $\ell = \ell'$. However, we obtain

$$\{\mathbf{x}_\ell\} \cap U(\mathbf{y}_\ell) = \{\mathbf{x}_\ell\} \cap U(\mathbf{x}^{\#}_{\ell-1} y_\ell) = \emptyset$$

implies (3.3) and thus $x^{\#} = y$. □

Consequently, by Lemma 3.4, we can define the map $T : (\mathcal{A}^\infty/\sim) \to \mathcal{M}$ by

$$T(\tilde{x}) = \{\mathbf{x}_\ell | \ell \in \mathbb{N}\}^\sim$$

where $x = (x_\ell)_{\ell \in \mathbb{N}}$ and $\{\mathbf{x}_\ell | \ell \in \mathbb{N}\}^\sim$ denotes the equivalence class of the Cauchy sequence $\{\mathbf{x}_\ell | \ell \in \mathbb{N}\}$.

Lemma 3.6. The map $T : (\mathcal{A}^\infty/\sim) \to \mathcal{M}$ is surjective.

Proof. Let $\{\mathbf{w}_n\}_{n\in\mathbb{N}} \subset \mathcal{W}$ be a Cauchy sequence. Then, since $\mathcal{A}$ is a finite set, there exists a subsequence $\{\mathbf{w}_{n(1,k)}\}_{k\in\mathbb{N}}$ such that the first letter of all $\mathbf{w}_{n(1,k)}; k \in \mathbb{N}$, is $x_1 \in \mathcal{A}$. Next we can extract a subsequence $\{\mathbf{w}_{n(2,k)}\}_{k\in\mathbb{N}} (\subset \{\mathbf{w}_{n(1,k)}\}_{k\in\mathbb{N}})$ such that the second letter of all $\mathbf{w}_{n(2,k)}; k \in \mathbb{N}$, is $x_2 \in \mathcal{A}$. Similarly we can extract a subsequence $\{\mathbf{w}_{n(j+1,k)}\}_{k\in\mathbb{N}} (\subset \{\mathbf{w}_{n(j,k)}\}_{k\in\mathbb{N}})$ such that the $(j+1)$-th letter of all $\mathbf{w}_{n(j+1,k)}; k \in \mathbb{N}$ is $x_{j+1} \in \mathcal{A}$. Define $\{\mathbf{w}^0_k\}_{k\in\mathbb{N}} = \{\mathbf{w}_{n(k,k)}\}_{k\in\mathbb{N}}$. Then, since $\{\mathbf{w}^0_k\}_{k\in\mathbb{N}}$ is a subsequence of $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$, it is a Cauchy sequence equivalent to $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$.

Define $x = x_1 x_2 \cdots x_n \cdots \in \mathcal{A}^\infty$ and $\mathbf{x}_k = x_1 x_2 \cdots x_k$. Then we have $T_0(x) = \{\mathbf{x}_k\}_{k\in\mathbb{N}}$ and by construction, for any $n \in \mathbb{N}$ the first n letters of $\mathbf{w}^0_n$ are $x_1 x_2 \cdots x_n$.

For any fixed $\mathbf{v} \in \mathcal{W}$, by the same argument as that in Lemma 3.3 and Proposition 2.20, we have

$$\lim_{n\to\infty} |k(\mathbf{v}, \mathbf{w}^0_n) - k(\mathbf{v}, \mathbf{x}_n)| = 0.$$

□

Lemma 3.7. $T : (\mathcal{A}^\infty/\sim) \to \mathcal{M}$ is a homeomorphism.

Proof. By Lemmas 3.5 and 3.6, we have that the map T is bijective. The continuity of T follows from the continuity of T_0, which is an easy consequence of Theorem A and (3.1). The continuity of T^{-1} follows from this and since $\mathcal{A}^\infty/\sim$ and M are compact. □

Remark 3.8. It goes without saying that we can also prove Theorem B using p_j which is defined in (2.19).

3.2 The Dirichlet Problem.

In this section, as a corollary to [5, 6], we shall solve the associated Dirichlet problem:

Let g be a continuous function on $\mathcal{M}$. Then

$$\begin{cases} ([P-I]f)(\mathbf{w}) = 0, & \mathbf{w} \in \mathcal{W} \\ \lim_{\mathbf{w}\to\xi} f(\mathbf{w}) = g(\xi), & \xi \in \mathcal{M}(= (\mathcal{A}^\infty/\sim) = \mathcal{P}) \end{cases}$$

has a unique solution in the class of uniformly continuous harmonic function space over $\mathcal{W}$, denoted by $\mathcal{H}_{C_u}(\mathcal{W})$.

The notion of a space of exits defined in Theorem 1.8 will play an important role in the proving the associated Dirichlet problem. Using the next corollary we first prove that $\mathcal{A}^\infty/\sim\ (= \mathcal{P})$ coincides with a space of exits.

By Theorem B and (2) of Theorem 1.8, we obtain

Corollary 3.9. There exists a Borel set $\mathcal{B} \subset \mathcal{M}$, called the *space of exits*, such that the following holds:

(1) The function $k(\cdot, z)$ is P-harmonic function on $\mathcal{W}$ for every $z \in \mathcal{B}$.

(2) For every P-excessive function $h \geq 0$ there exists a unique finite measure μ_h on $\overline{\mathcal{W}}$ such that

$$h(\mathbf{v}) = \int_{\mathcal{W}\cup\mathcal{B}} k(\mathbf{v}, y)d\mu_h(y) \text{ and } \mu_h(\mathcal{M}\setminus\mathcal{B}) = 0. \tag{3.4}$$

(3) $\mathbf{y} \in \mathcal{B}$ if and only if $\mu_{k(\cdot,\mathbf{y})}(\mathbf{x}) = \delta_{\mathbf{x},\mathbf{y}}$.

Note that the function $\mathbf{v} \mapsto k(\mathbf{v}, y)$ is P-excessive for every $y \in \mathcal{A}^\infty/\sim$.

Theorem C.

(1) The function $\mathbf{v} \mapsto k(\mathbf{v}, y)$ is P-harmonic on $\mathcal{W}$ for every $y \in \mathcal{A}^\infty/\sim$.

(2) $(\mathcal{A}^\infty/\sim) = \mathcal{B}$.

Proof.

(1) Let $y \in \mathcal{A}^\infty/\sim$ be fixed. Then, by Fubini's theorem, we have for any $\mathbf{v} \in \mathcal{W}$

$$\begin{aligned} Pk(\mathbf{v}, y) &= \sum_{\mathbf{u}\in\mathcal{W}} p(\mathbf{v}, \mathbf{u})k(\mathbf{v}, y) \\ &= \sum_{\mathbf{u}\in\mathcal{W}} p(\mathbf{v}, \mathbf{u}) \times \lim_{\mathbf{w}\to y} \frac{g(\mathbf{u}, \mathbf{w})}{g(\emptyset, \mathbf{w})} \\ &= \sum_{\mathbf{u}\in\mathcal{W}} \lim_{\mathbf{w}\to y} \left(\sum_{n=0}^{\infty} p(\mathbf{v}, \mathbf{u})p(n; \mathbf{u}, \mathbf{w})/g(\emptyset, \mathbf{w}) \right) \end{aligned}$$

$$
\begin{aligned}
&= \lim_{\mathbf{w}\to y}\left(\sum_{n=0}^{\infty} p(n+1;\mathbf{v},\mathbf{w})/g(\emptyset,\mathbf{w})\right)\\
&= \lim_{\mathbf{w}\to y}\frac{g(\mathbf{v},\mathbf{w})-\delta_{\mathbf{v},\mathbf{w}}}{g(\emptyset,\mathbf{w})}\\
&= k(\mathbf{v},y).
\end{aligned}
$$

(2) Let $y \in \mathcal{A}^\infty/\!\sim$. Since $k(\cdot, y)$ is P-harmonic on $\mathcal{W}$, by (3) of Corollary 3.9, the measure $\mu_{k(\cdot,y)}$ has its support in $\mathcal{A}^\infty/\!\sim$. Therefore it suffices to show that

$$(\mathcal{A}^\infty/\!\sim) \setminus \{y\} = \bigcup_{\mathbf{v}\in\mathcal{W};k(\mathbf{v},y)=0} \{\xi \in \mathcal{A}^\infty/\!\sim |k(\mathbf{v},\xi) > 0\}.$$

Indeed, if $\mathbf{v} \in \mathcal{W}$ and $k(\mathbf{v}, y) = 0$, then by (3.4) we have

$$0 = k(\mathbf{v}, y) = \int_{\mathcal{W}\cup(\mathcal{A}^\infty/\sim)} k(\mathbf{v},\xi)d\mu_{k(\mathbf{v},y)}(\xi),$$

so that $\mu_{k(\mathbf{v},y)}(\{\xi \in \overline{\mathcal{W}}|k(\mathbf{v},\xi) > 0\}) = 0$ and thus $\mu_{k(\mathbf{v},y)}((\mathcal{A}^\infty/\!\sim) \setminus \{y\}) = 0$.

Let $\xi, y \in \mathcal{A}^\infty/\sim$ such that $\xi \neq y$. Then we may assume that ξ and y have a representation

$$\xi = \mathbf{u}\xi_1\xi_2\cdots \quad , \quad y = \mathbf{u}y_1y_2\cdots$$

where $\mathbf{u} \in \mathcal{W}$ and $\xi_1 \neq y_1$. By Proposition 2.12, we have $\xi_1 = \sigma_d(y_1)$ where $d \in \{-1, 1\}$. If $y_2 \neq y_m$ for some $m \geq 3$, then the first letter of $y^\#$ is y_1 and hence

$$\{\mathbf{u}\xi_1\xi_2\cdots\xi_m\} \cap U(\mathbf{u}y_1y_2\cdots y_m) = \emptyset$$

which implies $k(\mathbf{w}\xi_1\xi_2\cdots\xi_m, y) = 0$. In addition, if $y_2 \in U(y_1)$, then by Lemma 2.16 we have $\mathbf{Anc}_m[\mathbf{u}y_1y_2^m] = U_0(\mathbf{u}y_1)$; so that we may assume $y = \mathbf{u}y_1\sigma_{2d}(y_1)^\infty$. On the other hand, if there exists $n \geq 2$ such that $\xi_n \neq \sigma_{-d}(y_1)$, then

$$\{\mathbf{u}\xi_1\xi_2\cdots\xi_n\} \cap U(\mathbf{u}y_1y_2\cdots y_n) = \emptyset.$$

Thus we have

$$\xi = \mathbf{u}\xi_1\xi_2\cdots = \mathbf{u}\sigma_d(y_1)\sigma_{-d}(y_1)^\infty = (\mathbf{u}y_1\sigma_{2d}(y_1)^\infty)^\# = y^\#,$$

which implies $\xi = y$ in $\mathcal{A}^\infty/\!\sim$. This is a contradiction. □

Therefore by Theorem 1.8 we also have shown

Corollary 3.10.

(1) For every bounded P-harmonic function $h \geq 0$, μ_h is absolutely continuous with respect to μ_1 with Radon-Nikodym derivative $\frac{d\mu_h}{d\mu_1}$ such that

$$h(\mathbf{v}) = \int_{\mathcal{A}^\infty/\sim} k(\mathbf{v},\xi)\frac{d\mu_h}{d\mu_1}(\xi)d\mu_1(\xi)$$

$$\lim_{n\to\infty} h(X_n) = \frac{d\mu_h}{d\mu_1}(X_\infty) \quad \mathrm{Pr}_{\mathbf{v}}\text{-a.e.} \quad \forall \mathbf{v} \in \mathcal{W}$$

$$\exists X_\infty \ni h(\mathbf{v}) = \mathbf{E}^{\mathbf{v}}\left[\frac{d\mu_h}{d\mu_1}(X_\infty)\right] \quad \forall \mathbf{v} \in \mathcal{W}.$$

Here $\{X_n | n \in \mathbb{N}_0\}$ denotes the associated Markov chain and $\mathrm{Pr}_{\mathbf{x}}$ is the probability measure concentrated on the paths starting from $\mathbf{x}$ given by

$$\mathrm{Pr}_{\mathbf{x}}[X_0 = \mathbf{x}_0, X_1 = \mathbf{x}_1, ..., X_n = \mathbf{x}_n] = \delta_{\mathbf{x},\mathbf{x}_0} p(\mathbf{x}_0,\mathbf{x}_1)p(\mathbf{x}_1,\mathbf{x}_2)\cdots p(\mathbf{x}_{n-1},\mathbf{x}_n).$$

(2) Conversely for every non-negative μ_1-integrable function f on $\mathcal{A}^\infty/\sim$

$$h_f(\mathbf{v}) := \int_{\mathcal{A}^\infty/\sim} k(\mathbf{v},\xi)f(\xi)d\mu_1(\xi) \tag{3.5}$$

defines a P-harmonic function on $\mathcal{W}$ and

$$f(X_\infty) = \lim_{n\to\infty} h_f(X_n) \quad \mathrm{Pr}_{\mathbf{v}}\text{-a.e.} \quad \forall \mathbf{v} \in \mathcal{W}.$$

Let ν be the *Bernoulli measure* on $\mathcal{A}^\infty$, that is, the product measure $\nu = \prod_{k=1}^\infty \nu_k$, where each ν_k is the uniform probability measure on $\mathcal{A}$. It is known that $\nu \circ \wp^{-1}$ is the Hausdorff measure on $\mathcal{P}$ where $\wp : \mathcal{A}^\infty \to \mathcal{A}^\infty/\sim$ denotes the canonical projection.

We also use the notation $C(A) = \{f | f : A \to \mathbb{R}, \text{ continuous}, A \subset \mathcal{A}^\infty/\sim\}$.

The following Corollaries 3.11 through 3.13 follow from [5].

Corollary 3.11. The harmonic measure μ_1 on $\mathcal{A}^\infty/\sim$ in Theorem 1.8, coincides with the normalised canonical Hausdorff measure $\nu \circ \wp^{-1}$.

Corollary 3.12. $\nu \circ \wp^{-1}$ is a *Radon measure* on $\mathcal{A}^\infty/\sim$ and *full*, i.e.

(1) $\nu \circ \wp^{-1}(A) = \sup\{\nu \circ \wp^{-1}(K) | K \subset A,\ K \text{ is compact subset of } \mathcal{A}^\infty/\sim\}$.

(2) For every non-empty open subset B of $\mathcal{A}^\infty/\sim$, we have that $\nu \circ \wp^{-1}(B)$ is strictly positive.

Corollary 3.13. Let h_f be defined in (3.5). Then we have $h_f \in \mathcal{H}_{C_u}(\mathcal{W} \setminus \{\emptyset\})$.

Corollary 3.14. Let f be a continuous function on $\mathcal{A}^\infty/\sim$. Then h_f can be extended to a continuous function on $\mathcal{W} \cup (\mathcal{A}^\infty/\sim)$, which coincides with f on $\mathcal{W} \cup (\mathcal{A}^\infty/\sim)$. In particular we have

$$\lim_{\mathbf{w}\to\xi} h_f(\mathbf{w}) = f(\xi)$$

for every $\xi \in \mathcal{A}^\infty/\sim$.

Proof. Since $\emptyset$ is an isolated point, h_f is uniformly continuous on the dense subset $\mathcal{W}$ of the compact metric space $\mathcal{W} \cup (\mathcal{A}^\infty/\sim)$ and extends to a continuous function $\overline{h_f}$ on $\mathcal{W} \cup (\mathcal{A}^\infty/\sim)$.

On the other hand by (2) of Corollary 3.10 we have

$$\lim_{n\to\infty} h_f(X_n) = f(X_\infty) \quad \text{Pr}_\emptyset\text{-a.e.}$$

and since $\nu \circ \wp^{-1} = \text{Pr}_\emptyset \circ X_\infty^{-1}$ we have

$$\overline{h_f}(\xi) = f(\xi) \quad (\text{Pr}_\emptyset \circ X_\infty^{-1})\text{-a.e.}$$

Since $\nu \circ \wp^{-1}$ is a Radon measure on $\mathcal{A}^\infty/\sim$ and full, we obtain that $\overline{h_f} \equiv f$ on $\mathcal{W} \cup (\mathcal{A}^\infty/\sim)$. ☐

We denote the map sending a bounded measurable function f on $\mathcal{A}^\infty/\sim$ to h_f by $\mathcal{I}$, that is, $\mathcal{I}(f) = h_f$.

Note that

$$\mathcal{I}(C(\mathcal{A}^\infty/\sim)) = \mathcal{H}_{C_u}(\mathcal{W}) \tag{3.6}$$

via Corollary 3.14.

We summerise our result in.

Theorem D. The Dirichlet problem for $P - I$ on $\mathcal{W}$,

$$\begin{cases} \{(P-I)f\}(\mathbf{w}) = 0, & \mathbf{w} \in \mathcal{W} \\ \lim_{\mathbf{w}\to\xi} f(\mathbf{w}) = g(\xi), & \xi \in \mathcal{M}(= (\mathcal{A}^\infty/\sim) = \mathcal{P}) \end{cases} \tag{D}$$

where $g \in C(\mathcal{A}^\infty/\sim)$, has a unique solution $f = h_g$ in $\mathcal{H}_{C_u}(\mathcal{W})$.

Proof. This follows from Corollary 3.14: Given a continuous function g, the function

$$h_g(\mathbf{v}) = \int_{\mathcal{M}} k(\mathbf{v}, \xi) g(\xi) d\mu_1(\xi)$$

is a solution. Hence it suffices to show that this solution of (D) is unique. We now consider

$$\begin{cases} ([P-I]F)(\mathbf{w}) = 0, & \mathbf{w} \in \mathcal{W} \\ \lim_{\mathbf{w}\to\xi} F(\mathbf{w}) = 0, & \xi \in \mathcal{M}. \end{cases} \tag{D*}$$

Then by (3.6), there exists some function $G \in C(\mathcal{A}^\infty/\sim) = C(\mathcal{M})$ such that

$$F(\mathbf{v}) = \int_{\mathcal{M}} k(\mathbf{v}, \xi) G(\xi) d\mu_1(\xi)$$

for any $\mathbf{v} \in \mathcal{W}$. Thus by (2) of Corollary 3.10 and (D*), we have $G \equiv 0$ on $\mathcal{M}$ which implies $F \equiv 0$ on $\mathcal{W}$. ☐

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REFERENCES

[1] M. T. Barlow, E. A. Perkins: Brownian Motion on the Sierpiński gasket. Probab. Theory Relat. Fields 79 (1988), 543–623.

[2] L. Barreira, Y. B. Pesin, J. Schmeling: On a General Concept of Multifractality: Multifractal Spectra for Dimensions, Entropies, and Lyapunov Exponents. Multifractal Rigidity. Chaos 7:1 (1997), 27–38.

[3] M. Denker, S. Koch: A Poisson formula for harmonic functions on the Sierpiński gasket. Forum Mathematicum. 12 (2000), 435–448

[4] M. Denker, H. Sato: Sierpiński gasket as a Martin boundary I: Martin kernels. To appear in Potential analysis.

[5] M. Denker, H. Sato: Sierpiński gasket as a Martin boundary II: The intrinsic metric. Publ. RIMS, Kyoto Univ 35 (1999), 769–794.

[6] M. Denker, H. Sato: Reflectons on harmonic analysis of the Sierpiński gasket. Preprint.

[7] J. L. Doob: Semimartingales and subharmonic functions. Trans. Amer. Math. Soc. 77 (1954), 86–121.

[8] J. L. Doob: Discrete potential theory and boundaries. J. Math. and Mech. 8 (1959), 433–458.

[9] E. B. Dynkin: Boundary theory of Markov processes (the discrete case). Russian Math. Surveys 24 (1969), 1–42.

[10] K. J. Falconer: Fractal geometry. Mathematical foundations and applications. John Wiley & Sons, 1990.

[11] W. Feller: Boundaries induced by non-negative matrices. Trans. Amer. Math. Soc. 83 (1956), 19–54.

[12] U. Freiberg, M. Zähle: Harmonic calculus on fractals – A measure geometric approach I. To appear in Potential analysis.

[13] M. Fukushima, T. Shima: On a spectral analysis for the Sierpiński gasket. Potential analysis 1 (1992), 1–35

[14] S. Goldstein: Random walk and diffusion on fractals. in H. Kesten (ed.), Percolation theory and ergodic theory of infinite particle systems, IMA Math. Appl. Springer, New York, 8 (1987), 121–129.

[15] M. Hata: On the structure of self-similarity. Japan J. Appl. Math. 2 (1985), 381–414.

[16] G. A. Hunt: Markov chains and Martin boundaries. Illinois J. Math. 4 (1960), 313–340.

[17] J. E. Hutchinson: Fractals and self-similarity. Indiana Univ. Math. J. 30 (1981), 713–747.

[18] V. A. Kaimanovich, A.M. Vershik: Random walks on discrete groups: boundary and entropy. Ann. Probab. 11 (1983), 457–490

[19] M. Kac: On some connections between probability theory and differential and integral equations. Proc. 2nd Berkely Symp. Math. Statist. Prob. (1951), 189–215

[20] S. Kakutani: Two-dimensional Brownian motion and harmonic functions. Proc. Imp. Acd. Tokyo 20 (1944), 704–714

[21] J. Kigami: A harmonic calculus on the Sierpiński space. Japan J. Appl. Math. 6 (1989), 259–290.
[22] J. Kigami: Harmonic calculus on p.c.f. self-similar sets. Trans. Amer. Math. Soc. 335 (1993), 721–755.
[23] A. W. Knapp: Connection between Brownian motion and potential theory. J. Math. Anal. and Appl. 12 (1965), 328–349.
[24] S. Kusuoka: A diffusion process on a fractal. in K. Ito, N. Ikeda (eds.), Probabilistic methods on mathematical physics, Proc. Taniguchi Symposium Kinokuniya, Tokyo (1987), 251–274.
[25] B. B. Mandelbrot: The Fractal geometry of nature. Freeman & Comp. New York, 1973.
[26] V. Metz: Potentialtheorie auf dem Sierpiński gasket. Math. Ann. 289 (1991), 207–237
[27] P. Ney, F. Spitzer: The Martin boundary for random walk. Trans. Amer. Math. Soc. 121 (1966), 116–132.
[28] C. Series: Martin boundaries of random walks on Fuchsian groups. Israel J. Math. 44 (1983), 221-242.
[29] T. Shima: On eigenvalue problems for the random walks on the Sierpiński pre-gaskets, Japan J. Indust. Appl. Math. 8 (1991), 124–141.
[30] W. Sierpiński: Sur une courbe dont tout point est un point de ramification. C. R. Acad. Sci. Paris 160 (1915), 302.
[31] S. Waddington: The Dirichlet problem for Laplacians on post critically finite self similar fractals. Preprint, Göttingen 1996.
[32] S. Waddington: The spectral zeta function for p.c.f. self-similar fractals. Preprint: Mathematica Gottingensis 6 (1996).
[33] W. Woess: Random walks on infinite graphs and groups. Cambridge Univ. Press, 2000

IMAI: GRADUATE SCHOOL OF MATHEMATICS, KYUSHU UNIVERSITY 33, FUKUOKA 812-8581, JAPAN
E-mail address: aimai@math.kyushu-u.ac.jp