On Gibbs Families for Fibrewise Expansive Systems

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Abstract

We prove that compact fibred systems, whose fibre maps are uniformly open, expansive and topologically exact, and whose fibres are continuous in the Hausdorff metric, possess, for every globally continuous, uniformly fibrewise locally constant potential, a unique Gibbs family of conditional measures on its fibres. If the system upon observation has only countably many fibres, then we associate to any continuous potential a Gibbs family, though, this time, this family may not be unique.

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Contents

Introduction

A fibred system is a dynamical system $T : Y \to Y$ on a metric space Y that forms a fibre bundle over another dynamical system $S : X \to X$, bundle which is preserved by the dynamics of T by means of a factor map $\pi : Y \to X$ with the property that $\pi \circ T = S \circ \pi$.

In 1999, M. Denker and M. Gordin [8] (see also [9]) showed that compact fibred systems, whose fibre maps are Ruelle expanding and topologically exact, admit, for every globally continuous, uniformly fibrewise Hölder continuous potential, a unique Gibbs family of conditional measures on its fibres. This result reduces to the celebrated Ruelle-Perron-Frobenius theorem in the non-fibred case (or, in other words, in the one-fibre case, that is, when X consists of only one point).

It is here considered whether such a family exists for systems having fibre maps that are expansive rather than Ruelle expanding. We will show that, whenever X is countable, compact fibred systems, whose fibre maps are uniformly open, expansive and topologically exact, and whose fibres are continuous in the Hausdorff metric, possess, for every continuous potential, a Gibbs family. Moreover, we will prove that, whatever the cardinality of X is, this class of systems admits a unique Gibbs family for every globally continuous, uniformly fibrewise locally constant potential.

Denker and Gordin derived the existence of a Gibbs family by means of relative transfer operators (also called Ruelle-Perron-Frobenius operators) and Birkhoff's theory of cones. They associated a transfer operator to each fibre. Each of these operators was defined on a dense cone of functions, in such a way that this bundle of cones contracted with respect to Hilbert's projective metrics under the application of the transfer operators. They then used the relative transfer operators to construct a sequence of globally defined operators that converged, thanks to the cones contraction, to an operator of conditional expectation. They subsequently projected this latter operator on each fibre and showed that the corresponding system of conditional probabilities constitutes the desired Gibbs family.

The approach we will use here is based upon Denker and Gordin's results. The main idea is the following: Knowing that Borel measures have no consciousness of the metric the underlying space Y is equipped with, we make a change of metric on Y, making sure that the topology be preserved, and show that the initially fibre expansive system is expanding along its fibres under the reign of the new metric. We then give a necessary and sufficient condition for the fibre maps to be Ruelle expanding. Finally, we find a family of potentials that retain their uniform fibrewise Hölder continuity under any compatible change of metric and apply Denker and Gordin's results, hence obtaining the wished Gibbs family.

The method of changing the metric for a more appropriate one was originally developed by E. Coven and W. Reddy [5] and relies on Frink's Metrization Lemma [10, 13]. They showed that positively expansive maps of compact manifolds are Ruelle expanding. The notion of Ruelle expandability is evidently due to D. Ruelle [19]. In his now famous book on thermodynamic formalism, Ruelle extracted the properties of expanding endomorphisms that are sufficient for topological dynamical systems to retain the ergodic theory of these maps. An expanding endomorphism of a compact differentiable manifold Y is, as introduced by M. Shub [22], a C^1 map $f : Y \to Y$ for which there is a compatible Riemannian metric and a constant $\Lambda > 1$ such that the induced norm $\|\cdot\|$ satisfies $\|Df^n(v)\| \geq \Lambda^n \|v\|$ for all $n \in \mathbb{N}$ and all tangent vectors v. Ruelle defined a continuous map $f : Y \to Y$ of a compact metrizable topological space Y as expanding (henceforth called Ruelle expanding) if there is a compatible metric d_Y for Y and constants $a > 0$ and $\Lambda > 1$ such that if $y_1, y_2' \in Y$ are such that $d_Y(f(y_1), y_2') < 2a$, then there is a unique $y_2 \in Y$ with $f(y_2) = y_2'$, $d_Y(y_1, y_2) < 2a$ and, furthermore, $d_Y(f(y_1), f(y_2)) \geq \Lambda\, d_Y(y_1, y_2)$. This notion is obviously stronger than the usual concept of expandability that is described by the existence of constants $a > 0$ and $\Lambda > 1$ such that $d_Y(f(y_1), f(y_2)) \geq \Lambda\, d_Y(y_1, y_2)$ whenever $d_Y(y_1, y_2) < 2a$. However, it was clear to Coven and Reddy (see [5]) that these two notions of expandability coincide exactly when f is an open map. They could thereby reduce their proof to the establishment of the existence of a compatible metric under which a positively expansive map of a compact manifold is expanding. They based their proof on the fact that such a map is a covering map and hence has a universal covering space. They could therefore lift the map to a homeomorphism.

Reddy [16] later remarked that the main ideas of the proof could be almost integrally transported to the case of a surjective map of a compact metric space. This time, though, he did not use the fact that such a map is a covering map.

Finally, M. Denker and M. Urbański [6] brought up a slightly different argument. Despite the fact that the chief ideas are the same as Reddy's ones and that their argument is lengthier and heavier, it has the advantage of being expressed in terms of the metrics $d_{n,Y}(y_1, y_2) = \max_{0\leq m<n} d_Y(f^m(y_1), f^m(y_2))$, where $n \in \mathbb{N}$. These metrics are sometimes called Bowen's metrics, since R. Bowen [2] used them extensively in defining topological entropy for non-compact dynamical systems. Using these metrics, we will extend Reddy's result. For this, we will, in analogy to Denker and Gordin's definition of Ruelle expandability along the fibres of fibred systems, introduce 'fibrewised' notions of (positive) expansiveness and expandability. These notions will be respectively called fibrewise expansiveness and fibrewise expandability, whereas Denker and Gordin's notion of expandability will be accordingly called fibrewise Ruelle expandability. We will show that fibrewise uniformly expansive uniformly continuous systems, whose fibres are uniformly continuous in the Hausdorff metric, are fibrewise expanding under a certain compatible metric. Our result generalizes Coven and Reddy's work in two different manners. Firstly, it applies not only to (positively) expansive maps of compact metric spaces, but also to the larger family of uniformly expansive uniformly continuous maps of non-compact spaces. In this regard, we further show that the new metric is uniformly topologically equivalent to the old one, so that the map retains its uniform continuity. Secondly, it applies to the larger class of systems whose expansiveness property lies along the fibres and is therefore weaker than global expansiveness. However, it is not known whether all fibrewise expansive systems are subject to this change of metric.

An analogous result holds for expansive homeomorphisms. Indeed, Reddy [17] and K. Sakai's [20] works revealed that every expansive homeomorphism of a compact space admits a hyperbolic metric. This means that there exists a compatible metric with respect to which the homeomorphism is contracting on sufficiently small local stable sets, whereas its inverse is contracting on the corresponding local unstable sets. Here again we will show that this result can be extended to the theory of invertible fibred systems and, in particular, that a uniformly expansive uniform homeomorphism of a non-compact space admits a hyperbolic metric. Moreover, in this latter case, the metric is, as in the non-invertible case, uniformly topologically equivalent to the original metric, hence preserving the uniform character of the homeomorphism. On the other hand, we will show that it suffices, whenever π is uniformly continuous, that the system be quasi invertible, that is, that the base transformation S be locally injective, for a change of metric to take place. This follows from the fact that one may always assume for purely topological problems that the map π is a strong contraction.

This right to pass from expansiveness (respectively, fibrewise expansiveness) to expandability (respectively, fibrewise expandability) at the cost of a change of metric has some interesting implications. In the non-fibred case, it was showed that an infinite compact metric space does not accept any positively expansive homeomorphism (see [1]). It was also proved, as a simple corollary to this result, that the closed unit interval does not support any expansive map. As a consequence of our result, we will show that no fibrewise surjective triangular map can be fibrewise expansive, much less expansive. This discloses another similarity between maps of the unit interval and triangular maps.

We will also shed some light on the relation between fibrewise expanding systems and fibrewise Ruelle expanding ones. As mentioned before, Coven and Reddy were aware that a compact dynamical system is Ruelle expanding if and only if it is open and expanding. However, we will show that a system which is fibrewise open and fibrewise expanding may not be fibrewise Ruelle expanding. This is due to the fact that fibrewise openness and uniform fibrewise openness are two distinct notions for fibred systems, even for compact ones. Indeed, we will show that a fibrewise expanding system is fibrewise Ruelle expanding if and only if it is uniformly fibrewise open. Since it turns out to be usually quite difficult to prove directly that a system is uniformly fibrewise open, we will give some conditions under which this property is fulfilled. In particular, it appears that it is not sufficient for the map T of a compact system $\mathcal{Y} = (Y, T, X, S, \pi)$ to be open to ensure that the system be uniformly fibrewise open. In fact, it does not even ensure that $\mathcal{Y}$ be fibrewise open. We will show, though, that if, in addition, S is locally injective, then the system is uniformly fibrewise open. We will also provide some examples of systems, for which S is locally injective everywhere except at one point, where it is locally two-to-one, that are not uniformly fibrewise open. We also investigate what properties the fibre maps must have for the system to be uniformly fibrewise open. We will prove that this is the case whenever each fibre map is a homeomorphism. This is a very strong requirement, but it turns out that there are some systems for which every fibre map is a homeomorphism, but one, which is a local homeomorphism, systems that are not uniformly fibrewise open. Finally, we impose some topological conditions on the fibres. We will look more closely to uniform fulc spaces, that is, those spaces whose fibres are uniformly locally connected have some uniformity between them. We show that every fibrewise open, fibrewise expanding uniform fulc compact system is fibrewise Ruelle expanding. Nevertheless, we provide some examples of fibrewise open, fibrewise expanding fulc compact systems that are not fibrewise Ruelle expanding. We also show that a compact skew-product is uniformly fulc if and only if one of its fibres (that are essentially all identical) is locally connected.

Having done that, we derive the existence of a Gibbs family for an important class of fibrewise expansive compact systems. The case X countable is completely tackled by noticing that the Lipschitz potentials are dense in the space of continuous potentials (this is a simple application of Stone-Weierstrass theorem). We can therefore approximate the potential imposed on Y by a sequence of Lipschitz potentials. This sequence gives rise, according to Denker and Gordin, to a sequence of Gibbs families. We will show that the 'accumulation points' of this latter form Gibbs families for the potential under consideration. Unfortunately, the uniqueness of the Gibbs family may be lost in the process. The case X uncountable proves to be much trickier and we adopt another strategy to partially solve it. We extract a family of potentials that keep their uniform fibrewise Hölder continuity under any compatible change of metric. This family regroups all uniformly fibrewise locally constant potentials. For each such potential, we deduce the existence of a unique Gibbs family.

Chapitre 1

Preliminaries

In this chapter, we introduce, among others, the basic concepts of *fibred systems, fibrewise expansiveness, fibrewise expandability and fibrewise Ruelle expandability.* These definitions are direct generalizations of well-known notions in the theory of dynamical systems.

1.1 Fibred Systems

1.1.1 Definition

Fibred systems constitute a special class of dynamical systems. Loosely speaking, they can be described as systems consisting of a fibre bundle (over another system) which is preserved under iteration.

Throughout this dissertation, a fibred system is a collection $\mathcal{Y} = (Y, T, X, S, \pi)$, where Y and X are metric spaces, $T : Y \to Y$ and $S : X \to X$ are continuous maps and $\pi : Y \to X$ is a continuous surjective map which satisfies $\pi \circ T = S \circ \pi$, that is, the following diagram commutes:

$$\begin{tikzcd} Y \arrow[r, "T"] \arrow[d, "\pi"'] & Y \arrow[d, "\pi"] \\ X \arrow[r, "S"] & X \end{tikzcd}$$

Thus T preserves the fibres $Y_x := \pi^{-1}(\{x\})$. The restriction of T^n ($n \in I\!N$) to the fibre Y_x will be denoted by T_x^n, so $T_x^n : Y_x \to Y_{S^n(x)}$. The metric on Y will be denoted by d_Y, while the metric imposed upon X will be denoted by d_X.

We will often be in the presence of two different metrics for Y and be willing to 'compare' them. Let us recall that two metrics d and ρ for a space Y are topologically equivalent (or compatible) if they induce the same topology on Y. This is equivalent to the exigence that the identity map $Id : (Y, d) \to (Y, \rho)$ be a homeomorphism. Recall also that d and ρ are said to be uniformly topologically equivalent (or uniformly compatible) if the identity map Id is a uniform homeomorphism, that is, if both Id and its inverse are uniformly continuous maps.

A fibred system $\mathcal{Y}$ is a skew-product if $Y = X \times Z$, where Z is a metric space and $T(y) = T(x, z) = (S(x), T_x(z))$. Skew-products form an important subclass of fibred systems, as the reader will see later.

A fibred system is said to be compact if Y is a compact space (note that this also implies that X is compact). It is also said to be uniformly continuous if T has this property with respect to d_Y.

We will also say that a system $\mathcal{Y}' = (Y' \subseteq Y, T|_{Y'}, \pi(Y'), S|_{\pi(Y')}, \pi|_{Y'})$ is a subsystem of a system $\mathcal{Y} = (Y, T, X, S, \pi)$ if $T(Y') \subseteq Y'$. In the compact case, we will additionally require that Y' be closed in order to ensure properties inheritance.

Closely related to fibred systems are fibred spaces. Fibred spaces constitute the underlying space structures of fibred systems, in the same manner as topological spaces or metric spaces form the space structures of topological or metric dynamical systems. Strictly speaking, a fibred space is a triple (Y, X, π), where Y and X are metric spaces and $\pi : Y \to X$ is a continuous surjective map. In analogy to fibred systems, a fibred space (Y, X, π) will be called compact if Y has this property.

A fibred space can be viewed as a space that consists of a fibre bundle over another space. One can also interpret it as a fibred system from which the dynamics has been wiped out.

1.1.2 Morphisms

A fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is homomorphic to another fibred system $\mathcal{Y}' = (Y', T', X', S', \pi')$ if there exist two continuous maps $\phi : Y \to Y'$ and $\psi : X \to X'$ such that $\phi \circ T = T' \circ \phi$ and $\psi \circ \pi = \pi' \circ \phi$ (notice that this implies $\psi \circ S = S' \circ \psi$). The pair of maps (ϕ, ψ) is therefore called a homomorphism from $\mathcal{Y}$ into $\mathcal{Y}'$. The situation may be represented schematically by the commutative diagram:

$$
\begin{tikzcd}
 & Y \arrow[r, "\phi"] & Y' & \\
Y \arrow[r, "T"] \arrow[d, "\pi"'] & Y \arrow[u, "T"] \arrow[r, "\phi"] \arrow[d, "\pi"] & Y' \arrow[u, "T'"'] \arrow[r, "T'"] \arrow[d, "\pi'"'] & Y' \arrow[d, "\pi'"] \\
X \arrow[r, "S"] & X \arrow[r, "\psi"] \arrow[d, "S"'] & X' \arrow[r, "S'"] \arrow[d, "S'"] & X' \\
 & X \arrow[r, "\psi"] & X' &
\end{tikzcd}
$$

In such circumstances, each fibre of $\mathcal{Y}$ is mapped by ϕ into a fibre of $\mathcal{Y}'$.

A fibred system $\mathcal{Y}'$ is a factor of a system $\mathcal{Y}$ if there exists a homomorphism (ϕ, ψ) from $\mathcal{Y}$ to $\mathcal{Y}'$ such that ϕ is a surjective map (note that this also ensures that ψ is surjective). We thereby say that (ϕ, ψ) is a factor pair of maps from $\mathcal{Y}$ onto $\mathcal{Y}'$.

Finally, two fibred systems $\mathcal{Y}$ and $\mathcal{Y}'$ are called isomorphic (or fibrewise conjugate) if there exists a factor pair of maps (ϕ, ψ) from $\mathcal{Y}$ onto $\mathcal{Y}'$ such that both maps ϕ and ψ are homeomorphisms. The couple (ϕ, ψ) is therefore called a isomorphism (or, alternatively, a fibrewise conjugacy) from $\mathcal{Y}$ onto $\mathcal{Y}'$. Remark also that (ϕ^{-1}, ψ^{-1}) is an isomorphism from $\mathcal{Y}'$ onto $\mathcal{Y}$. Moreover, note that the fibres of $\mathcal{Y}$ are then in a one-to-one correspondence with and homeomorphic to the fibres of $\mathcal{Y}'$.

1.1.3 Composition, Invertibility and Quasi Invertibility

The composition of two fibred systems of the form $\mathcal{Y} = (Y, T, X, S, \pi)$ and $\mathcal{Y}' = (Y, T', X, S', \pi)$ is defined by $\mathcal{Y}' \circ \mathcal{Y} = (Y, T' \circ T, X, S' \circ S, \pi)$. This operation is depicted by the following commutative diagram:

$$
\begin{tikzcd}
Y \arrow[r, "T"] \arrow[d, "\pi"'] & Y \arrow[r, "T'"] \arrow[d, "\pi"] & Y \arrow[d, "\pi"] \\
X \arrow[r, "S"] & X \arrow[r, "S'"] & X
\end{tikzcd}
$$

A fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ will be referred to as invertible if the maps T and S are homeomorphisms. This implies in particular that $\mathcal{Y}^{-1} := (Y, T^{-1}, X, S^{-1}, \pi)$

forms a fibred system. For an invertible fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$, it is easy to see that $\mathcal{Y}^{-1} \circ \mathcal{Y} = Id = \mathcal{Y} \circ \mathcal{Y}^{-1}$, where $Id = (Y, Id_Y, X, Id_X, \pi)$. In fact, a fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is invertible if and only if there exists a system $\mathcal{Y}' = (Y, T', X, S', \pi)$ satisfying $\mathcal{Y}' \circ \mathcal{Y} = Id = \mathcal{Y} \circ \mathcal{Y}'$.

Finally, we will say that a fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is quasi invertible if T is a homeomorphism and S locally injective.

1.2 Fibrewise Expansiveness

The concept of expansion has drawn the attention and aroused the interest of a great many mathematicians. Amidst the large variety of such dynamical behaviors, expansiveness has turned out to be a rather weak but useful notion. It has hence been extensively studied.

This notion was introduced by W.R. Utz [23] in 1950. He used the term *unstable homeomorphisms* to describe these maps but Gottschalk and Hedlund [11] later suggested the term *expansive homeomorphisms*, which has been used ever since. Five years after Utz, R.F. Williams [26] investigated positive expansiveness of maps. Among all the other early contributors, we would like to mention the works realized by B.F. Bryant [3] and H.B. Keynes and J.B. Robertson [14].

We now introduce a concept of expansiveness along the fibres of a system.

Definition 1.1 *A fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is said to be (uniformly) fibrewise expansive if there exists a constant $\epsilon > 0$ such that the following holds:*
If $y_1, y_2 \in Y$, $\pi(y_1) = \pi(y_2)$ and $y_1 \neq y_2$, then

$$\sup_{n \in \mathbb{Z}_+} d_Y\Big(T^n(y_1), T^n(y_2)\Big) > \epsilon,$$

that is,

$$\sup_{n \in \mathbb{N}} d_{n,Y}(y_1, y_2) > \epsilon,$$

where

$$d_{n,Y}(y_1, y_2) = \max_{0 \leq m < n} d_Y\Big(T^m(y_1), T^m(y_2)\Big)$$

are new metrics defined on Y (though the notation does not show the dependence on T, it is crucial to remember that these metrics arise from the dynamics of the system (Y, T)). The open ball centred at y and of radius r in the metric $d_{n,Y}$ is $\cap_{m=0}^{n-1} T^{-m}(B_Y(T^m(y), r))$ and will be denoted by $B_{n,Y}(y, r)$ (remark that $d_{1,Y} = d_Y$ and hence $B_{1,Y}(y, r) = B_Y(y, r)$).

Equivalently, $\mathcal{Y}$ is fibrewise expansive if there is $\epsilon > 0$ such that $y_1, y_2 \in Y$, $\pi(y_1) = \pi(y_2)$ and $\sup_{n \in \mathbb{Z}_+} d_Y(T^n(y_1), T^n(y_2)) \leq \epsilon$ implies that $y_1 = y_2$. We call ϵ a fibrewise expansive constant. We will also say under these circumstances that $\mathcal{Y}$ is fibrewise ϵ-expansive.

It is clear that an expansive fibred system is fibrewise expansive. Though the converse is not true, we now show that if S is expansive, so is a fibrewise expansive system $\mathcal{Y} = (Y, T, X, S, \pi)$.

Theorem 1.2 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibrewise expansive system for which π is uniformly continuous. If S is expansive, so is T.*

Proof. Let $\epsilon > 0$ be a fibrewise expansive constant for the system and let $0 < \delta < \epsilon$. By uniform continuity of π, there exists $0 < \zeta < \epsilon$ such that

$$d_Y(y_1, y_2) < \zeta \Rightarrow d_X\Big(\pi(y_1), \pi(y_2)\Big) < \delta.$$

If T were not expansive, there would exist points $y_\delta, \tilde{y}_\delta \in Y, \pi(y_\delta) \neq \pi(\tilde{y}_\delta)$ with

$$\sup_{n \in \mathbb{Z}_+} d_Y\Big(T^n(y_\delta), T^n(\tilde{y}_\delta)\Big) < \zeta.$$

This would imply that

$$\sup_{n \in \mathbb{Z}_+} d_X\Big(S^n(\pi(y_\delta)), S^n(\pi(\tilde{y}_\delta))\Big) = \sup_{n \in \mathbb{Z}_+} d_X\Big(\pi(T^n(y_\delta)), \pi(T^n(\tilde{y}_\delta))\Big) \leq \delta.$$

Hence there would be $x_\delta (:= \pi(y_\delta)) \neq \tilde{x}_\delta (:= \pi(\tilde{y}_\delta)) \in X$ such that

$$\sup_{n \in \mathbb{Z}_+} d_X\Big(S^n(x_\delta), S^n(\tilde{x}_\delta)\Big) \leq \delta.$$

Since this would be possible for any $0 < \delta < \epsilon$, one would conclude that S is not expansive. This would lead to a contradiction. $\square$

Note that the expansiveness of S does not in general constrain T to be expansive. Nor does the expansiveness of T force S to be expansive.

We will later investigate in detail the properties of fibrewise expansiveness. For now, let us define the notion of fibrewise uniform expansiveness.

1.3 Fibrewise Uniform Expansiveness

In 1962, B.F. Bryant [4] remarked that there is a certain type of uniformity associated with the expansiveness of compact dynamical systems. The uniformity lies in the minimum rate at which the points of the system leave their mutual neighbourhood of expansiveness. This rate depends solely on the distance separating points. This behavior was formalized and studied by M. Sears [21] eleven years later. Its generalization to fibred systems can be stated as follows.

Definition 1.3 *A fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is called (uniformly) fibrewise uniformly expansive if there is $\epsilon > 0$ with the property that for all $0 < \tilde{\epsilon} \leq \epsilon$ there exists $N = N(\tilde{\epsilon}) \in I\!N$ such that the following holds:*
If $y_1, y_2 \in Y$, $\pi(y_1) = \pi(y_2)$ and $d_Y(y_1, y_2) > \tilde{\epsilon}$, then $d_{N,Y}(y_1, y_2) > \epsilon$.
The constant ϵ is then called a fibrewise uniform expansive constant for the system and this latter is said to be fibrewise uniformly ϵ-expansive.

One possible characterization of fibrewise uniform expansiveness is the following.

Lemma 1.4 *A fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is fibrewise uniformly ϵ-expansive if and only if*

$$\lim_{n\to\infty} \sup_{y\in Y} \operatorname{diam}_Y \left(B_{n,Y}(y,\epsilon) \cap Y_{\pi(y)}\right) = 0.$$

Proof. This can be easily seen from Definition 1.3. □

Corollary 1.5 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibrewise uniformly ϵ-expansive fibred system. Then there exists $M \in I\!N$ such that*

$$\sup_{y\in Y} \operatorname{diam}_Y \left(B_{M+1,Y}(y,\epsilon) \cap Y_{\pi(y)}\right) < \frac{\epsilon}{8},$$

that is, for any $y \in Y$,

$$B_{M+1,Y}(y,\epsilon) \cap Y_{\pi(y)} \subseteq B_Y\left(y, \frac{\epsilon}{8}\right).$$

It is evident that a fibrewise uniformly expansive system is fibrewise expansive. As in the non-fibred case or, in other words, the one-fibre case, the compactness of the system guarantees that these two notions are equivalent (see [21]).

Lemma 1.6 *Every compact fibrewise expansive system is fibrewise uniformly expansive.*

Proof. Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibrewise expansive compact fibred system with fibrewise expansive constant $\epsilon > 0$.

First, notice that for any $y \in Y$ and $\delta > 0$

$$B_{n_1,Y}(y,\delta) \subseteq B_{n_2,Y}(y,\delta)$$

whenever $n_1 \geq n_2$ (since $d_{n_1,Y} \geq d_{n_2,Y}$). Then

$$B_{n_1,Y}(y,\delta) \cap Y_{\pi(y)} \subseteq B_{n_2,Y}(y,\delta) \cap Y_{\pi(y)}.$$

This implies that

$$\operatorname{diam}_Y \left(B_{n_1,Y}(y,\delta) \cap Y_{\pi(y)}\right) \leq \operatorname{diam}_Y \left(B_{n_2,Y}(y,\delta) \cap Y_{\pi(y)}\right),$$

and hence

$$\sup_{y\in Y} \operatorname{diam}_Y \left(B_{n_1,Y}(y,\delta)\cap Y_{\pi(y)}\right) \le \sup_{y\in Y} \operatorname{diam}_Y \left(B_{n_2,Y}(y,\delta)\cap Y_{\pi(y)}\right).$$

The sequence $(\sup_{y\in Y}\operatorname{diam}_Y(B_{n,Y}(y,\delta)\cap Y_{\pi(y)}))_{n\in\mathbb{N}}$ is therefore decreasing and hence converges to the infimum of its set of values.

Let

$$\eta := \lim_{n\to\infty}\sup_{y\in Y} \operatorname{diam}_Y \left(B_{n,Y}(y,\epsilon)\cap Y_{\pi(y)}\right) = \inf_{n\in\mathbb{N}}\sup_{y\in Y} \operatorname{diam}_Y \left(B_{n,Y}(y,\epsilon)\cap Y_{\pi(y)}\right).$$

If $\eta>0$, then for each $n\in\mathbb{N}$, there would exist $y_n,\tilde{y}_n\in Y$ such that $\tilde{y}_n\in B_{n,Y}(y_n,\epsilon)\cap Y_{\pi(y_n)}$ and $d_Y(\tilde{y}_n,y_n)\ge \eta/2$. By taking a subsequence if necessary, we may assume, without loss of generality, that the sequences $(\tilde{y}_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ converge to, say, $\tilde{y}$ and y respectively. Then

$$d_Y(\tilde{y},y)=\lim_{n\to\infty} d_Y(\tilde{y}_n,y_n)\ge\frac{\eta}{2}>0.$$

Hence $\tilde{y}\ne y$. But $\pi(\tilde{y}_n)=\pi(y_n)$. The continuity of π then ensures that $\pi(\tilde{y})=\pi(y)$.

Moreover, for $m\in\mathbb{Z}_+$, the continuity of T^m implies that

$$\begin{aligned} d_Y\left(T^m(\tilde{y}),T^m(y)\right) &= \lim_{n\to\infty} d_Y\left(T^m(\tilde{y}_n),T^m(y_n)\right)\\ &\le \sup_{n\in\mathbb{N}} d_{n,Y}(\tilde{y}_n,y_n)\\ &\le \epsilon. \end{aligned}$$

Since $m\in\mathbb{Z}_+$ was chosen arbitrarily, we deduce that

$$\sup_{m\in\mathbb{Z}_+} d_Y\left(T^m(\tilde{y}),T^m(y)\right)\le\epsilon.$$

The fibrewise expansiveness of the system would then imply that $\tilde{y}=y$. But we know that this is not the case. Hence $\eta=0$ and the result follows from Lemma 1.4. □

Nonetheless, note that these two notions may differ for non-compact systems (again, see [21]).

1.4 Fibred Generators

The original concept of generator was introduced by V.A. Rokhlin [18] in ergodic theory. Its topological analogue is due to Keynes and Robertson [14]. They characterized expansive homeomorphisms as being those homeomorphisms that can be associated with a generator. As the concept of generator has proved to be useful, we are naturally led to generalize it.

Definition 1.7 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a compact fibred system (respectively a compact invertible fibred system).*

A finite open cover $\mathcal{U}$ of Y is called one-sided (respectively two-sided) fibred generator if for every sequence (resp. bisequence) $(U_i)_{i \in I}$ (with I being equal to $\mathbb{Z}_+$ and $\mathbb{Z}$ respectively) of atoms of $\mathcal{U}$, the set $(\cap_{i\in I} T^{-i}(\overline{U_i})) \cap Y_x = \cap_{i \in I} T^{-i}(\overline{U_i} \cap Y_{S^i(x)})$ consists of at most one point for every $x \in X$.

$\mathcal{U}$ is said to be a weak one-sided (respectively two-sided) fibred generator if for every sequence (resp. bisequence) $(U_i)_{i\in I}$ (with I chosen as above) of atoms of $\mathcal{U}$, the set $(\cap_{i \in I} T^{-i}(U_i)) \cap Y_x$ consists of at most one point for every $x \in X$.

Clearly, a fibred generator is a weak fibred generator. Though the converse is not true, one can show, as expected from Keynes and Robertson's work, that a fibred system admitting a weak fibred generator also has a fibred generator.

Unless otherwise stated, $\mathcal{Y}$ is assumed to be a compact fibred system (resp. a compact invertible fibred system) for the remainder of this section. It is also understood that the terms *one-sided* (resp. *two-sided*) and *sequence* (resp. *bisequence*) relate to a fibred system (resp. an invertible fibred system). Moreover, for an invertible fibred system, fibrewise expansiveness is defined in terms of all (backward and forward) iterates, whereas we will refer to positive fibrewise expansiveness if we just want to look at the forward iterates. In other words, an invertible fibred system $\mathcal{Y}$ is said to be fibrewise expansive if there exists $\epsilon > 0$ such that if $y_1, y_2 \in Y$, $\pi(y_1) = \pi(y_2)$ and $y_1 \neq y_2$, then $\sup_{n \in \mathbb{N}} d^{\pm}_{n,Y}(y_1, y_2) > \epsilon$, where $d^{\pm}_{n,Y}(y_1, y_2) = \max_{-n<m<n} d_Y(T^m(y_1), T^m(y_2))$. We will specify that $\mathcal{Y}$ is positively fibrewise expansive if it is fibrewise expansive when considered as a non-invertible system, that is, satisfies Definition 1.1.

The proofs of the following results go along the same lines as Keynes and Robertson [14] and Walters [24].

Lemma 1.8 *If $\mathcal{Y}$ has a weak fibred generator, then it has a fibred generator.*

Proof. Suppose that $\mathcal{U}$ is a weak fibred generator for $\mathcal{Y}$ and let $\delta > 0$ be a Lebesgue number for $\mathcal{U}$. Let $\mathcal{V}$ be a finite open cover of sets V_i such that $\mathrm{diam}_Y(\overline{V_i}) < \delta$. If $(V_{i_n})_{n \in I}$ is a sequence (resp. a bisequence) in $\mathcal{V}$, then for all $n \in I$ (I being either $\mathbb{Z}_+$ or $\mathbb{Z}$), there exists j_n such that $\overline{V_{i_n}} \subseteq U_{j_n}$. Hence

$$\cap_{n \in I} T^{-n}(\overline{V_{i_n}}) \bigcap Y_x \subseteq \cap_{n\in I} T^{-n}(\overline{U_{j_n}}) \bigcap Y_x,$$

which is either empty or a single point for any $x \in X$. So $\mathcal{V}$ is a fibred generator. □

Theorem 1.9 *$\mathcal{Y}$ is fibrewise expansive if and only if it has a weak fibred generator and hence if and only if it admits a fibred generator.*

Proof. By Lemma 1.8, it suffices to show that $\mathcal{Y}$ is fibrewise expansive if and only if it admits a fibred generator.

Let $\delta > 0$ be a fibrewise expansive constant for $\mathcal{Y}$ and $\mathcal{U}$ be a finite cover by open balls of radius $\delta/2$. Suppose that $y_1, y_2 \in \cap_{n\in I} T^{-n}(\overline{U_n}) \cap Y_x$ for some $U_n \in \mathcal{U}$ and some $x \in X$. Then $d_Y(T^n(y_1), T^n(y_2)) < \delta$ for all $n \in I$ and $\pi(y_1) = \pi(y_2)$. The fibrewise expansiveness of the system therefore implies that $y_1 = y_2$. Hence $\mathcal{U}$ is a fibred generator.

Conversely, suppose that $\mathcal{U}$ is a fibred generator. Let $\delta > 0$ be a Lebesgue number for $\mathcal{U}$. If $y_1, y_2 \in Y$ are such that $\pi(y_1) = \pi(y_2)$ and $d_Y(T^n(y_1), T^n(y_2)) \leq \delta$ for all $n \in I$, then there exists for each n an atom $U_n \in \mathcal{U}$ with $T^n(y_1), T^n(y_2) \in U_n$ and so $y_1, y_2 \in \cap_{n\in I} T^{-n}(U_n) \cap Y_{\pi(y_i)}$. Since $\mathcal{U}$ is a fibred generator, we deduce that $y_1 = y_2$. Hence $\mathcal{Y}$ is fibrewise δ-expansive. $\square$

As usual, the notion of generator can be used to deduce the following properties (again, compare with [24], page 139).

Corollary 1.10 *The following assertions hold:*

(i) Fibrewise expansiveness is independent of the metric on Y as long as the metric induces the topology of Y. However, the fibrewise expansive constant may change. Furthermore, fibrewise expansiveness does in general depend on the metric on Y when $\mathcal{Y}$ is non-compact.

(ii) More generally, fibrewise expansiveness is a fibrewise conjugacy invariant, that is, if $\mathcal{Y}$ and $\mathcal{Y}'$ are two isomorphic (or fibrewise conjugate) fibred systems, then $\mathcal{Y}$ is fibrewise expansive if and only if $\mathcal{Y}'$ is.

(iii) If $k \neq 0$, then $\mathcal{Y}$ is fibrewise expansive if and only if all of its iterates $\mathcal{Y}^k = (Y, T^k, X, S^k, \pi)$ are fibrewise expansive (the case $k < 0$ is only considered when $\mathcal{Y}$ is invertible).

(iv) A subsystem $\mathcal{Y}'$ of a fibrewise expansive system $\mathcal{Y}$ is fibrewise expansive.

Proof.

(i) This follows from the fact that the concept of generator does not depend on the metric.

(ii) $\mathcal{U}$ is a fibred generator for $\mathcal{Y}$ if and only if $\phi(\mathcal{U})$ is a fibred generator for $\mathcal{Y}'$.

(iii) If $\mathcal{U}$ is a fibred generator for $\mathcal{Y}$, then $\vee_{j=0}^{k-1} T^{-j}(\mathcal{U})$ is a fibred generator for $\mathcal{Y}^k$. Moreover, any fibred generator for $\mathcal{Y}^k$ is a fibred generator for $\mathcal{Y}$.

(iv) This follows directly from the definition of fibrewise expansiveness. One can also argue that the restriction of a fibred generator for the system to any of its subsystems is a fibred generator for the chosen subsystem.

Keynes and Robertson showed that generators completely determine the topology of the underlying space upon which a dynamical system lives (see [14]). Let us now investigate what information fibred generators give.

Lemma 1.11 *Let $\mathcal{Y}$ be a compact fibred system (respectively a compact invertible fibred system) and let $\mathcal{U}$ be a one-sided (resp. two-sided) fibred generator for $\mathcal{Y}$. Then for every $\epsilon > 0$, there exists $N = N(\epsilon) \in I\!N$ such that each set of the form $\vee_{n=0}^{N} T^{-n}(\mathcal{U}) \cap V_{1/N}(Y_x)$ (respectively $\vee_{n=-N}^{N} T^{-n}(\mathcal{U}) \cap V_{1/N}(Y_x)$) has diameter less than ϵ.*

Conversely, for each $N \in I\!N$, there exists $\epsilon = \epsilon(N) > 0$ such that $d_Y(y_1, y_2) < \epsilon$ implies that

$$y_1, y_2 \in \cap_{n=0}^{N} T^{-n}(U_n) \bigcap V_{1/N}(Y_x)$$

for some $U_0, \ldots, U_N \in \mathcal{U}$ and some $x \in X$ (n varying from $-N$ to N in the invertible case).

Here $V_\delta(A)$ stands for the open δ-neighbourhood of the set A with respect to the metric d_Y imposed on Y.

Proof. We only give a proof for the non-invertible case. The reader may supply a similar argument for invertible systems.

Suppose that the first statement is not true. Then there is $\epsilon > 0$ such that for each $m \in I\!N$, there are $x_m \in X$, $U_m^{(k)} \in \mathcal{U}$ $(0 \le k \le m)$ and $y_m^{(1)}, y_m^{(2)} \in \cap_{k=0}^{m} T^{-k}(U_m^{(k)}) \cap V_{1/m}(Y_{x_m})$ with $d_Y(y_m^{(1)}, y_m^{(2)}) \ge \epsilon$. Since X and Y are compact spaces, there exist subsequences (for clarity, we will identify these subsequences with the original sequences) of $(x_m)_{m \in I\!N}$, $(y_m^{(1)})_{m \in I\!N}$ and $(y_m^{(2)})_{m \in I\!N}$ that converge to, say, $x, y^{(1)}$ and $y^{(2)}$ respectively. We then have that $d_Y(y^{(1)}, y^{(2)}) \ge \epsilon$ and hence $y^{(1)} \ne y^{(2)}$.

Furthermore, since $y_m^{(1)}, y_m^{(2)} \in V_{1/m}(Y_{x_m})$, there exist $\tilde{y}_m^{(1)}, \tilde{y}_m^{(2)} \in Y_{x_m}$ such that $d_Y(\tilde{y}_m^{(i)}, y_m^{(i)}) < 1/m$. Hence

$$\pi(y^{(i)}) = \pi(\lim_{m \to \infty} y_m^{(i)}) = \pi(\lim_{m \to \infty} \tilde{y}_m^{(i)}) = \lim_{m \to \infty} \pi(\tilde{y}_m^{(i)}) = \lim_{m \to \infty} x_m = x.$$

Thus, $y^{(1)} \ne y^{(2)}$ and $y^{(1)}, y^{(2)} \in Y_x$.

Now, consider the sequence of sets $(U_m^{(0)})_{m \in I\!N}$. Infinitely many of them coincide since $\mathcal{U}$ is finite. Thus, there is $U_0 \in \mathcal{U}$ such that $y_m^{(1)}, y_m^{(2)} \in U_0$ for infinitely many m's and hence $y^{(1)}, y^{(2)} \in \overline{U_0}$. Similarly, for each k, infinitely many $U_m^{(k)}$ coincide and we obtain $U_k \in \mathcal{U}$ with $y^{(1)}, y^{(2)} \in T^{-k}(\overline{U_k})$. Therefore $y^{(1)}, y^{(2)} \in \cap_{k=0}^{\infty} T^{-k}(\overline{U_k}) \cap Y_x$, hence contradicting the fact that $\mathcal{U}$ is a fibred generator.

To prove the converse, let $N \in \mathbb{N}$ and $\delta > 0$ be a Lebesgue number for $\mathcal{U}$. Choose $0 < \epsilon < 1/N$ such that $d_Y(y_1, y_2) < \epsilon$ implies that $d_{N+1,Y}(y_1, y_2) < \delta$. If $y_1, y_2 \in Y$ are such that $d_Y(y_1, y_2) < \epsilon$, then for every $0 \leq n < N+1$, there is some $U_n \in \mathcal{U}$ with $T^n(y_1), T^n(y_2) \in U_n$. Therefore $y_1, y_2 \in \cap_{n=0}^{N} T^{-n}(U_n) \cap V_{1/N}(Y_{\pi(y_i)})$ for $i = 1, 2$. □

This shows that one can generate the topology of Y by projecting a fibred generator on neighbourhoods of the fibres of the system upon observation.

Keynes and Robertson also showed that expansive homeomorphisms are factors of subshifts of finite shifts. This family of subshifts therefore plays a special rôle in the theory of dynamical systems.

We now obtain an analogue for fibred systems. The following result states that a system $\mathcal{Y} = (Y, T, X, S, \pi)$ which has a fibred generator (or, in other words, that is fibrewise expansive) is a factor of a subsystem of a skew-product. This skew-product consists of the dynamical system (X, S) that forms the base of $\mathcal{Y}$ and of a full finite shift space.

Theorem 1.12 *Suppose that a compact fibred system (respectively a compact invertible fibred system) $\mathcal{Y} = (Y, T, X, S, \pi)$ has a fibred generator. Then $\mathcal{Y}$ is a factor of a closed subsystem W of the skew-product $(X \times \{1, \ldots, d\}^I, S \times \sigma, X, S, \pi_X)$, where I is equal to $\mathbb{Z}_+$ (respectively $\mathbb{Z}$), $\{1, \ldots, d\}^I$ is the one-sided (respectively two-sided) full d-shift space, σ is the one-sided (respectively two-sided) left shift map and π_X is the projection on X.*

Proof. We will here give a proof for the invertible case. The non-invertible situation can be treated in a similar manner.

Let $\mathcal{U} = \{U_1, \ldots, U_d\}$ be a two-sided fibred generator for the compact invertible system $\mathcal{Y}$. Let $x \in X$ and define

$$Z_x := \left\{ z \in \mathcal{A}^{\mathbb{Z}} \,\middle|\, \exists\, (U_{z_i})_{i \in \mathbb{Z}}, U_{z_i} \in \mathcal{U} \text{ such that } \cap_{i=-\infty}^{\infty} T^{-i}(\overline{U_{z_i}}) \bigcap Y_x \neq \emptyset \right\},$$

where $\mathcal{A} = \{1, \ldots, d\}$. Note that Z_x is not necessarily invariant, although $\sigma(Z_x) = Z_{S(x)}$. Define also

$$W := \bigcup_{x \in X} \{x\} \times Z_x$$

and notice that $(S \times \sigma)(W) = W$. Hence W is an invertible subsystem of the invertible skew-product $(X \times \{1, \ldots, d\}^{\mathbb{Z}}, S \times \sigma, X, S, \pi_X)$.

Now let $\phi : W \to Y$ be defined by

$$\phi(x, z) = y \text{ if } y \in \cap_{i=-\infty}^{\infty} T^{-i}(\overline{U_{z_i}}) \bigcap Y_x.$$

One can easily see that ϕ is well-defined, is surjective and, if ψ is the identity map on X, then $\pi \circ \phi = \psi \circ \pi_X$ and $T \circ \phi = \phi \circ (S \times \sigma)$.

We now want to show that W is closed and ϕ continuous. For this, suppose $((x^j, z^j))_{j\in\mathbb{N}}$ is a sequence in W converging to a point (x, z). For each $i \in \mathbb{Z}$, we have $\lim_{n\to\infty} z_i^j = z_i$, and hence there exists $j_i \in \mathbb{N}$ such that $z_i^j = z_i$ for all $j \geq j_i$. Thus, $\phi(x^j, z^j) \in T^{-i}(\overline{U_{z_i}})$ for all $j \geq j_i$. Since $T^{-i}(\overline{U_{z_i}})$ is compact, by considering a subsequence if necessary, we may assume that $\lim_{j\to\infty} \phi(x^j, z^j) = y$, where $y \in T^{-i}(\overline{U_{z_i}})$ for every $i \in \mathbb{Z}$. Thus, $y \in \cap_{i=-\infty}^{\infty} T^{-i}(\overline{U_{z_i}}) \neq \emptyset$. Furthermore, as $\pi(\phi(x^j, z^j)) = x^j$ for all j, the continuity of π implies that

$$\pi(y) = \pi\Big(\lim_{j\to\infty} \phi(x^j, z^j)\Big) = \lim_{j\to\infty} \pi(\phi(x^j, z^j)) = \lim_{j\to\infty} x^j = x.$$

Hence, $y \in \cap_{i=-\infty}^{\infty} T^{-i}(\overline{U_{z_i}}) \cap Y_x \neq \emptyset$ and $\phi(x, z) = y$. Therefore $(x, z) \in W$ and W is closed.

Remark also that every convergent subsequence of $(\phi(x^j, z^j))_{j\in\mathbb{N}}$ must converge to y, as otherwise $\cap_{i=-\infty}^{\infty} T^{-i}(\overline{U_{z_i}}) \cap Y_x$ would contain more than one point. We conclude that $\lim_{j\to\infty} \phi(x^j, z^j) = \phi(x, z)$, so that ϕ is continuous.

This shows that $\mathcal{Y}$ is a factor of W. □

Corollary 1.13 *Suppose that a compact fibred system (respectively a compact invertible fibred system) $\mathcal{Y} = (Y, T, X, S, \pi)$, with Y a zero-dimensional space, has a fibred generator. Then $\mathcal{Y}$ is embedded in a skew-product of the form $(X \times \{1, \ldots, d\}^I, S \times \sigma, X, S, \pi_X)$, where I is equal to $\mathbb{Z}_+$ (respectively $\mathbb{Z}$), $\{1, \ldots, d\}^I$ is the one-sided (respectively two-sided) full d-shift space, σ is the one-sided (respectively two-sided) left shift map and π_X is the natural projection on X.*

Proof. Since Y is zero-dimensional, we can choose a clopen (that is, closed-open) generator $\mathcal{V}$. Let $\mathcal{U}$ be the partition generated by $\mathcal{V}$. Then $\mathcal{U}$ is a clopen generator too. Now apply Theorem 1.12 with the generator $\mathcal{U}$ and remark that the map ϕ is injective due to the $\mathcal{U}$-partitioning of the space. Hence ϕ is a continuous bijective map from a compact metric space onto another one. Therefore ϕ is a homeomorphism and (ϕ, ψ) is an isomorphism from an invertible subsystem $(X \times \{1, \ldots, d\}^I, S \times \sigma, X, S, \pi_X)$ onto $\mathcal{Y} = (Y, T, X, S, \pi)$. This completes the proof. □

The foregoing two results show that skew-products play a particular rôle in the theory of fibred systems. In particular, they show that a fibrewise expansive system $\mathcal{Y} = (Y, T, X, S, \pi)$ is a factor of a subsystem of the skew-product of its base (X, S) with a full finite shift space. This family of skew-products plays in the theory of fibred systems the rôle performed by the shift spaces in the theory of dynamical systems.

1.5 Fibrewise Expanding Systems

We now define a stronger notion of expansion along the fibres of a system.

Definition 1.14 *A fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is said to be (uniformly) fibrewise expanding if there exist constants $a > 0$ and $\Lambda > 1$ such that the following holds:*
If $y_1, y_2 \in Y, \pi(y_1) = \pi(y_2)$ and $d_Y(y_1, y_2) < 2a$, then

$$d_Y\Big(T(y_1), T(y_2)\Big) \geq \Lambda\, d_Y(y_1, y_2).$$

Such a system is also referred to as being fibrewise (a, Λ)-expanding under the metric d_Y.

Evidently, an expanding fibred system (under some metric) is fibrewise expanding (with respect to that same metric) but the converse is obviously false.

It is also easy to see that a fibrewise expanding system is fibrewise uniformly expansive. We will show in the next chapter that these two notions coincide, up to a change of metric, for a huge class of fibred systems.

1.6 Fibrewise Ruelle Expanding Systems

In 1978, D. Ruelle [19] introduced a notion of expansion that allows the recovery of the ergodic properties of expanding endomorphisms of differentiable manifolds. In 1999, M. Denker and M. Gordin [8] initiated the investigation of systems that present such a behavior along their fibres.

Definition 1.15 *A fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is called (uniformly) fibrewise Ruelle expanding if there exist constants $a > 0$ and $\Lambda > 1$ such that the following condition holds:*
If $y_1, y_2' \in Y$, $\pi(T(y_1)) = \pi(y_2')$ and $d_Y(T(y_1), y_2') < 2a$, then there exists a unique $y_2 \in Y$ such that $T(y_2) = y_2', \pi(y_2) = \pi(y_1)$ and $d_Y(y_1, y_2) < 2a$. Furthermore,

$$d_Y\Big(T(y_1), T(y_2)\Big) \geq \Lambda\, d_Y(y_1, y_2).$$

Such a system is called fibrewise Ruelle (a, Λ)-expanding under the metric d_Y.

In contradistinction to the notions of expansivity and expandability, it is not true, in general, that a Ruelle expanding fibred system is fibrewise Ruelle expanding. We will later give some examples of such situations.

Evidently, a fibrewise Ruelle (a, Λ)-expanding system is fibrewise (a, Λ)-expanding. We will show in the next chapter that the converse does not hold and will find general conditions under which it does.

1.7 Fibrewise Openness

We now review some concepts of topology and introduce new notions that will play a preponderant rôle in our work.

Let Y and Y' be topological spaces and $T : Y \to Y'$ a continuous map from Y into Y'. The function T is said to be open if it maps every open set U of Y to an open set $T(U)$ of Y'. When Y and Y' are both metric spaces, the openness of T can be rephrased as follows. A function $T : Y \to Y'$ from a metric space Y into a metric space Y' is open if for every $y \in Y$ and $\delta > 0$, there exists $\epsilon_y(\delta) > 0$ such that $T(B_Y(y, \delta)) \supseteq B_{Y'}(T(y), \epsilon_y(\delta))$.

If, in addition, one can choose the $\epsilon_y(\delta)$'s in such a way that $\inf_{y \in Y} \epsilon_y(\delta) > 0$ for every $\delta > 0$, then one says that T is uniformly open. In other words, a function $T : Y \to Y'$ is uniformly open if for every $\delta > 0$ there exists $\epsilon(\delta) > 0$ such that $T(B_Y(y, \delta)) \supseteq B_{Y'}(T(y), \epsilon(\delta))$ for all $y \in Y$.

Evidently, a uniformly open map is open. The converse is not true though, but the two notions coincide whenever the domain of T, namely Y, is compact.

Lemma 1.16 *Let Y and Y' be metric spaces, Y being compact. Then any continuous open map $T : Y \to Y'$ is uniformly open.*

Proof. Suppose this is not the case. Then there exist $\delta > 0$ and a sequence $(y_n)_{n \in I\!N}$ in Y such that $T(B_Y(y_n, \delta)) \not\supseteq B_{Y'}(T(y_n), 1/n)$.

Without loss of generality, we may assume that this sequence converges to, say, y. The continuity of T then ensures that the sequence $(T(y_n))_{n \in I\!N}$ converges to $T(y)$ in Y'.

Let $\epsilon > 0$ be such that $T(B_Y(y, \delta/2)) \supseteq B_{Y'}(T(y), \epsilon)$. Let also $N \geq 2/\epsilon$ be such that $d_Y(y_n, y) < \delta/2$ and $d_{Y'}\big(T(y_n), T(y)\big) < \epsilon/2$ for all $n \geq N$.

Then

$$T\Big(B_Y(y_n, \delta)\Big) \supseteq T\Big(B_Y(y, \delta/2)\Big) \supseteq B_{Y'}\Big(T(y), \epsilon\Big) \supseteq B_{Y'}\Big(T(y_n), \epsilon/2\Big) \supseteq B_{Y'}\Big(T(y_n), 1/n\Big)$$

for all $n \geq N$. This contradicts the definition of the y_n's. □

We now define some notions of openness along the fibres of a system.

Definition 1.17 *A fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is called fibrewise open if the fibre map $T_x : Y_x \to Y_{S(x)}$ is open for every $x \in X$.*

The system $\mathcal{Y}$ is said to be fibrewise uniformly open if each fibre map T_x is uniformly open.

Clearly, a fibrewise uniformly open system is fibrewise open, but the converse does not hold in general.

The following is an immediate consequence of Lemma 1.16 and the fact that a compact fibred system has compact fibres.

Lemma 1.18 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibrewise open compact fibred system. Then $\mathcal{Y}$ is fibrewise uniformly open, that is, for each $x \in X$ and every $\delta > 0$, there exists $\epsilon_x(\delta) > 0$ such that for all $y \in Y_x$*

$$T\Big(B_Y(y,\delta) \cap Y_x\Big) \supseteq B_Y\Big(T(y), \epsilon_x(\delta)\Big) \cap Y_{S(x)}.$$

Unfortunately, it may happen that the infimum of the supremum of the constants $\{\epsilon_x(\delta)\}_{x\in X}$ that can be chosen be 0 for some δ. Indeed, consider the following example of a fibred system which should be regarded to as being embedded in $\mathbb{R}^2$. (All fibred systems presented here as examples should be looked upon in this way.)

Example 1.19 *Let $X = [0,1]$, $Y = \cup_{x\in X} Y_x$, where $Y_x = \{(x,0),(x,x)\}$, and $\pi : Y \to X$ with $\pi(x,y) = x$, the projection along the y-axis in $\mathbb{R}^2$.*

Define also $S : X \to X$ by

$$S(x) = \begin{cases} 2x & \text{if } x \le 1/2 \\ 2-2x & \text{if } x \ge 1/2 \end{cases}$$

and $T : Y \to Y$ by

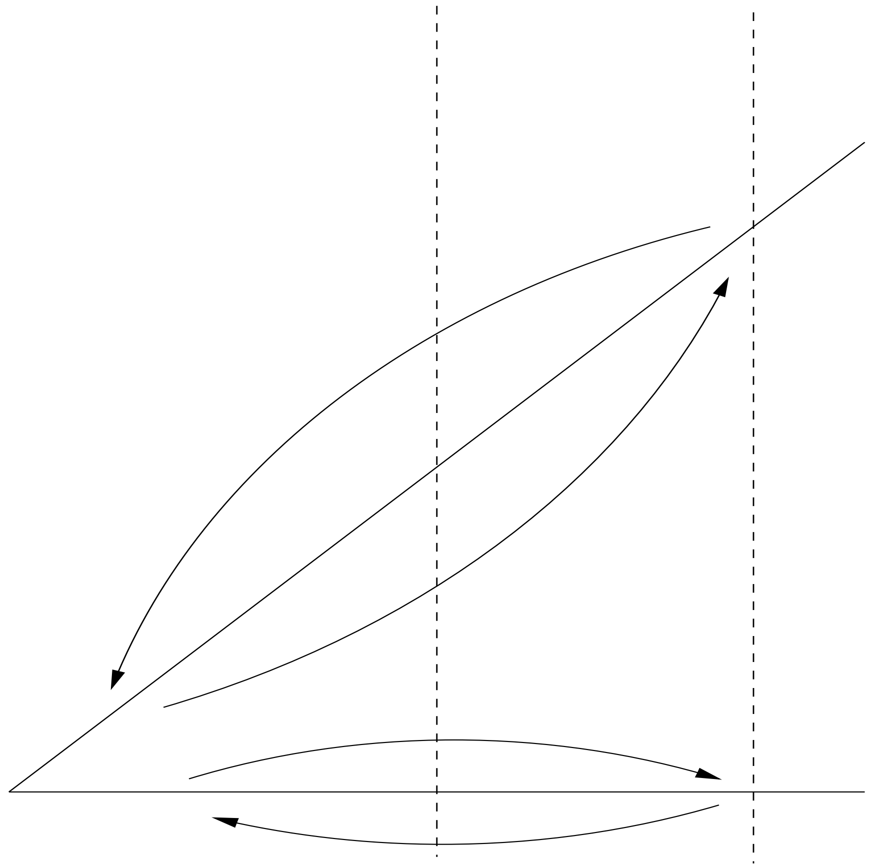
$$\begin{aligned} T(x,y) &= (2x, 2y) \quad \text{if } x \le 1/2 \\ T(x,0) &= (2-2x, 0) \quad \text{if } x \ge 1/2 \\ T(x,x) &= (2-2x, 2-2x) \quad \text{if } x \ge 1/2. \end{aligned}$$

Note that the fibres of the system coincide with the intersection of vertical lines (represented by dashed lines in the forthcoming figure) with Y.

We now verify that $\mathcal{Y}$ is a fibrewise open compact fibred system (and hence fibrewise uniformly open). We leave it to the reader to supply the arguments showing that $\mathcal{Y}$ really forms a compact fibred system and we concentrate on the fibrewise openness.

It is readily seen that $\mathcal{Y}$ is fibrewise open since the fibres are finite discrete subspaces. More precisely, given $\delta > 0$, we obtain that $\epsilon_x^{\sup}(\delta) := \sup \epsilon_x(\delta) = S(x)$ whenever $\delta \le x < 1$, whereas $\epsilon_x^{\sup}(\delta) = \infty$ whenever $x < \delta$ or $x = 1$. (Remark that this only happens when the image of a relative ball of radius δ covers the image fiber). Therefore, given $0 < \delta < 1$, we deduce that $\inf_{x\in X} \epsilon_x^{\sup}(\delta) = \inf_{\delta \le x < 1} S(x) = 0$.

Moreover, the reader may show that this system is fibrewise (a, Λ)-expanding with $1 < \Lambda \le 2$ and $0 < 2a \le 2/(\Lambda + 2)$. However, this system is not fibrewise Ruelle



expanding. Indeed, given $0 < a < 1/2$, choose $x > \max(2a, 1-a)$, $y_1 = (x,x)$ and $y_2' = (2-2x, 0)$. Then $\pi(T(y_1)) = 2-2x = \pi(y_2')$ and $d_Y(T(y_1), y_2') = 2-2x < 2a$. The only point $y_2 \in Y$ satisfying $T(y_2) = y_2' = (2-2x, 0)$ and $\pi(y_2) = \pi(y_1) = x$ is the point $y = (x, 0)$. But this point does not satisfy $d_Y(y_1, y_2) < 2a$, for $d_Y(y_1, y) = x > 2a$. On the other hand, if $a \geq 1/2$, then choose $x > 2/3$ and y_1, y_2' as above. Then $\pi(T(y_1)) = \pi(y_2')$ and $d_Y(T(y_1), y_2') < 2a$. This time there is a unique point $y_2 \in Y$ that satisfies $T(y_2) = y_2'$, $\pi(y_2) = \pi(y_1)$ and $d_Y(y_1, y_2) < 2a$. This point is $y_2 = (x, 0)$. Nevertheless, $d_Y(T(y_1), T(y_2)) = 2-2x < x = d_Y(y_1, y_2)$. This proves that the system is not fibrewise Ruelle expanding.

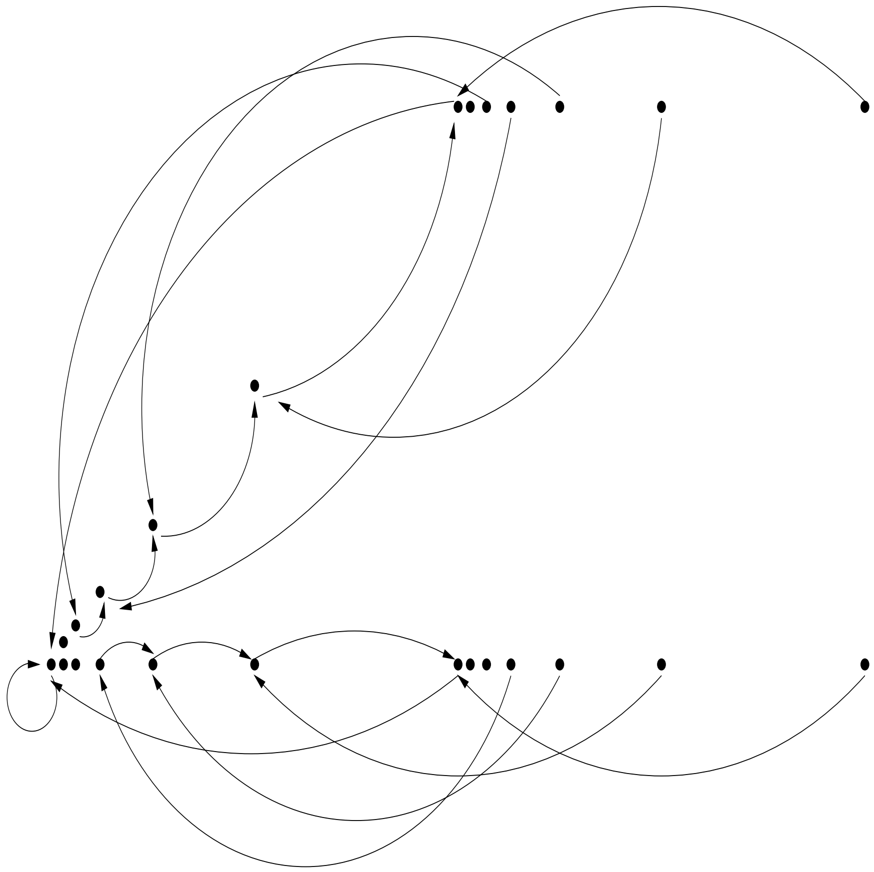
A similar study reveals that the following somehow similar system constitutes another example of a fibrewise uniformly open, fibrewise expanding, but not fibrewise Ruelle expanding, compact fibred system. As in the previous example, the fibres lie on vertical lines of $I\!R^2$.

Example 1.20 *Set*

$$X = \left\{1/2^s, 1 + 1/2^s\right\}_{s \in Z\!\!\!Z_+ \cup \{\infty\}}$$

and $Y = \cup_{x \in X} Y_x$*, where*

$$Y_x = \begin{cases} \{(x,0),(x,x)\} & \textit{if} \quad x \leq 1 \\ \{(x,0),(x,1)\} & \textit{if} \quad x \geq 1. \end{cases}$$



Once again, let $\pi : Y \to X$ be the projection along the y-axis. Define also $S : X \to X$ by

$$S(x) = \begin{cases} 2x & \text{if } x \leq 1/2 \\ x-1 & \text{if } x \geq 1 \end{cases}$$

and $T : Y \to Y$ by

$$\begin{array}{rcll} T(x,y) & = & (2x,2y) & \text{if } x \leq 1/2 \\ T(x,0) & = & (x-1,0) & \text{if } x \geq 1 \\ T(x,1) & = & (x-1,x-1) & \text{if } x \geq 1. \end{array}$$

The dynamics associated to this fibred system is depicted by the above graphical representation.

Hence the situation of openness may differ considerably from one fibre to another. Thus we feel the necessity of defining a notion of uniform fibrewise openness or, equivalently, uniform fibrewise uniform openness.

Definition 1.21 *Let $\mathcal{Y} = (Y,T,X,S,\pi)$ be a fibred system. $\mathcal{Y}$ is called uniformly fibrewise (uniformly) open if for every $\delta > 0$, there exists $\epsilon = \epsilon(\delta) > 0$ such that for all $y \in Y$*

$$T\Big(B_Y(y,\delta) \cap Y_{\pi(y)}\Big) \supseteq B_Y\Big(T(y),\epsilon\Big) \cap Y_{S(\pi(y))}.$$

Equivalently, $\mathcal{Y}$ is uniformly fibrewise open if it is fibrewise uniformly open and one can choose the $\epsilon_x(\delta)$'s (appearing in Lemma 1.18) in such a way that $\inf_{x\in X}\epsilon_x(\delta)>0$ for every $\delta>0$.

This means in particular that the systems considered in Examples 1.19 and 1.20 are not uniformly fibrewise open.

Theorem 1.22 *Fibrewise (uniform) openness and uniform fibrewise openness are fibrewise conjugacy invariants for the class of compact fibred systems, that is, if two compact fibred systems are isomorphic to each other, then one of them is fibrewise (uniformly) open (resp. uniformly fibrewise open) if and only if the other one is.*

Proof. We will give an argument for the preservation of uniform fibrewise openness. It can be proved in a similar way that fibrewise (uniform) openness is preserved under an isomorphism. Let $\mathcal{Y}=(Y,T,X,S,\pi)$ and $\mathcal{Y}'=(Y',T',X',S',\pi')$ be compact fibred systems that are fibrewise conjugate to each other. This means that there exist two homeomorphisms $\phi:Y\to Y'$ and $\psi:X\to X'$ such that $\phi\circ T=T'\circ\phi$ and $\psi\circ\pi=\pi'\circ\phi$ (recall that this implies $\psi\circ S=S'\circ\psi$). Assume also that $\mathcal{Y}$ is uniformly fibrewise open and let $\delta>0$.

As ϕ is a homeomorphism on a compact metric space, ϕ and ϕ^{-1} are uniformly open. Hence there exists $\zeta>0$ such that for all $y'\in Y'$

$$\phi^{-1}\Big(B_{Y'}(y',\delta)\Big)\supseteq B_Y\Big(\phi^{-1}(y'),\zeta\Big).$$

Consequently,

$$\begin{aligned}\phi^{-1}\Big(B_{Y'}(y',\delta)\cap Y'_{\pi'(y')}\Big) &= \phi^{-1}\Big(B_{Y'}(y',\delta)\Big)\cap\phi^{-1}(Y'_{\pi'(y')})\\ &= \phi^{-1}\Big(B_{Y'}(y',\delta)\Big)\cap Y_{\pi(\phi^{-1}(y'))}\\ &\supseteq B_Y\Big(\phi^{-1}(y'),\zeta\Big)\cap Y_{\pi(\phi^{-1}(y'))}.\end{aligned}$$

Since $\mathcal{Y}$ is uniformly fibrewise open, there exists $\xi>0$ such that for any $y'\in Y'$

$$\begin{aligned}T\Big(B_Y(\phi^{-1}(y'),\zeta)\cap Y_{\pi(\phi^{-1}(y'))}\Big) &\supseteq B_Y\Big(T(\phi^{-1}(y')),\xi\Big)\cap Y_{S(\pi(\phi^{-1}(y')))}\\ &= B_Y\Big(T(\phi^{-1}(y')),\xi\Big)\cap Y_{\pi(T(\phi^{-1}(y')))}.\end{aligned}$$

As ϕ is uniformly open, there also exists $\epsilon>0$ such that for all $y\in Y$

$$\phi\Big(B_Y(y,\xi)\Big)\supseteq B_{Y'}\Big(\phi(y),\epsilon\Big).$$

Therefore, for all $y'\in Y'$,

$$\begin{aligned}\phi\Big(B_Y(T(\phi^{-1}(y')),\xi)\cap Y_{\pi(T(\phi^{-1}(y')))}\Big) &= \phi\Big(B_Y(T(\phi^{-1}(y')),\xi)\Big)\cap\phi\Big(Y_{\pi(T(\phi^{-1}(y')))}\Big)\\ &\supseteq B_{Y'}\Big(\phi(T(\phi^{-1}(y'))),\epsilon\Big)\cap Y'_{\pi'(\phi(T(\phi^{-1}(y'))))}\\ &= B_{Y'}\Big(T'(y'),\epsilon\Big)\cap Y'_{\pi'(T'(y'))}\\ &= B_{Y'}\Big(T'(y'),\epsilon\Big)\cap Y'_{S'(\pi'(y'))}.\end{aligned}$$

It follows that for all $y' \in Y'$

$$\begin{aligned}
T'\Big(B_{Y'}(y',\delta) \cap Y'_{\pi'(y')}\Big) &= \phi\Big(T\Big(\phi^{-1}\Big(B_{Y'}(y',\delta) \cap Y'_{\pi'(y')}\Big)\Big)\Big) \\
&\supseteq \phi\Big(T\Big(B_Y(\phi^{-1}(y'),\zeta) \cap Y_{\pi(\phi^{-1}(y'))}\Big)\Big) \\
&\supseteq \phi\Big(B_Y(T(\phi^{-1}(y')),\xi) \cap Y_{\pi(T(\phi^{-1}(y')))}\Big) \\
&\supseteq B_{Y'}\Big(T'(y'),\epsilon\Big) \cap Y'_{S'(\pi'(y'))}.
\end{aligned}$$

This completes the proof. □

Thus, these notions are, within the family of compact fibred systems, independent of the metric considered (as long as the metric is compatible with the original topology).

Remark that the assumption of compactness of the system is crucial. This notion generally depends on the metric the space is equipped with in the non-compact case.

1.8 The Uniform FULC Property

We proceed on with our revision of topological concepts and introduction of new notions that will later play an important rôle.

A topological space Y is said to be locally connected at a point y of Y if y has a neighbourhood base consisting of connected sets or, in other words, a connected neighbourhood base. The space Y is locally connected if it is locally connected at each of its points. Note that a metric space Y is locally connected at y if and only if for every $\epsilon > 0$, there is a $\delta = \delta(y, \epsilon) > 0$ such that any point $y' \in Y$ with $d_Y(y', y) < \delta$ lies together with y in a connected set of diameter less than ϵ.

The concept of local path-connectedness is also well-known. A space Y is called locally path-connected at a point y of Y if y admits a path-connected neighbourhood base. It is locally path-connected if it is locally path-connected at each of its points. For metric spaces, this definition can be rephrased as follows. A metric space Y is locally path-connected at a point y if for every $\epsilon > 0$, there is a $\delta = \delta(y, \epsilon) > 0$ such that any point $y' \in Y$ with $d_Y(y', y) < \delta$ can be joined to y via a path of diameter less than ϵ.

A frequently used abbreviation for the phrase *locally connected* are the letters 'lc', whereas the letters 'lpc' are often used to refer to the phrase *locally path-connected.*

Evidently, a lpc space is lc. Though the converse is not true, a lc compact metric space is lpc and, perhaps most surprisingly, even more. We will come back to this nebulous statement in a moment.

These two concepts of 'local connectivity' can be readily altered into uniform notions analogous to uniform continuity, uniform openness or any other similar notion that arises in the theory of uniform spaces. One says that a metric space Y is uniformly locally connected (abbreviated 'ulc') if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that any pair of points y, y' of Y with $d_Y(y, y') < \delta$ lie together in a connected subset of Y of diameter less than ϵ.

Obviously, a ulc space is lc. Though the converse does not hold in general, a lc compact metric space is ulc.

In analogy to the definition of ulc spaces, one says that a metric space Y is uniformly locally path-connected (abbreviated 'ulpc') provided that for every $\epsilon > 0$, there exists a $\delta > 0$ such that any pair of points y, y' of Y with $d_Y(y, y') < \delta$ can be joined together by means of a path of diameter less than ϵ.

A ulpc space is indubitably a lpc space. Furthermore, a lpc compact metric space is a ulpc space.

On the other hand, a ulpc space is ulc. The converse is certainly not true in general, but a ulc compact metric space is ulpc. In fact, a lc compact metric space is ulpc.

Theorem 1.23 *Let Y be a compact metric space. If Y is locally connected, then it is uniformly locally path-connected.*

Proof. For convenience, we reproduce the proof (and correct a small mistake) that appears in Aoki and Hiraide [1].

Let $\epsilon > 0$ be given. Since Y is assumed to be a locally connected compact metric space, it is uniformly locally connected. This implies that there exists a sequence of numbers $(\delta_n)_{n\in\mathbb{N}}$, $\delta_n > 0$, such that any two points $y_1, y_2 \in Y$ with $d_Y(y_1, y_2) < \delta_n$ lie together in a connected set of diameter less than $\epsilon/2^n$.

We show that any two points $y(0), y(1) \in Y$ with $d_Y(y(0), y(1)) < \delta_0$ can be joined by means of a path of diameter at most 4ϵ.

To do this we choose a sequence of denominators $1 = k_0 < k_1 < k_2 < \ldots$, each of which divides the next. For each fraction of the form i/k_n between 0 and 1 we choose an intermediate point $y(i/k_n)$ satisfying two conditions:

(1) Any two consecutive fractions i/k_n and $(i+1)/k_n$ correspond to points $y(i/k_n)$ and $y((i+1)/k_n)$ which have a distance less than δ_n.

Thus any two such points are contained in a connected set $C(i, k_n)$ which has diameter less than $\epsilon/2^n$. The second condition is the following:

(2) Each point of the form $y(j/k_{n+1})$, where j/k_{n+1} lies between i/k_n and $(i+1)/k_n$, belongs to the set $C(i, k_n)$, and thus

$$d_Y\Big(y(j/k_{n+1}), y(i/k_n)\Big) < \epsilon/2^n \text{ and } d_Y\Big(y(j/k_{n+1}), y((i+1)/k_n)\Big) < \epsilon/2^n.$$

By induction on n the construction of such denominators k_n and intermediate points $y(i/k_n)$ is straightforward. Thus, we may suppose that $y(r)$ has been defined for a dense set of rational numbers r in $[0, 1]$.

Next we prove that the correspondence $r \mapsto y(r)$ is uniformly continuous. Let r and r' be any two rational numbers such that $y(r)$ and $y(r')$ are defined. Suppose that $|r - r'| < 1/(2k_n)$. We can hence choose i/k_n as close as possible to both r and r'. Then we have

$$d_Y\Big(y(r), y(i/k_n)\Big) < 2\epsilon/2^n \text{ and } d_Y\Big(y(r'), y(i/k_n)\Big) < 2\epsilon/2^n,$$

and so

$$d_Y\Big(y(r), y(r')\Big) < 4\epsilon/2^n.$$

This proves uniform continuity. Consequently, there is a unique continuous extension $t \mapsto y(t)$ which is defined for all $t \in [0,1]$. We have constructed the required path of diameter at most 4ϵ from $y(0)$ to $y(1)$. Therefore Y is uniformly locally path-connected. □

We may now go on with the definition of new concepts that specifically pertain to the space structure of fibred systems. In a similar manner as one can uniformize a given property on a space, one can 'fibrewise' it over a fibred space.

Definition 1.24 *A fibred space (Y, X, π) is said to be fibrewise locally connected (abbreviated 'flc') if each of its fibres is locally connected (lc).*

Similarly, the space (Y, X, π) is called fibrewise locally path-connected (abbreviated 'flpc') if each of its fibres is locally path-connected (lpc).

Clearly, a flpc space is flc. From what we said before, it is not really astonishing that a flc compact fibred space be flpc. In fact, it is even fulpc. But, for this to make sense, we must establish the following definition.

Definition 1.25 *A fibred space (Y, X, π) is called fibrewise uniformly locally connected (resp. fibrewise uniformly locally path-connected) (abbreviated 'fulc', resp. 'fulpc') if each fibre of the space is uniformly locally connected (ulc) (resp. uniformly locally path-connected (ulpc)).*

Similar remarks as for the other properties so far presented can be made about fulc and fulpc spaces. A fulpc space is fulc and a fulc compact space is fulpc.

Both definitions can also be restated in the following way. A fibred space (Y, X, π) is fulc provided that for every $x \in X$ and $\epsilon > 0$, there is $\delta_x(\epsilon) > 0$ such that any pair of points $y_1, y_2 \in Y_x$ with $d_Y(y_1, y_2) < \delta_x(\epsilon)$ lies in a connected subset of Y_x whose diameter is less than ϵ.

Similarly, a fibred space (Y, X, π) is fulpc if for every $x \in X$ and $\epsilon > 0$, there exists a $\delta_x(\epsilon) > 0$ such that any pair of points $y_1, y_2 \in Y_x$ with $d_Y(y_1, y_2) < \delta_x(\epsilon)$ can be joined via a path in Y_x of diameter less than ϵ.

Finally, the fibres of a space may even have a stronger topological structure.

Definition 1.26 *A fibred space (Y, X, π) is referred to as being uniformly fulc (resp. uniformly fulpc) if for every $\epsilon > 0$, there is a $\delta > 0$ such that any two points $y_1, y_2 \in Y$ with $\pi(y_1) = \pi(y_2)$ and $d_Y(y_1, y_2) < \delta$ lie together in a connected subset of (resp. can be joined via a path in) $Y_{\pi(y_i)}$ of diameter less than ϵ.*

Equivalently, a space is uniformly fulc (resp. uniformly fulpc) if it is fulc (resp. fulpc) and for all $\epsilon > 0$, one can choose the $\delta_x(\epsilon)$'s (that appeared previously) in such a way that $\inf_{x\in X} \delta_x(\epsilon) > 0$.

It is unequivocal that a uniformly fulpc space is uniformly fulc and that a uniformly fulc compact space is uniformly fulpc. As this fact will later have some importance, we state it as a theorem.

Theorem 1.27 *Let (Y, X, π) be a compact fibred space. If (Y, X, π) is uniformly fulc, then it is uniformly fulpc.*

Proof. The watershed of the proof of Theorem 1.23 is that whenever two points are close enough, one can join them via a path whose diameter is small enough and independent of the two points considered. This path is constructed via the existence of sufficiently uniformly small connected sets. The theorem is an immediate corollary to that proof. $\square$

A uniformly fulc (resp. uniformly fulpc) space is undoubtedly fulc (resp. fulpc). However, the converse does not hold in general. There are some fulpc spaces which are not uniformly fulc.

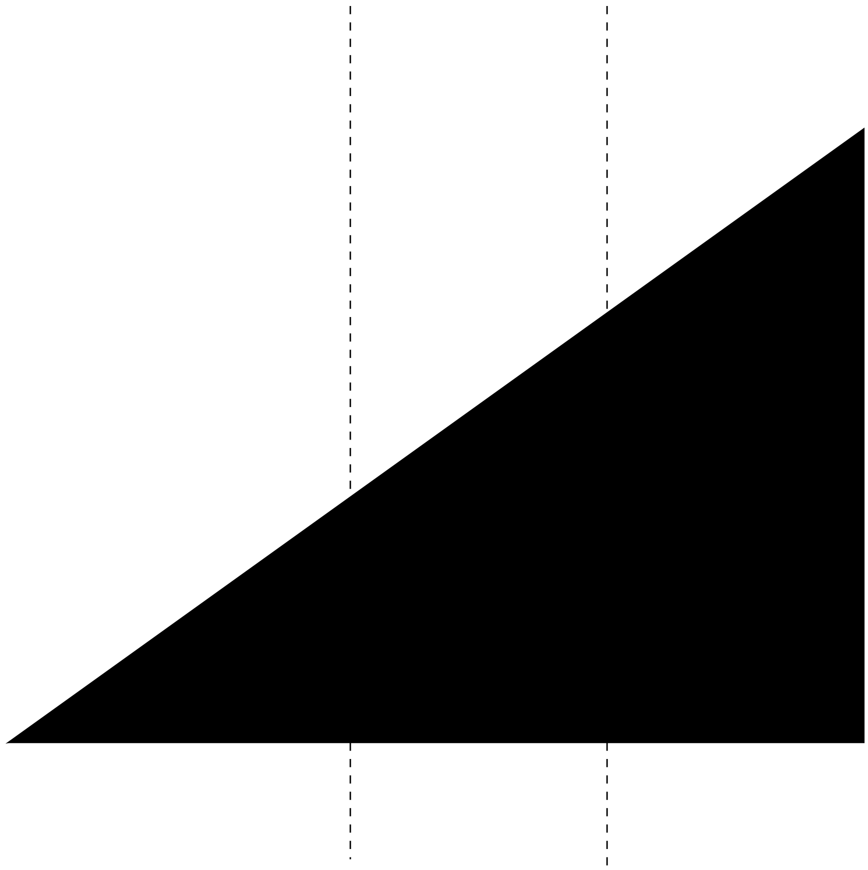
Example 1.28 *Let $X = [0, 1]$, $Y = \cup_{x\in X} Y_x$, where $Y_x = \{(x, 0), (x, x)\}$ and define $\pi : Y \to X$ by $\pi(x, y) = x$, the projection along the y-axis in $I\!R^2$. Thus the fibres of the space coincide with the non-empty intersections of the vertical lines of $I\!R^2$ with Y.*

The reader may seem familiar with this fibred space. It is indeed the underlying space structure of the fibred system considered in Example 1.19.

This space is obviously fulpc. Furthermore, for every $0 < \epsilon < 1$ and each $0 < x < \epsilon$, the supremum of the $\delta_x(\epsilon)$'s that can be chosen is $\delta_x^{\sup}(\epsilon) = x$. Hence $\inf_{x\in X} \delta_x^{\sup}(\epsilon) \leq \inf_{0<x<\epsilon} x = 0$ and this space is not uniformly fulpc.

But if each of the fibres of this space is 'filled in', then the resulting space is uniformly fulpc.

Example 1.29 *Let $X = [0, 1]$, $Y = \{(x, y) \in I\!R^2 \,|\, x \in X, y \in [0, x]\}$ or, more explicitly, $Y = \cup_{x\in X} Y_x$, where $Y_x = \{(x, y) \in I\!R^2 \,|\, y \in [0, x]\}$. As usual, $\pi : Y \to X$ is the projection along the y-axis in $I\!R^2$. Thus the fibres of the space are, as before, the non-empty intersections of the vertical lines of $I\!R^2$ with Y (see figure at the top of the next page).*



Again, it is clear that this space is fulpc. Moreover, the reader may verify that, given $\epsilon > 0$, one can choose $\delta_x(\epsilon) = \epsilon$ for every $x \in X$. Hence $\inf_{x\in X} \delta_x(\epsilon) = \epsilon$. Therefore (Y, X, π) is a uniformly fulpc space.

There are some important cases in which a fulc (resp. fulpc) space is uniformly fulc (resp. uniformly fulpc). To describe one of those situations, let us define an equivalence relation on the fibres of the system (or, equivalently, on X), saying that two fibres Y_{x_1} and Y_{x_2} are *identical* if there is an isometry between them. This equivalence relation generates a partition of the space X into equivalence classes. We then have the following result.

Theorem 1.30 *Let (Y, X, π) be a fibred space for which the corresponding set $X/\!\sim$ is finite. Then (Y, X, π) is uniformly fulc (resp. uniformly fulpc) if and only if it is fulc (resp. fulpc).*

In particular, a skew-product space $(Y = X \times Z, X, \pi)$ is uniformly fulc (resp. uniformly fulpc) if and only if Z is ulc (resp. ulpc).

Proof. We give a proof for the fulc case and leave to the reader the task of convincing himself that a similar argument applies to the fulpc situation.

Let $[x]$ denote the equivalence class of x with respect to the equivalence relation $\sim$. By hypothesis, the fibre Y_x is ulc. Hence, given $\epsilon > 0$, there exists $\delta_x(\epsilon) > 0$ such that any pair of points $y_1, y_2 \in Y_x$ with $d_Y(y_1, y_2) < \delta_x(\epsilon)$ lies in a connected subset of Y_x whose diameter is less than ϵ. Let $x' \in X$ be such that $[x'] = [x]$, that is, the fibres $Y_{x'}$ and Y_x are identical. Let also $y'_1, y'_2 \in Y_{x'}$ be such that $d_Y(y'_1, y'_2) < \delta_x(\epsilon)$. Since $Y_{x'}$ and Y_x are identical, there is an isometry $I_{x',x} : Y_{x'} \to Y_x$ between them. This implies that $d_Y(I_{x',x}(y'_1), I_{x',x}(y'_2)) = d_Y(y'_1, y'_2) < \delta_x(\epsilon)$. Therefore there exists a connected subset C of Y_x of diameter less than ϵ that contains both $I_{x',x}(y'_1)$ and $I_{x',x}(y'_2)$. It follows that $I^{-1}_{x',x}(C)$ is a connected subset of $Y_{x'}$ of diameter less than ϵ that contains y'_1 and y'_2. One hence obtains that one can choose $\delta_{x'}(\epsilon) = \delta_x(\epsilon)$. This means that the $\delta_.(\epsilon)$'s can be chosen so that they are constant on each equivalence class. One can therewith define a map $\tilde{\delta}_.(\epsilon)$ from $X/\sim$ into $\mathbb{R}^*_+$. If $X/\sim$ is finite, then the minimum of $\tilde{\delta}_.(\epsilon)$ is a strictly positive number and by taking δ to be this minimum, it follows immediately that the space is uniformly fulc.

The second part of the statement follows from the simple observation that $X/\sim$ reduces to one point whenever (Y, X, π) is a skew-product space. □

Remarks.

(1) In the compact case, the theorem can be rephrased as follows. Let (Y, X, π) be a compact fibred space for which the corresponding set $X/\sim$ is finite. Then (Y, X, π) is uniformly fulc (or, equivalently, uniformly fulpc) if and only if it is flc (or, equivalently, flpc). In particular, a compact skew-product space $(Y = X \times Z, X, \pi)$ is uniformly flc if and only if Z is lc.

(2) The result does not generally hold when $X/\sim$ is infinite. Examples 1.28 and 1.29 prove it.

(3) One could define a weaker relation on the fibres of a space by requiring that two fibres be equivalent if there exists a bi-Lipshitz map between them. One may then ask whether the previous theorem holds or not. The answer to this question must be negative. For, if it were holding, the spaces considered in Examples 1.28 and 1.29, that consist of only two such equivalence classes, would be uniformly fulc. But we know this is not the case.

Finally, we show an invariance theorem.

Theorem 1.31 *Fulc-ness and uniform fulc-ness are isomorphism invariants for the category of compact fibred spaces, that is, if two compact fibred spaces are isomorphic to each other, then one of them has one of the above cited properties if and only if the other one has it too.*

Proof. We give a proof of the invariance of uniform fulc-ness and leave it to the reader to persuade himself that fulc-ness is, under these circumstances, invariant under an isomorphism too.

Let (Y, X, π) and (Y', X', π') be compact fibred spaces that are isomorphic to each other. This means that there exist two homeomorphisms $\phi : Y \to Y'$ and $\psi : X \to X'$ such that $\psi \circ \pi = \pi' \circ \phi$ (recall that this implies that the fibers of the two spaces are homeomorphic in pairs and that they are in a one-to-one correspondence). Assume also that $\mathcal{Y}$ is uniformly fulc and let $\epsilon > 0$.

As ϕ is a homeomorphism on a compact metric space, ϕ and ϕ^{-1} are uniformly open. Hence there exists $\delta > 0$ such that for all $y' \in Y'$

$$\phi^{-1}\Big(B_{Y'}(y', \epsilon)\Big) \supseteq B_Y\Big(\phi^{-1}(y'), \delta\Big).$$

Consequently

$$\begin{aligned}
\phi^{-1}\Big(B_{Y'}(y', \epsilon) \cap Y'_{\pi'(y')}\Big) &= \phi^{-1}\Big(B_{Y'}(y', \epsilon)\Big) \cap \phi^{-1}(Y'_{\pi'(y')}) \\
&= \phi^{-1}\Big(B_{Y'}(y', \epsilon)\Big) \cap Y_{\pi(\phi^{-1}(y'))} \\
&\supseteq B_Y\Big(\phi^{-1}(y'), \delta\Big) \cap Y_{\pi(\phi^{-1}(y'))}.
\end{aligned}$$

Since $\mathcal{Y}$ is uniformly fulc, there exist $\xi > 0$ and a relative connected neighbourhood $V(\phi^{-1}(y'))$ of $\phi^{-1}(y')$ in the fibre $Y_{\pi(\phi^{-1}(y'))}$ that satisfies

$$\begin{aligned}
B_Y\Big(\phi^{-1}(y'), \xi\Big) \cap Y_{\pi(\phi^{-1}(y'))} &\subseteq V(\phi^{-1}(y')) \\
&\subseteq B_Y\Big(\phi^{-1}(y'), \delta\Big) \cap Y_{\pi(\phi^{-1}(y'))} \\
&\subseteq \phi^{-1}\Big(B_{Y'}(y', \epsilon) \cap Y'_{\pi'(y')}\Big).
\end{aligned}$$

As ϕ is uniformly open, there also exists $\zeta > 0$ such that for all $y \in Y$

$$\phi\Big(B_Y(y, \xi)\Big) \supseteq B_{Y'}\Big(\phi(y), \zeta\Big).$$

It follows that

$$\begin{aligned}
B_{Y'}(y', \zeta) \cap Y'_{\pi'(y')} &= B_{Y'}\Big(\phi(\phi^{-1}(y')), \zeta\Big) \cap Y'_{\pi'(\phi(\phi^{-1}(y')))} \\
&\subseteq \phi\Big(B_Y(\phi^{-1}(y'), \xi) \cap Y_{\pi(\phi^{-1}(y'))}\Big) \\
&\subseteq \phi(V(\phi^{-1}(y'))) \\
&\subseteq \phi\Big(\phi^{-1}\Big(B_{Y'}(y', \epsilon) \cap Y'_{\pi'(y')}\Big)\Big) \\
&= B_{Y'}(y', \epsilon) \cap Y'_{\pi'(y')}.
\end{aligned}$$

We have hence found a relative connected neighbourhood $\phi(V(\phi^{-1}(y')))$ of y' in the fibre $Y'_{\pi'(y')}$. $\square$

1.9 Fibrewise Topological Exactness

The concept of exactness is well-known in ergodic theory. There are few mathematicians, though, that are familiar with its topological counterpart.

Definition 1.32 *A topological dynamical system (Y, T) is said to be topologically exact if for every open set U of Y, there exists $M = M(U) \in \mathbb{Z}_+$ such that for every $m \geq M$*

$$T^m(U) = Y.$$

Remarks.

(1) Topological exactness of a system is a conjugacy invariant, in that it is preserved by any topological conjugacy.

(2) A topologically exact system is surjective.

(3) Since topological exactness implies surjectivity of the map, it is sufficient that the condition be satisfied for some m.

(4) Topological exactness can be characterized as follows.

A system is topologically exact if and only if there exists a base $\mathcal{B}$ for Y and for each member B of $\mathcal{B}$, there is $M = M(B) \in \mathbb{Z}_+$ such that for every $m \geq M$

$$T^m(B) = Y.$$

(5) The topological exactness of a compact metric dynamical system can also be defined as follows.

A compact metric dynamical system is topologically exact if for every $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{Z}_+$ such that for every $y \in Y$ and $n \geq N$

$$T^n\Big(B_Y(y, \epsilon)\Big) = Y.$$

Note again that it suffices that the latter equation hold for N for the condition to be fulfilled. Notice also that this characterization does not hold (in general) for a non-compact system. It may also depend on the metric in this latter case and is a stronger requirement than topological exactness.

A generalization of this notion was introduced by Denker and Gordin [8] for fibred systems. They defined it in the following way.

Definition 1.33 *A fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is topologically exact along fibres if for every $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{Z}_+$ such that for every $y \in Y$ and $n \geq N$*

$$T^n\Big(B_Y(y, \epsilon) \cap Y_{\pi(y)}\Big) = Y_{S^n(\pi(y))}.$$

Remarks.

(1) It is obvious that this notion reduces to topological exactness for systems that have only one fibre (that is, non-fibred systems or, in other words, metric dynamical systems).

(2) Topological exactness along fibres of a system does not necessarily imply that this system is fibrewise surjective (meaning that each fibre map $T_x : Y_x \to Y_{S(x)}$ is surjective). Nor does it imply that the system is surjective. But it does imply that there exists $N \in \mathbb{N}$ such that $\mathcal{Y}$ is fibrewise surjective on $\cup_{n \geq N} S^n(X)$. It further implies that the fibred systems $\mathcal{Y}^n = (Y, T^n, X, S^n, \pi)$ are fibrewise surjective for $n \geq N$. This means in particular that the requirement $n \geq N$ can make, by contradistinction with the non-fibred case, a difference. However, if S is surjective, then a system that is topologically exact along fibres is both surjective and fibrewise surjective.

Denker and Gordin [8] were interested in fibred systems that are topologically exact along fibres but did not study the property itself. It seems natural to extend to fibred systems the original definition of topological exactness in the way they did. But is this property metric independent? Or, even better, is it a fibrewise conjugacy invariant?

Theorem 1.34 *Topological exactness along fibres is a fibrewise conjugacy invariant for the class of compact systems, that is, if two compact fibred systems are isomorphic to each other, then one of them has this property if and only if the other one does.*

In particular, topological exactness along fibres of a compact system is independent of the metric considered as far as the metric is compatible with the topology.

Proof. Let $\mathcal{Y} = (Y, T, X, S, \pi)$ and $\mathcal{Y}' = (Y', T', X', S', \pi')$ be compact fibred systems that are fibrewise conjugate to each other. This means that there exist two homeomorphisms $\phi : Y \to Y'$ and $\psi : X \to X'$ such that $\phi \circ T = T' \circ \phi$ and $\psi \circ \pi = \pi' \circ \phi$ (recall that this implies $\psi \circ S = S' \circ \psi$). Assume that $\mathcal{Y}$ is topologically exact along fibres and let $\epsilon > 0$.

As ϕ is a homeomorphism on a compact metric space, ϕ and ϕ^{-1} are uniformly open. Hence there exists $\delta > 0$ such that for all $y' \in Y'$

$$\phi^{-1}\Big(B_{Y'}(y', \epsilon)\Big) \supseteq B_Y\Big(\phi^{-1}(y'), \delta\Big).$$

Consequently,

$$\begin{aligned}\phi^{-1}\Big(B_{Y'}(y',\epsilon)\cap Y'_{\pi'(y')}\Big) &= \phi^{-1}\Big(B_{Y'}(y',\epsilon)\Big)\cap\phi^{-1}(Y'_{\pi'(y')})\\ &= \phi^{-1}\Big(B_{Y'}(y',\epsilon)\Big)\cap Y_{\pi(\phi^{-1}(y'))}\\ &\supseteq B_Y\Big(\phi^{-1}(y'),\delta\Big)\cap Y_{\pi(\phi^{-1}(y'))}.\end{aligned}$$

Since $\mathcal{Y}$ is topologically exact along fibres, there exists $N\in\mathbb{Z}_+$ such that for all $y'\in Y'$

$$T^n\Big(B_Y(\phi^{-1}(y'),\delta)\cap Y_{\pi(\phi^{-1}(y'))}\Big)=Y_{S^n(\pi(\phi^{-1}(y')))}.$$

It follows that

$$T^n\Big(\phi^{-1}\Big(B_{Y'}(y',\epsilon)\cap Y'_{\pi'(y')}\Big)\Big)=Y_{S^n(\pi(\phi^{-1}(y')))}$$

and therefore

$$\begin{aligned}(T')^n\Big(B_{Y'}(y',\epsilon)\cap Y'_{\pi'(y')}\Big) &= \phi\Big(T^n\Big(\phi^{-1}\Big(B_{Y'}(y',\epsilon)\cap Y'_{\pi'(y')}\Big)\Big)\Big)\\ &= \phi\Big(Y_{S^n(\pi(\phi^{-1}(y')))}\Big)\\ &= \phi\Big(Y_{\pi(T^n(\phi^{-1}(y')))}\Big)\\ &= Y'_{\pi'(\phi(T^n(\phi^{-1}(y'))))}\\ &= Y'_{\pi'((T')^n(y'))}\\ &= Y'_{(S')^n(\pi'(y'))}.\end{aligned}$$

This completes the proof. □

Remark. The assumption of compactness of the system is crucial. Topological exactness along fibres may depend on the metric a non-compact system is endowed with.

Now, we would like to address the problem of defining a more general notion that we will call fibrewise topological exactness. We would like this notion to be expressed in terms of open sets only.

First, observe that topological exactness along fibres implies that for every open set U of Y and every $x\in X$ such that $U\cap Y_x\neq\emptyset$, there exists $N=N(U,x)\in\mathbb{Z}_+$ such that for every $n\geq N$

$$T^n(U\cap Y_x)=Y_{S^n(x)}.$$

However, one can see that there is not necessarily any uniformity over the fibres. This leads us to the following definition.

Definition 1.35 *A fibred system $\mathcal{Y}=(Y,T,X,S,\pi)$ is called (uniformly) fibrewise topologically exact if there exists a base $\mathcal{B}$ for Y and for each member B of $\mathcal{B}$, there is $M=M(B)\in\mathbb{Z}_+$ such that for every $m\geq M$ and every $x\in X$ for which $B\cap Y_x\neq\emptyset$*

$$T^m(B\cap Y_x)=Y_{S^m(x)}.$$

Remarks.

(1) Fibrewise topological exactness of a system does not necessarily imply that the system is fibrewise surjective. Nor does it imply that the system is surjective. But it does imply, when the system is compact, that there exists $N \in I\!N$ such that $\mathcal{Y}$ is fibrewise surjective on $\cup_{n\geq N} S^n(X)$ and that the fibred systems $\mathcal{Y}^n = (Y, T^n, X, S^n, \pi)$ are fibrewise surjective for $n \geq N$. Hence, if S is surjective, then a fibrewise topologically exact compact system is surjective and fibrewise surjective.

(2) Note that the requirement that a base of the space has a given property is much weaker than requiring that all open sets share this property.

(3) It is easy to see that this definition reduces to the usual definition of topological exactness in the non-fibred case.

We now establish various relations between the two notions of 'fibred exactness' previously defined.

On the one hand, we show that fibrewise topological exactness is a stronger requirement than topological exactness along fibres for the class of compact systems.

Proposition 1.36 *A fibrewise topologically exact compact system is topologically exact along fibres.*

Proof. Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibrewise topologically exact compact system. Let also $\epsilon > 0$ and $y \in Y$. Since $\mathcal{Y}$ is fibrewise topologically exact, there exist an open neighbourhood U_y of y with $B_Y(y, \delta_y) \subseteq U_y \subseteq B_Y(y, \epsilon/2)$ (for some $0 < \delta_y \leq \epsilon/2$) and $M_y \in Z\!\!\!Z_+$ such that for every $x \in X$ for which $U_y \cap Y_x \neq \emptyset$ and every $m \geq M_y$

$$T^m(U_y \cap Y_x) = Y_{S^m(x)}.$$

This implies that, for any point $\tilde{y} \in B_Y(y, \delta_y)$,

$$\emptyset \neq U_y \cap Y_{\pi(\tilde{y})} \subseteq B_Y(y, \epsilon/2) \cap Y_{\pi(\tilde{y})} \subseteq B_Y(\tilde{y}, \epsilon) \cap Y_{\pi(\tilde{y})}$$

and hence

$$T^m\Big(B_Y(\tilde{y}, \epsilon) \cap Y_{\pi(\tilde{y})}\Big) = Y_{S^m(\pi(\tilde{y}))}$$

for every point $\tilde{y} \in B_Y(y, \delta_y)$ and every $m \geq M_y$.

We carry on this procedure at each point $y \in Y$. As Y is compact, we can extract from the open cover $\{B_Y(y, \delta_y)\}_{y\in Y}$ a finite subcover $\{B_Y(y_i, \delta_{y_i})\}_{i=1}^k$. Choosing $M = \max_{1\leq i\leq k} M_{y_i}$, we conclude that for every $y \in Y$ and every $m \geq M$

$$T^m\Big(B_Y(y, \epsilon) \cap Y_{\pi(y)}\Big) = Y_{S^m(\pi(y))}.$$

This proves that $\mathcal{Y}$ is topologically exact along fibres. $\square$

On the other hand, we prove that topological exactness along fibres is stronger an exigency than fibrewise topological exactness when π is an open map.

Proposition 1.37 *A system $\mathcal{Y} = (Y, T, X, S, \pi)$ that is topologically exact along fibres and for which π is an open map is fibrewise topologically exact.*

Proof. Let $\epsilon > 0$ and $y \in Y$. Then the set

$$\begin{aligned} U(y, \epsilon) &:= \pi^{-1}\left[\pi\left(B_Y(y, \epsilon/2)\right)\right] \cap B_Y(y, \epsilon) \\ &= \left(\bigcup_{\tilde{y} \in B_Y(y, \epsilon/2)} Y_{\pi(\tilde{y})}\right) \cap B_Y(y, \epsilon) \\ &= \bigcup_{\tilde{y} \in B_Y(y, \epsilon/2)} \left(Y_{\pi(\tilde{y})} \cap B_Y(y, \epsilon)\right) \\ &\supseteq \bigcup_{\tilde{y} \in B_Y(y, \epsilon/2)} \left(B_Y(\tilde{y}, \epsilon/2) \cap Y_{\pi(\tilde{y})}\right) \end{aligned}$$

is an open neighbourhood of y, which is smaller than $B_Y(y, \epsilon)$ and contains an open ball of radius $\epsilon/2$ in each fibre that it intersects. Indeed, let $x \in X$ be such that $U(y, \epsilon) \cap Y_x \neq \emptyset$. Then $x \in \pi(B_Y(y, \epsilon/2))$. Now choose $y' \in B_Y(y, \epsilon/2)$ such that $\pi(y') = x$. Then

$$U(y, \epsilon) \cap Y_x \supseteq B_Y(y', \epsilon/2) \cap Y_{\pi(y')}.$$

The result follows directly from the definition of topological exactness along fibres. $\square$

Chapitre 2

On the Relations between Different Fibrewise Expansion Behaviors

2.1 Fibrewise Expansive vs. Fibrewise Expanding Systems

In 1980, E.M. Coven and W.L. Reddy [5] proved that a map of a compact manifold onto itself is positively expansive if and only if it is expanding in the sense of Ruelle with respect to some topologically equivalent metric. Their proof relied upon the fact that such a map is a covering map. Indeed, they made extensive use of the existence of a universal covering space to lift the map to a homeomorphism.

Reddy [16] later remarked that this result also applies to expansive mappings of compact metrizable spaces.

Finally, M. Denker and M. Urbański [6] gave a slightly different proof that appears, following our work, to be more suitable for generalization to (non necessarily compact) fibred systems.

Before presenting this generalization, we would like to remind the reader about Frink's Metrization Lemma, lemma on which all the approaches exploited so far (as well as ours) are based.

Lemma 2.1 *Let Y be a topological space. Suppose $(U_n)_{n\in\mathbb{Z}_+}$ to be a nested sequence of open symmetric neighbourhoods of the diagonal $\Delta = \Delta(Y)$ of $Y \times Y$ such that*

$$U_0 = Y \times Y,$$

$$\bigcap_{n\in\mathbb{Z}_+} U_n = \Delta$$

and, for all $n \in \mathbb{N}$,

$$U_n \circ U_n \circ U_n \subseteq U_{n-1},$$

where, given a set $W \subseteq Y \times Y$, one defines $W \circ W = \{(y_1, y_2) \in Y \times Y \,|\, \exists\, y \in Y$ with $(y_1, y), (y, y_2) \in W\}$.

Then there exists a compatible metric ρ for Y such that

$$U_n \subseteq \left\{(y_1, y_2) \in Y \times Y \,\middle|\, \rho(y_1, y_2) < 2^{-n}\right\} \subseteq U_{n-1}.$$

This lemma has important consequences in the scrutiny of problems that involve the presence of a topology on a system, via a metric the system is equipped with, but that do not depend on that metric. This is the case for concepts like expansiveness and fibrewise expansiveness, as well as for many other notions or questions that arise in topological dynamics (for instance, the notion of distality).

We now show that one may always consider, in such situations, that π is a strong contraction, that is, there exists $0 < \lambda < 1$ such that $d_X(\pi(y_1), \pi(y_2)) \leq \lambda\, d_Y(y_1, y_2)$ for all $y_1, y_2 \in Y$.

Lemma 2.2 *Let (Y, X, π) be a fibred space. Then there exists a topologically equivalent metric ρ for Y with respect to which π is a strong contraction and the identity map $Id : (Y, \rho) \to (Y, d_Y)$ uniformly continuous. Moreover, if $\pi : (Y, d_Y) \to (X, d_X)$ is uniformly continuous, then Id is a uniform homeomorphism and hence ρ is uniformly topologically equivalent to d_Y.*

Proof. Let $0 < \delta < 1/4$. Define

$$U_0 = Y \times Y$$

and

$$U_n = \left\{(y_1, y_2) \in Y \times Y \,\middle|\, d_X\Big(\pi(y_1), \pi(y_2)\Big) < 3^{-n}\,\delta,\ d_Y(y_1, y_2) < 3^{-n}\right\}$$

for $n \in \mathbb{N}$.

The sequence $(U_n)_{n \in \mathbb{Z}_+}$ obviously satisfies all the hypotheses of Frink's Metrization Lemma except the relation $U_n \circ U_n \circ U_n \subseteq U_{n-1}$ for $n > 1$. So, fix $n > 1$ and let $(y_1, y_4) \in U_n \circ U_n \circ U_n$. Then there exist points $y_2, y_3 \in Y$ such that $(y_1, y_2), (y_2, y_3), (y_3, y_4) \in U_n$. This means that, for $i = 1, 2, 3$,

$$d_X\Big(\pi(y_i), \pi(y_{i+1})\Big) < 3^{-n}\,\delta$$

and

$$d_Y(y_i, y_{i+1}) < 3^{-n}.$$

Hence

$$d_X\Big(\pi(y_1), \pi(y_4)\Big) < 3^{-(n-1)}\,\delta$$

and

$$d_Y(y_1, y_4) < 3^{-(n-1)}.$$

This shows that $(y_1, y_4) \in U_{n-1}$. Therefore $U_n \circ U_n \circ U_n \subseteq U_{n-1}$.

We can accordingly apply Frink's Lemma and hence obtain a compatible metric ρ such that

$$U_n \subseteq \left\{(y_1, y_2) \in Y \times Y \,\middle|\, \rho(y_1, y_2) < 2^{-n}\right\} \subseteq U_{n-1}.$$

We would now like to show that π is a strong contraction under ρ. If $y_1, y_2 \in Y$ are such that $\pi(y_1) = \pi(y_2)$, then there is nothing to prove. So, assume that $y_1, y_2 \in Y$ are such that $\pi(y_1) \neq \pi(y_2)$. Then there is a (unique) $n \in \mathbb{Z}_+$ such that $(y_1, y_2) \in U_n \backslash U_{n+1}$. This implies that

$$d_X\Big(\pi(y_1), \pi(y_2)\Big) < 3^{-n}\,\delta$$

and

$$\rho(y_1, y_2) \geq 2^{-(n+2)}.$$

Therefore

$$d_X\Big(\pi(y_1), \pi(y_2)\Big) < 3^{-n}\,\delta < 2^{-n}\,\delta = 4\delta \cdot 2^{-(n+2)} \leq 4\delta\rho(y_1, y_2).$$

Hence, by choice of δ, the map π is a strong contraction.

Furthermore, note that

$$\rho(y_1, y_2) < 2^{-n} \Rightarrow (y_1, y_2) \in U_{n-1} \Rightarrow d_Y(y_1, y_2) < 3^{-(n-1)}.$$

This proves that the map $Id : (Y, \rho) \to (Y, d_Y)$ is uniformly continuous.

Assume now that π is uniformly continuous under d_Y. Then for every $n \in \mathbb{N}$, there exists $0 < \zeta_n < 3^{-n}$ such that

$$d_Y(y_1, y_2) < \zeta_n \Rightarrow d_X\Big(\pi(y_1), \pi(y_2)\Big) < 3^{-n}\,\delta.$$

It follows that

$$d_Y(y_1, y_2) < \zeta_n \Rightarrow (y_1, y_2) \in U_n \Rightarrow \rho(y_1, y_2) < 2^{-n}.$$

Hence $Id : (Y, d_Y) \to (Y, \rho)$ is a uniform homeomorphism. □

With this result at hand, we can now give a direct proof of Theorem 1.2:

Theorem 2.3 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibrewise expansive system for which π is uniformly continuous. If S is expansive, so is T.*

Moreover, if $\mathcal{Y}$ is fibrewise uniformly expansive and S is uniformly expansive, then T is uniformly expansive.

Proof. We only prove the first statement, for the second one can be proved in a similar manner.

Using Lemma 2.2, we replace the original metric d_Y by a uniformly topologically equivalent one (that we will also denote by d_Y) with respect to which π is a contraction. Let $\epsilon > 0$ be a fibrewise expansive constant for $\mathcal{Y}$ and $\delta > 0$ an expansive constant for S. Finally, let $\alpha = \min(\epsilon, \delta)$ and $y_1, y_2 \in Y$ be such that $y_1 \neq y_2$.

First case: y_1, y_2 are such that $\pi(y_1) = \pi(y_2)$.

Since $\mathcal{Y}$ is fibrewise ϵ-expansive, we have that

$$\sup_{n\in\mathbb{Z}_+} d_Y\Big(T^n(y_1), T^n(y_2)\Big) > \epsilon \geq \alpha.$$

Second case: y_1, y_2 are such that $\pi(y_1) \neq \pi(y_2)$.

Since S is δ-expansive, we obtain that

$$\sup_{n\in\mathbb{Z}_+} d_X\Big(S^n(\pi(y_1)), S^n(\pi(y_2))\Big) > \delta \geq \alpha.$$

As π is a contraction, it follows that

$$\begin{aligned}\sup_{n\in\mathbb{Z}_+} d_Y\Big(T^n(y_1), T^n(y_2)\Big) &\geq \sup_{n\in\mathbb{Z}_+} d_X\Big(\pi(T^n(y_1)), \pi(T^n(y_2))\Big) \\ &= \sup_{n\in\mathbb{Z}_+} d_X\Big(S^n(\pi(y_1)), S^n(\pi(y_2))\Big) \\ &> \alpha.\end{aligned}$$

We have hence showed that, given any two points $y_1, y_2 \in Y$, $y_1 \neq y_2$, it holds that

$$\sup_{n\in\mathbb{Z}_+} d_Y\Big(T^n(y_1), T^n(y_2)\Big) > \alpha.$$

So, T is α-expansive. □

Let us turn our attention to the problem we are really concerned with and that we briefly alluded to in the introduction to this section. We would like to determine whether a fibrewise expansive compact system can be, in view of a certain compatible metric, fibrewise expanding.

The following result extends Coven and Reddy's work in that it applies not only to expansive compact dynamical systems, but also to all uniformly expansive, uniformly continuous dynamical systems. In fact, it is even more general, for it claims that a large subclass of fibrewise uniformly expansive, uniformly continuous systems can be conceived as fibrewise expanding under some compatible metric. Indeed, it turns out that all those systems that have uniformly continuous fibres can undergo this change of metric. It is not known, though, if the whole class of fibrewise uniformly expansive, uniformly continuous systems can support such a change.

Theorem 2.4 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibrewise uniformly expansive, uniformly continuous system, whose fibres are uniformly continuous in the Hausdorff metric induced by d_Y. Then there exists a topologically equivalent metric ρ for Y with respect to which the system is fibrewise expanding and π a strong contraction. Moreover, if $\pi : (Y, d_Y) \to (X, d_X)$ is uniformly continuous and X a discrete space, then ρ is uniformly topologically equivalent to d_Y and T retains its uniform continuity under ρ.*

Proof. The proof is divided into nine steps.

Step 1. We define a convenient sequence of (strictly) positive real parameters that decreases monotonically to 0 and that will ensure a good control on the fibres.

As T is uniformly continuous, so are all of its iterates T^m ($m \in \mathbb{Z}_+$). Hence, given $\tilde{\epsilon} > 0$, there exists $\delta_m'''(\tilde{\epsilon}) > 0$ such that

$$d_Y(y_1, y_2) < \delta_m'''(\tilde{\epsilon}) \Rightarrow d_Y\Big(T^m(y_1), T^m(y_2)\Big) < \tilde{\epsilon}.$$

For $n \in \mathbb{N}$, let $\delta_n''(\tilde{\epsilon}) := \min_{0 \le m < n} \delta_m'''(\tilde{\epsilon})$. It follows that

$$d_Y(y_1, y_2) < \delta_n''(\tilde{\epsilon}) \Rightarrow d_{n,Y}(y_1, y_2) < \tilde{\epsilon}.$$

The continuity assumption on the fibres is rigorously expressed via the continuity of the map

$$\begin{array}{rccl} \overline{\pi}: & X & \to & \mathcal{F}(\mathcal{Y}) \\ & x & \mapsto & \overline{\pi}(x) = Y_x, \end{array}$$

where $\mathcal{F}(\mathcal{Y})$ is the set of all fibres of $\mathcal{Y}$ endowed with the Hausdorff metric induced by d_Y, namely, the distance between $Y_{x_1}, Y_{x_2} \in \mathcal{F}(\mathcal{Y})$ is given by

$$d_Y^H(Y_{x_1}, Y_{x_2}) \;:=\; \max\left(\sup_{y_1 \in Y_{x_1}} d_Y\Big(y_1, Y_{x_2}\Big), \sup_{y_2 \in Y_{x_2}} d_Y\Big(Y_{x_1}, y_2\Big)\right).$$

The uniform continuity of $\overline{\pi}$ means that for each $n \in \mathbb{N}$, there exists $\delta_n'(\tilde{\epsilon}) > 0$ such that

$$d_X(x_1, x_2) < \delta_n'(\tilde{\epsilon}) \Rightarrow d_Y^H\Big(Y_{x_1}, Y_{x_2}\Big) < \delta_n''(\tilde{\epsilon}/2).$$

The desired parameters are defined by recursion as

$$\delta_1(\tilde{\epsilon}) = \min\left(\tilde{\epsilon}, \delta_1'(\tilde{\epsilon}), \delta_1''(\tilde{\epsilon})\right)$$

and

$$\delta_n(\tilde{\epsilon}) = \min\left(\delta_n'(\tilde{\epsilon}), \delta_n''(\tilde{\epsilon}), \frac{\delta_{n-1}(\tilde{\epsilon})}{3}\right)$$

for $n \ge 2$.

Step 2. We here define a suitable nested sequence of sets.

For $n \in I\!N$, let

$$V_n(\tilde{\epsilon}) = \left\{ (y_1, y_2) \in Y \times Y \,\Big|\, d_X\Big(\pi(y_1), \pi(y_2)\Big) < \frac{1}{3}\,\delta_{n+1}(\tilde{\epsilon}),\ d_{n+1,Y}(y_1, y_2) < \tilde{\epsilon} \right\}.$$

Then let

$$U_0 = Y \times Y$$

and

$$U_n = V_{nM}(\epsilon/4),$$

where $\epsilon < 2$ is a fibrewise uniform expansive constant for $\mathcal{Y}$ and $M \in I\!N$ is chosen, following Corollary 1.5, in such a way that for any $y \in Y$, we have $B_{M+1,Y}(y, \epsilon) \cap Y_{\pi(y)} \subseteq B_Y(y, \epsilon/8)$.

Step 3. $U_n \circ U_n \circ U_n \subseteq U_{n-1}$ for all $n \in I\!N$.

It is evident that $U_1 \circ U_1 \circ U_1 \subseteq U_0$. Now, let $(y_1, y_4) \in U_n \circ U_n \circ U_n$ for some $n > 1$. Then there exist $y_2, y_3 \in Y$ such that $(y_1, y_2), (y_2, y_3), (y_3, y_4) \in U_n$. Hence, for $j = 1, 2, 3$,

$$d_X\Big(\pi(y_j), \pi(y_{j+1})\Big) < \frac{1}{3}\,\delta_{nM+1}(\epsilon/4)$$

and

$$d_{nM+1,Y}(y_j, y_{j+1}) < \frac{\epsilon}{4}.$$

This implies that

$$\begin{aligned}
d_X\Big(\pi(y_1), \pi(y_4)\Big) &\leq d_X\Big(\pi(y_1), \pi(y_2)\Big) + d_X\Big(\pi(y_2), \pi(y_3)\Big) + d_X\Big(\pi(y_3), \pi(y_4)\Big) \\
&< 3\,\frac{1}{3}\,\delta_{nM+1}(\epsilon/4) \\
&= \delta_{nM+1}(\epsilon/4) \\
&\leq \frac{1}{3^M}\,\delta_{(n-1)M+1}(\epsilon/4).
\end{aligned}$$

Therefore

$$d_X\Big(\pi(y_1), \pi(y_4)\Big) \;<\; \frac{1}{3}\,\delta_{(n-1)M+1}(\epsilon/4). \tag{2.1}$$

But we do not know whether $\pi(y_1) = \pi(y_4)$ or not. Because of the uniform continuity of the fibres, we can find a point $\tilde{y}_4 \in Y_{\pi(y_1)}$ with $d_Y(\tilde{y}_4, y_4) < \delta''_{nM+1}(\epsilon/8)$. Indeed, since $d_X(\pi(y_1), \pi(y_4)) < \delta_{nM+1}(\epsilon/4) \leq \delta'_{nM+1}(\epsilon/4)$, we have that

$$d_Y^H\Big(Y_{\pi(y_1)}, Y_{\pi(y_4)}\Big) < \delta''_{nM+1}(\epsilon/8),$$

that is,

$$\max\left(\sup_{y \in Y_{\pi(y_1)}} d_Y\Big(y, Y_{\pi(y_4)}\Big),\ \sup_{y' \in Y_{\pi(y_4)}} d_Y\Big(Y_{\pi(y_1)}, y'\Big) \right) < \delta''_{nM+1}(\epsilon/8).$$

Since $y_4 \in Y_{\pi(y_4)}$, there exists $\tilde{y}_4 \in Y_{\pi(y_1)}$ such that

$$d_Y(\tilde{y}_4, y_4) < \delta''_{nM+1}(\epsilon/8).$$

The choice of the constant $\delta''_{nM+1}(\epsilon/8)$ then implies that

$$d_{nM+1,Y}(\tilde{y}_4, y_4) < \frac{\epsilon}{8}.$$

Consequently,

$$\begin{aligned} d_{nM+1,Y}(y_1, \tilde{y}_4) &\leq d_{nM+1,Y}(y_1, y_2) + d_{nM+1,Y}(y_2, y_3) \\ &\quad + d_{nM+1,Y}(y_3, y_4) + d_{nM+1,Y}(y_4, \tilde{y}_4) \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{8} \\ &< \epsilon. \end{aligned}$$

Hence, for $0 \leq j < (n-1)M + 1$, we obtain that

$$d_{M+1,Y}\Big(T^j(y_1), T^j(\tilde{y}_4)\Big) < \epsilon$$

or, equivalently,

$$T^j(\tilde{y}_4) \in B_{M+1,Y}\Big(T^j(y_1), \epsilon\Big) \cap Y_{\pi(T^j(y_1))}.$$

By the way M was chosen, we deduce that

$$T^j(\tilde{y}_4) \in B_Y\Big(T^j(y_1), \frac{\epsilon}{8}\Big),$$

that is,

$$d_Y\Big(T^j(y_1), T^j(\tilde{y}_4)\Big) < \frac{\epsilon}{8}$$

for all $0 \leq j < (n-1)M + 1$.

We therefore observe that

$$d_{(n-1)M+1,Y}(y_1, \tilde{y}_4) < \frac{\epsilon}{8}$$

and thus

$$\begin{aligned} d_{(n-1)M+1,Y}(y_1, y_4) &\leq d_{(n-1)M+1,Y}(y_1, \tilde{y}_4) + d_{(n-1)M+1,Y}(\tilde{y}_4, y_4) \\ &\leq d_{(n-1)M+1,Y}(y_1, \tilde{y}_4) + d_{nM+1,Y}(\tilde{y}_4, y_4) \\ &< \frac{\epsilon}{8} + \frac{\epsilon}{8} \\ &= \frac{\epsilon}{4}. \end{aligned}$$

Hence

$$d_{(n-1)M+1,Y}(y_1, y_4) < \frac{\epsilon}{4}. \tag{2.2}$$

So, by 2.1 and 2.2, we conclude that $(y_1, y_4) \in U_{n-1}$.

Step 4. The sets U_n are open symmetric subsets of $Y \times Y$.

This is obvious for $n = 0$. So, fix $n \in I\!N$ and let $(y_1, y_2) \in U_n$. Then

$$d_X\Big(\pi(y_1), \pi(y_2)\Big) < \frac{1}{3}\,\delta_{nM+1}(\epsilon/4)$$

and

$$d_{nM+1,Y}(y_1, y_2) < \frac{\epsilon}{4}.$$

Since π is continuous on Y, there exists, for each $\tilde{\epsilon} > 0$ and $y \in Y$, a constant $\omega_y(\tilde{\epsilon}) > 0$ such that

$$d_Y(y', y) < \omega_y(\tilde{\epsilon}) \Rightarrow d_X\Big(\pi(y'), \pi(y)\Big) < \tilde{\epsilon}.$$

Now let

$$\gamma := \frac{1}{2}\left(\frac{1}{3}\,\delta_{nM+1}(\epsilon/4) - d_X\Big(\pi(y_1), \pi(y_2)\Big)\right) > 0,$$

$$\xi := \frac{1}{2}\left(\frac{\epsilon}{4} - d_{nM+1,Y}(y_1, y_2)\right) > 0$$

and

$$\zeta := \min\Big(\omega_{y_1}(\gamma), \omega_{y_2}(\gamma), \delta_{nM+1}(\xi)\Big) > 0.$$

Finally, suppose that $(y_1', y_2') \in Y \times Y$ is such that

$$d_{Y\times Y}\Big((y_1', y_2'), (y_1, y_2)\Big) := d_Y(y_1', y_1) + d_Y(y_2', y_2) < \zeta.$$

Then

$$\max_{j=1,2} d_Y(y_j', y_j) < \zeta.$$

On the one hand, this implies that

$$\max_{j=1,2} d_X\Big(\pi(y_j'), \pi(y_j)\Big) < \gamma$$

and hence

$$\begin{aligned}
d_X\Big(\pi(y_1'), \pi(y_2')\Big) &\le d_X\Big(\pi(y_1'), \pi(y_1)\Big) + d_X\Big(\pi(y_1), \pi(y_2)\Big) + d_X\Big(\pi(y_2), \pi(y_2')\Big) \\
&< \gamma + d_X\Big(\pi(y_1), \pi(y_2)\Big) + \gamma \\
&= 2\,\gamma + d_X\Big(\pi(y_1), \pi(y_2)\Big) \\
&= \frac{1}{3}\,\delta_{nM+1}(\epsilon/4). \qquad (2.3)
\end{aligned}$$

On the other hand, we obtain that

$$\max_{j=1,2} d_{nM+1,Y}(y_j', y_j) < \xi,$$

and consequently

$$\begin{aligned} d_{nM+1,Y}(y_1', y_2') &\leq d_{nM+1,Y}(y_1', y_1) + d_{nM+1,Y}(y_1, y_2) + d_{nM+1,Y}(y_2, y_2') \\ &< \xi + d_{nM+1,Y}(y_1, y_2) + \xi \\ &= 2\,\xi + d_{nM+1,Y}(y_1, y_2) \\ &= \frac{\epsilon}{4}. \end{aligned} \tag{2.4}$$

The strict inequalities 2.3 and 2.4 show that $(y_1', y_2') \in U_n$. We have hence showed that

$$B_{Y\times Y}\Big((y_1, y_2), \zeta\Big) \subseteq U_n,$$

that is, since $(y_1, y_2) \in U_n$ was chosen arbitrarily, that the set U_n is open.

It is also clear that the V_n's, and hence the U_n's, are symmetric by definition.

Step 5. $\bigcap_{n\in\mathbb{Z}_+} U_n = \Delta(Y)$.

It is clear that

$$\Delta(Y) \subseteq \bigcap_{n\in\mathbb{Z}_+} U_n. \tag{2.5}$$

Now, let $(y_1, y_2) \in (Y \times Y)\backslash\Delta(Y)$. If $\pi(y_1) \neq \pi(y_2)$, then, choosing $N \in \mathbb{N}$ such that

$$d_X\Big(\pi(y_1), \pi(y_2)\Big) \geq \frac{1}{3}\,\delta_{NM+1}(\epsilon/4),$$

we observe that

$$(y_1, y_2) \notin V_{NM}(\epsilon/4) = U_N.$$

Hence

$$(y_1, y_2) \notin \bigcap_{n\in\mathbb{Z}_+} U_n.$$

Suppose now that $\pi(y_1) = \pi(y_2)$. Because of the fibrewise ϵ-expansiveness of the system, there exists $N \in \mathbb{N}$ such that $d_{NM+1,Y}(y_1, y_2) > \epsilon$. This implies that $(y_1, y_2) \notin U_N$ and hence $(y_1, y_2) \notin \cap_{n\in\mathbb{Z}_+} U_n$.

So, we conclude that

$$\Big[\Big(Y \times Y\Big)\backslash\Delta(Y)\Big] \bigcap \left[\bigcap_{n\in\mathbb{Z}_+} U_n\right] = \emptyset$$

or, equivalently,

$$\bigcap_{n\in\mathbb{Z}_+} U_n \subseteq \Delta(Y). \tag{2.6}$$

Therefore relations 2.5 and 2.6 show that

$$\bigcap_{n\in\mathbb{Z}_+} U_n = \Delta(Y).$$

Step 6. There exists a topologically equivalent metric ρ' for Y such that

$$U_n \subseteq \Big\{(y_1, y_2) \in Y \times Y \,\Big|\, \rho'(y_1, y_2) < 2^{-n}\Big\} \subseteq U_{n-1}$$

for all $n \in \mathbb{N}$. Moreover, whenever π is uniformly continuous, the identity map $Id : (Y, d_Y) \to (Y, \rho')$ is uniformly continuous. If, in addition, X is a discrete space, then ρ' and d_Y are uniformly topologically equivalent (that is, Id is a uniform homeomorphism) and T retains its uniform continuity under ρ'.

Steps 3, 4 and 5 show that the sequence $(U_n)_{n\in\mathbb{Z}_+}$ satisfies all the hypotheses of Frink's Metrization Lemma. We hence obtain a compatible metric ρ' satisfying the set inclusions indicated above.

If π is uniformly continuous, then there is, for every $\tilde{\epsilon} > 0$, a constant $0 < \omega(\tilde{\epsilon}) < \tilde{\epsilon}$ such that

$$d_Y(y', y) < \omega(\tilde{\epsilon}) \Rightarrow d_X\Big(\pi(y'), \pi(y)\Big) < \tilde{\epsilon}.$$

Letting $y_1, y_2 \in Y$ be such that $d_Y(y_1, y_2) < \omega(\frac{1}{3}\delta_{nM+1}(\epsilon/4))$, we obtain that

$$d_X\Big(\pi(y_1), \pi(y_2)\Big) < \frac{1}{3}\delta_{nM+1}(\epsilon/4)$$

and

$$d_{nM+1,Y}(y_1, y_2) < \frac{\epsilon}{4}.$$

Therefore $(y_1, y_2) \in V_{nM}(\epsilon/4) = U_n$ and hence $\rho'(y_1, y_2) < 2^{-n}$. This proves that the identity map $Id : (Y, d_Y) \to (Y, \rho')$ is uniformly continuous whenever π is.

Assume now that X is a discrete space and let $\alpha = \min_{x_1,x_2\in X} d_X(x_1, x_2) > 0$. Let $m, n \in \mathbb{N}$, with n being such that $\frac{1}{3}\delta_{(n-1)M+1}(\epsilon/4) \leq \alpha$ and $\mathrm{diam}_Y(B_{(n-1)M+1,Y}(y, \epsilon) \cap Y_{\pi(y)}) < 1/m$ for all $y \in Y$ (for the fulfillment of this latter requirement, we use Lemma 1.4), and suppose that $y_1, y_2 \in Y$ are such that $\rho'(y_1, y_2) < 2^{-n}$. Then $(y_1, y_2) \in U_{n-1} = V_{(n-1)M}(\epsilon/4)$. This means that $d_X(\pi(y_1), \pi(y_2)) < \frac{1}{3}\delta_{(n-1)M+1}(\epsilon/4) \leq \alpha$ and $d_{(n-1)M+1,Y}(y_1, y_2) < \epsilon/4$. The former inequality implies that $\pi(y_1) = \pi(y_2)$. We conclude from this and the latter inequality that $d_Y(y_1, y_2) < 1/m$. Therefore the identity map $Id : (Y, \rho') \to (Y, d_Y)$ is uniformly continuous.

Step 7. There exist $N \in \mathbb{N}$, $a' > 0$ and $\Lambda' > 1$ such that $\mathcal{Y}^N = (Y, T^N, X, S^N, \pi)$ is a fibrewise (a', Λ')-expanding system under the metric ρ'.

Suppose that $y_1, y_2 \in Y$ are such that $\pi(y_1) = \pi(y_2)$ and $0 < \rho'(y_1, y_2) < 2^{-4}$. Then there exists a (unique) $n \geq 3$ such that $(y_1, y_2) \in U_n \backslash U_{n+1}$. This implies in particular that

$$2^{-(n+2)} \leq \rho'(y_1, y_2) < \min(2^{-4}, 2^{-n})$$

and

$$d_{nM+1,Y}(y_1, y_2) < \frac{\epsilon}{4} \leq d_{(n+1)M+1,Y}(y_1, y_2).$$

Consequently, there exists $nM+1 \le j < (n+1)M+1$ such that

$$d_Y\Big(T^j(y_1), T^j(y_2)\Big) \ge \frac{\epsilon}{4}.$$

Now we express j as $j = 3M + i$, where i thereby satisfies

$$0 \le (n-3)M = nM - 3M < i \le (n+1)M - 3M = (n-2)M.$$

It follows that

$$d_Y\Big(T^i(T^{3M}(y_1)), T^i(T^{3M}(y_2))\Big) = d_Y\Big(T^j(y_1), T^j(y_2)\Big) \ge \frac{\epsilon}{4}.$$

This implies that

$$d_{i+1,Y}\Big(T^{3M}(y_1), T^{3M}(y_2)\Big) \ge \frac{\epsilon}{4}$$

and hence

$$\Big(T^{3M}(y_1), T^{3M}(y_2)\Big) \notin V_i(\epsilon/4) \supseteq V_{(n-2)M}(\epsilon/4) = U_{n-2}.$$

Therefore

$$\begin{aligned} \rho'\Big(T^{3M}(y_1), T^{3M}(y_2)\Big) &\ge 2^{-(n-1)} \\ &= 2 \cdot 2^{-n} \\ &> 2\,\rho'(y_1, y_2). \end{aligned}$$

So, whenever $\rho'(y_1, y_2) < 1/16$,

$$\rho'\Big(T^{3M}(y_1), T^{3M}(y_2)\Big) \ge 2\,\rho'(y_1, y_2),$$

that is, $\mathcal{Y}^{3M}$ is fibrewise (a', Λ')-expanding with constants $2a' = 1/16$ and $\Lambda' = 2$.

Step 8. There exists a compatible metric ρ for Y under which $\mathcal{Y}$ is fibrewise expanding. Moreover, whenever π is uniformly continuous and X discrete, ρ is uniformly compatible with ρ' and hence with d_Y.

In fact, we show the following lemma, which is a topological extension of Mather's result on adapted metrics (see [15]). It is also a generalization of the so-called Inheritance Lemma (see, for instance, [5, 16]).

Lemma 2.5 *The iterates $\mathcal{Y}^n = (Y, T^n, X, S^n, \pi)$ of a fibrewise expanding uniformly continuous system $\mathcal{Y} = (Y, T, X, S, \pi)$ are fibrewise expanding with respect to the same metric.*

On the other hand, if an iterate $\mathcal{Y}^n$ of $\mathcal{Y}$ is fibrewise expanding, then there exists a compatible metric with respect to which $\mathcal{Y}$ is fibrewise expanding. Furthermore, if T is uniformly continuous, then there is a uniformly compatible metric with respect to which $\mathcal{Y}$ is fibrewise expanding.

Proof. Assume that $\mathcal{Y} = (Y, T, X, S, \pi)$ is fibrewise (a, Λ)-expanding with respect to some metric ρ and that T is uniformly continuous under this latter. Fix $n \in I\!N$. Then there exists $a_n > 0$ such that

$$\rho(y_1, y_2) < 2a_n \Rightarrow \sup_{0 \le m < n} \rho\Big(T^m(y_1), T^m(y_2)\Big) < 2a.$$

Therefore

$$\rho\Big(T^n(y_1), T^n(y_2)\Big) \ge \Lambda^n \, \rho(y_1, y_2)$$

whenever $\pi(y_1) = \pi(y_2)$ and $\rho(y_1, y_2) < 2a_n$. This shows that $\mathcal{Y}^n$ is fibrewise (a_n, Λ^n)-expanding under ρ whenever T is uniformly continuous.

Suppose now that there is a metric ρ_n for Y with respect to which $\mathcal{Y}^n$ is fibrewise (a_n, Λ_n)-expanding and define the following metric on Y:

$$\rho(y_1, y_2) = \sum_{m=0}^{n-1} \frac{1}{\Lambda^m} \, \rho_n\Big(T^m(y_1), T^m(y_2)\Big),$$

where $\Lambda = (\Lambda_n)^{1/n}$.

Then

$$\begin{aligned}
\frac{1}{\Lambda}\, \rho\Big(T(y_1), T(y_2)\Big) &= \frac{1}{\Lambda} \sum_{m=0}^{n-1} \frac{1}{\Lambda^m} \, \rho_n\Big(T^m(T(y_1)), T^m(T(y_2))\Big) \\
&= \sum_{m=1}^{n} \frac{1}{\Lambda^m} \, \rho_n\Big(T^m(y_1), T^m(y_2)\Big) \\
&= \sum_{m=1}^{n-1} \frac{1}{\Lambda^m} \, \rho_n\Big(T^m(y_1), T^m(y_2)\Big) + \frac{1}{\Lambda_n} \, \rho_n\Big(T^n(y_1), T^n(y_2)\Big) \\
&\ge \sum_{m=1}^{n-1} \frac{1}{\Lambda^m} \, \rho_n\Big(T^m(y_1), T^m(y_2)\Big) + \rho_n(y_1, y_2) \\
&= \rho(y_1, y_2)
\end{aligned}$$

whenever $\pi(y_1) = \pi(y_2)$ and $\rho_n(y_1, y_2) < 2a_n$.

Hence

$$\rho\Big(T(y_1), T(y_2)\Big) \ge \Lambda \, \rho(y_1, y_2)$$

whenever $\pi(y_1) = \pi(y_2)$ and $\rho(y_1, y_2) < 2a_n$, that is, $\mathcal{Y}$ is fibrewise (a, Λ)-expanding under the metric ρ with constants $a = a_n$ and $\Lambda = (\Lambda_n)^{1/n}$.

It is rather clear from its definition that ρ is topologically equivalent to ρ_n. So, let us show that ρ and ρ_n are uniformly topologically equivalent whenever T is uniformly continuous with respect to ρ_n.

On the one hand,

$$\rho(y_1, y_2) \ge \rho_n(y_1, y_2). \tag{2.7}$$

On the other hand, since T is uniformly continuous with respect to ρ_n, so are all of its iterates. Thus, given $\epsilon > 0$, there exists $\delta_n = \delta_n(\epsilon) > 0$ such that

$$\rho_n(y_1, y_2) < \delta_n \Rightarrow \sup_{0 \leq m < n} \rho_n\Big(T^m(y_1), T^m(y_2)\Big) < \Big(1 \Big/ \sum_{m=0}^{n-1} \frac{1}{\Lambda^m}\Big)\epsilon,$$

from which it follows that

$$\rho_n(y_1, y_2) < \delta_n \Rightarrow \rho(y_1, y_2) < \epsilon. \tag{2.8}$$

Relations 2.7 and 2.8 show that the metrics ρ and ρ_n are uniformly topologically equivalent whenever T is uniformly continuous with respect to ρ_n. This completes the proof of the lemma. □

Step 8 follows immediately from this lemma since we have already noted that ρ' and d_Y are uniformly topologically equivalent and T retains its uniform continuity under ρ' whenever π is uniformly continuous and X discrete.

Step 9. π is a strong contraction with respect to both ρ' and ρ.

If $y_1, y_2 \in Y$ are such that $\pi(y_1) = \pi(y_2)$, then there is nothing to prove. So assume that $y_1, y_2 \in Y$ are such that $\pi(y_1) \neq \pi(y_2)$. This means in particular that $y_1 \neq y_2$. Hence there is a (unique) $n \in \mathbb{Z}_+$ such that $(y_1, y_2) \in U_n \backslash U_{n+1}$. This implies that $\rho'(y_1, y_2) \geq 2^{-(n+2)}$.

By definition of the constants $\delta_n(\tilde{\epsilon})$, we obtain by induction that $\delta_n(\tilde{\epsilon}) \leq 3^{-(n-1)}\delta_1(\tilde{\epsilon})$. Since $(y_1, y_2) \in U_n = V_{nM}(\epsilon/4)$, we conclude that

$$\begin{aligned}
d_X\Big(\pi(y_1), \pi(y_2)\Big) &< \frac{1}{3}\delta_{nM+1}(\epsilon/4) \\
&\leq \frac{1}{3}\frac{1}{3^{nM}}\delta_1(\epsilon/4) \\
&\leq \frac{1}{3^{nM+1}}\frac{\epsilon}{4} \\
&\leq \frac{1}{2^{nM+1}}\frac{\epsilon}{4} \\
&\leq \frac{1}{2^{n+1}}\frac{\epsilon}{4} \\
&\leq \frac{\epsilon}{2}\rho'(y_1, y_2).
\end{aligned}$$

Since we have chosen $\epsilon < 2$, we conclude that π is a strong contraction under ρ' and, since $\rho' \leq \rho$, under ρ. This completes the proof of the theorem. □

For compact fibred systems, we have the following result.

Corollary 2.6 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a compact fibred system which is fibrewise expansive and whose fibres are continuous in the Hausdorff metric. Then there exists a (uniformly) topologically equivalent metric for Y with respect to which the system is fibrewise expanding and π is a strong contraction.*

Proof. We showed in Lemma 1.6 that the notions of fibrewise expansiveness and fibrewise uniform expansiveness are the same for compact fibred systems. Similarly, the assumptions of uniform continuity on T and the fibres simply boil down to requirements of continuity, due to the compactness of the system. $\square$

For ordinary dynamical systems, we obtain a generalization of Coven and Reddy's result.

Corollary 2.7 *If (Y, T) is a uniformly expansive uniformly continuous dynamical system, then there exists a uniformly compatible metric for Y with respect to which the system is expanding and remains uniformly continuous.*

Proof. We have already mentioned that an ordinary dynamical system can be conceived as a fibred system consisting of a unique fibre (or, in other words, X is reduced to a single point). This means that the notion of fibrewise uniform expansiveness coincides with that of uniform expansiveness. That the fibres are uniformly continuous in the Hausdorff metric is trivial since there is only one fibre. For the same reason, π is constant and thereby uniformly continuous. $\square$

2.2 The Invertible and Quasi Invertible Cases

One notion that has proved to be useful, particularly in the invertible case, is that of stable and unstable sets.

Let (Y, T) be an invertible dynamical system (that is, T is a homeomorphism of Y) and let $\epsilon > 0$. Recall that the local ϵ-stable set $W^s_\epsilon(y, d_Y)$ and local ϵ-unstable set $W^u_\epsilon(y, d_Y)$ of a point $y \in Y$ are defined as

$$W^s_\epsilon(y, d_Y) = \left\{ \tilde{y} \in Y \Big| \sup_{n \in \mathbb{Z}_+} d_Y\Big(T^n(\tilde{y}), T^n(y)\Big) \leq \epsilon \right\}$$

$$W^u_\epsilon(y, d_Y) = \left\{ \tilde{y} \in Y \Big| \sup_{n \in \mathbb{Z}_+} d_Y\Big(T^{-n}(\tilde{y}), T^{-n}(y)\Big) \leq \epsilon \right\}.$$

Definition 2.8 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibred system for which T is a homeomorphism and let $\epsilon > 0$. Define the local fibrewise ϵ-stable set $W^s_\epsilon(y, d_Y, \pi)$ and local fibrewise ϵ-unstable set $W^u_\epsilon(y, d_Y, \pi)$ of y by*

$$W^s_\epsilon(y, d_Y, \pi) = W^s_\epsilon(y, d_Y) \cap Y_{\pi(y)}$$

$$W^u_\epsilon(y, d_Y, \pi) = W^u_\epsilon(y, d_Y) \cap Y_{\pi(y)}.$$

Knowing that local ϵ-stable and ϵ-unstable sets play an important rôle in the theory of dynamical systems, it is quite conceivable that local fibrewise ϵ-stable and fibrewise ϵ-unstable sets play a similar rôle for fibred systems and have similar properties. Indeed, this seems to be the case.

Lemma 2.9 *A compact invertible fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is fibrewise ϵ-expansive with respect to a metric d_Y if and only if for every $\gamma > 0$, there is $N = N(\gamma) \in \mathbb{Z}_+$ such that for all $y \in Y$ and all $n \geq N$*

$$T^n\Big(W^s_\epsilon(y, d_Y, \pi)\Big) \subseteq W^s_\gamma\Big(T^n(y), d_Y, \pi\Big)$$

and

$$T^{-n}\Big(W^u_\epsilon(y, d_Y, \pi)\Big) \subseteq W^u_\gamma\Big(T^{-n}(y), d_Y, \pi\Big).$$

Proof. Suppose that the claim concerning the local fibrewise ϵ-stable sets is not true. Then there exist $\gamma > 0$, sequences $(m_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}_+$, $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ in Y, and points $y, z \in Y$ such that

$$z_n \in W^s_\epsilon(y_n, d_Y, \pi) \Big\backslash T^{-m_n} W^s_\gamma\Big(T^{m_n}(y_n), d_Y, \pi\Big)$$

and

$$\lim_{n\to\infty} T^{m_n}(y_n) = y,\ \lim_{n\to\infty} T^{m_n}(z_n) = z \text{ and } m_n \uparrow \infty.$$

Since $z_n \in W^s_\epsilon(y_n, d_Y, \pi)$, we have that $\pi(T^{m_n}(y_n)) = \pi(T^{m_n}(z_n))$ and

$$d_Y\Big(T^i \circ T^{m_n}(y_n), T^i \circ T^{m_n}(z_n)\Big) \leq \epsilon$$

for all $i \geq -m_n$. Consequently, $\pi(y) = \pi(z)$ and $d_Y(T^i(y), T^i(z)) \leq \epsilon$ for all $i \in \mathbb{Z}$. Hence $y = z$ because of the fibrewise ϵ-expansiveness.

On the other hand,

$$d_Y(y, z) = \lim_{n\to\infty} d_Y\Big(T^{m_n}(y_n), T^{m_n}(z_n)\Big) \geq \gamma.$$

Therefore $y \neq z$ and this is a contradiction.

Suppose now that the claim regarding the local fibrewise ϵ-unstable sets is false. Then there exist $\gamma > 0$, sequences $(m_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}_+$, $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ in Y, and points $y, z \in Y$ such that

$$z_n \in W^u_\epsilon(y_n, d_Y, \pi) \Big\backslash T^{m_n} W^u_\gamma\Big(T^{-m_n}(y_n), d_Y, \pi\Big)$$

and

$$\lim_{n\to\infty} T^{-m_n}(y_n) = y,\ \lim_{n\to\infty} T^{-m_n}(z_n) = z \text{ and } m_n \uparrow \infty.$$

Since $z_n \in W^u_\epsilon(y_n, d_Y, \pi)$, we have that $\pi(T^{-m_n}(y_n)) = \pi(T^{-m_n}(z_n))$ and

$$d_Y\Big(T^i \circ T^{-m_n}(y_n), T^i \circ T^{-m_n}(z_n)\Big) \le \epsilon$$

for all $i \le m_n$. Consequently, $\pi(y) = \pi(z)$ and $d_Y\Big(T^i(y), T^i(z)\Big) \le \epsilon$ for all $i \in \mathbb{Z}$. Hence $y = z$.

On the other hand,

$$d_Y(y, z) = \lim_{n\to\infty} d_Y\Big(T^{-m_n}(y_n), T^{-m_n}(z_n)\Big) \ge \gamma.$$

Therefore $y \neq z$ and this is a contradiction.

Conversely, if $d_Y(T^n(y), T^n(z)) \le \epsilon$ for all $n \in \mathbb{Z}$ and $\pi(y) = \pi(z)$, we have for any $\gamma > 0$ and $|n| \ge N = N(\gamma)$

$$z = T^{-n}\Big(T^n(z)\Big) \in W^\sigma_\gamma\Big(T^{-n}(T^n(y)), d_Y, \pi\Big) = W^\sigma_\gamma\Big(y, d_Y, \pi\Big).$$

Since γ is arbitrary, the latter relation implies that $y = z$. □

In fact, we obtain a somewhat similar statement for quasi invertible systems (that is, when the base transformation S is only locally injective). This time, though, the characterization is given up to a change of metric.

Lemma 2.10 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a compact quasi invertible fibred system. Then $\mathcal{Y}$ is fibrewise expansive if and only if there exist a metric d_Y for Y and $\epsilon > 0$ such that for every $\gamma > 0$, there is $N = N(\gamma) \in \mathbb{Z}_+$ with the property that for all $y \in Y$ and all $n \ge N$*

$$T^n\Big(W^s_\epsilon(y, d_Y, \pi)\Big) \subseteq W^s_\gamma\Big(T^n(y), d_Y, \pi\Big)$$

and

$$T^{-n}\Big(W^u_\epsilon(y, d_Y, \pi)\Big) \subseteq W^u_\gamma\Big(T^{-n}(y), d_Y, \pi\Big).$$

Proof. We can, according to Lemma 2.2, change the original metric d_Y on Y for a topologically equivalent one (also denoted by d_Y) with respect to which π is a (strong) contraction. This permits T to remain a homeomorphism and the system to retain its fibrewise expansiveness.

Since S is locally injective on the compact metric space X, it is uniformly locally injective, that is, there exists $\delta > 0$ such that $x_1, x_2 \in X$, $d_X(x_1, x_2) < \delta$ implies $S(x_1) \neq S(x_2)$. (We will prove this fact later.)

Now, let $0 < \eta < \delta$ and $z \in W^u_\eta(y, d_Y, \pi)$. Then

$$d_X\Big(\pi(T^{-1}(z)), \pi(T^{-1}(y))\Big) \le d_Y\Big(T^{-1}(z), T^{-1}(y)\Big) \le \eta < \delta$$

and

$$d_X\Big(S(\pi(T^{-1}(z))), S(\pi(T^{-1}(y)))\Big) = d_X\Big(\pi(z), \pi(y)\Big) = 0.$$

Therefore $\pi(T^{-1}(z)) = \pi(T^{-1}(y))$. By induction on n, we obtain that $\pi(T^{-n}(z)) = \pi(T^{-n}(y))$ for all $n \in \mathbb{N}$.

From this, we see that, using the 'new' metric, we can apply the proof of Theorem 2.9 by making the additional restriction $\epsilon < \delta$. $\Box$

Now, let (Y, T) be an invertible dynamical system. Recall that the stable set $W^s(y, d_Y)$ and unstable set $W^u(y, d_Y)$ of a point $y \in Y$ are defined as

$$W^s(y, d_Y) = \left\{ \tilde{y} \in Y \Big| \lim_{n\to\infty} d_Y\Big(T^n(\tilde{y}), T^n(y)\Big) = 0 \right\}$$

and

$$W^u(y, d_Y) = \left\{ \tilde{y} \in Y \Big| \lim_{n\to\infty} d_Y\Big(T^{-n}(\tilde{y}), T^{-n}(y)\Big) = 0 \right\}.$$

Evidently, the notion of stable and unstable sets can also be 'fibrewised'.

Definition 2.11 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibred system for which T is a homeomorphism. Define the fibrewise stable set $W^s(y, d_Y, \pi)$ and fibrewise unstable set $W^u(y, d_Y, \pi)$ of y by*

$$W^s(y, d_Y, \pi) = W^s(y, d_Y) \cap Y_{\pi(y)}$$

and

$$W^u(y, d_Y, \pi) = W^u(y, d_Y) \cap Y_{\pi(y)}.$$

We then have a relativised version of a well-known result (see [1]).

Lemma 2.12 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibrewise ϵ-expansive compact invertible fibred system. Then*

$$W^s(y, d_Y, \pi) = \bigcup_{n\in\mathbb{Z}_+} T^{-n} W^s_\epsilon\Big(T^n(y), d_Y, \pi\Big)$$

and

$$W^u(y, d_Y, \pi) = \bigcup_{n\in\mathbb{Z}_+} T^{n} W^u_\epsilon\Big(T^{-n}(y), d_Y, \pi\Big).$$

Proof. If $y' \in \cup_{n\in\mathbb{Z}_+} T^{-n} W^s_\epsilon(T^n(y), d_Y, \pi)$, then there is $n \in \mathbb{Z}_+$ with $T^n(y') \in W^s_\epsilon(T^n(y), d_Y, \pi)$. Thus for $\gamma > 0$, there is $N = N(\gamma) \in \mathbb{Z}_+$ such that for all $m \geq N$

$$T^{m+n}(y') \in T^m W^s_\epsilon\Big(T^n(y), d_Y, \pi\Big) \subseteq W^s_\gamma\Big(T^{m+n}(y), d_Y, \pi\Big)$$

and so $d_Y(T^{m+n}(y'), T^{m+n}(y)) \leq \gamma$ for $m \geq N$. Moreover, $\pi(T^n(y')) = \pi(T^n(y))$. Therefore $y' \in W^s(y, d_Y, \pi)$.

Conversely, if $y' \in W^s(y, d_Y, \pi)$ and $\epsilon > 0$, then we can find $N \in \mathbb{Z}_+$ such that $d_Y(T^n(y'), T^n(y)) \leq \epsilon$ for $n \geq N$. Thus $d_Y(T^i \circ T^N(y'), T^i \circ T^N(y)) \leq \epsilon$ for all $i \in \mathbb{Z}_+$. Therefore $T^N(y') \in W^s_\epsilon(T^N(y), d_Y, \pi)$ and so

$$y' \in T^{-N} W^s_\epsilon\Big(T^N(y), d_Y, \pi\Big) \subseteq \bigcup_{n\in\mathbb{Z}_+} T^{-n} W^s_\epsilon\Big(T^n(y), d_Y, \pi\Big). \Box$$

The following theorem constitutes a generalization of Reddy [17] and Sakai's [20] works. It is the two-sided variation of Theorem 2.4.

Theorem 2.13 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be an invertible fibred system for which T is a uniform homeomorphism and whose fibres are uniformly continuous in the Hausdorff metric. Suppose also that this system is fibrewise uniformly expansive. Then there exist a topologically equivalent metric ρ for Y and constants $\nu > 0$, $C > 0$ and $0 < \lambda < 1$ such that*

$$\tilde{y} \in W_\nu^s(y, \rho, \pi) \Rightarrow \rho\Big(T^n(\tilde{y}), T^n(y)\Big) \leq C\lambda^n \rho(\tilde{y}, y)$$

and

$$\tilde{y} \in W_\nu^u(y, \rho, \pi) \Rightarrow \rho\Big(T^{-n}(\tilde{y}), T^{-n}(y)\Big) \leq C\lambda^n \rho(\tilde{y}, y)$$

for all $n \in \mathbb{Z}_+$. Furthermore, π is a strong contraction with respect to ρ. Such a metric is said to be fibrewise hyperbolic.

Moreover, whenever $\pi : (Y, d_Y) \to (X, d_X)$ is uniformly continuous, so are the identity map $Id : (Y, d_Y) \to (Y, \rho)$ and the fibres (in the Hausdorff metric induced by the new metric ρ). If, in addition, X is a discrete space, then ρ and d_Y are uniformly topologically equivalent (that is, Id is a uniform homeomorphism) and T retains its uniform continuity under ρ.

Proof. Since the ideas are essentially the same as the ones presented in the proof of Theorem 2.4, we only sketch the argument and put emphasis on the parts that differ greatly from the ones given in the non-invertible case.

Steps 1 to 6. In analogy to the non-invertible situation, we define convenient sequences of numbers and sets. This time we take into account the invertibility of the system by considering both backward and forward iterates of T. One can again show that the sequence of sets hence defined satisfies all the assumptions of Frink's Lemma. (The reader may want to supply the relevant arguments.)

Hence there exists a compatible metric ρ for Y such that

$$U_n \subseteq \Big\{(y_1, y_2) \in Y \times Y \,\Big|\, \rho(y_1, y_2) < 2^{-n}\Big\} \subseteq U_{n-1}.$$

As in the non-invertible case, one can show that whenever π is uniformly continuous, so is the identity map $Id : (Y, d_Y) \to (Y, \rho)$. Therefore the fibres are uniformly continuous in the Hausdorff metric induced by ρ. If, in addition, X is a discrete space, one can prove that Id is a uniform homeomorphism and hence ρ is uniformly topologically equivalent to d_Y and T retains its uniform continuity under ρ.

Steps 7 and 8. We show that ρ is the desired metric.

The proof is a straightforward generalization of a well-known argument (see, for instance, [17], [20] or [1]).

Let $\tilde{\epsilon} > 0$ and $E \subseteq Y \times Y$ and denote by $yE := \{\tilde{y} \in Y \mid (y, \tilde{y}) \in E\}$. Then

$$T\Big(W^s_{\tilde{\epsilon}}(y, d_Y, \pi) \bigcap yV_n(\tilde{\epsilon})\Big) = W^s_{\tilde{\epsilon}}\Big(T(y), d_Y, \pi\Big) \bigcap T(y)V_{n+1}(\tilde{\epsilon}),$$

$$T^{-1}\Big(W^u_{\tilde{\epsilon}}(y, d_Y, \pi) \bigcap yV_n(\tilde{\epsilon})\Big) = W^u_{\tilde{\epsilon}}\Big(T^{-1}(y), d_Y, \pi\Big) \bigcap T^{-1}(y)V_{n+1}(\tilde{\epsilon})$$

for all $n \in I\!N$. It follows by induction that

$$T^N\Big(W^s_{\tilde{\epsilon}}(y, d_Y, \pi) \bigcap yV_n(\tilde{\epsilon})\Big) = W^s_{\tilde{\epsilon}}\Big(T^N(y), d_Y, \pi\Big) \bigcap T^N(y)V_{n+N}(\tilde{\epsilon}),$$

$$T^{-N}\Big(W^u_{\tilde{\epsilon}}(y, d_Y, \pi) \bigcap yV_n(\tilde{\epsilon})\Big) = W^u_{\tilde{\epsilon}}\Big(T^{-N}(y), d_Y, \pi\Big) \bigcap T^{-N}(y)V_{n+N}(\tilde{\epsilon})$$

for any $N \in I\!N$.

We now prove that ρ is a hyperbolic metric on each fibre. First of all, note that $W^s_\eta(y, \rho, \pi) \subseteq W^s_{\epsilon/4}(y, d_Y, \pi)$ for any $y \in Y$ and $\eta < 1/4$. So, fix $\nu < 1/4$ and let $\tilde{y} \in W^s_\nu(y, \rho, \pi) \backslash \{y\}$. We show that

$$\rho\Big(T^{3M}(y), T^{3M}(\tilde{y})\Big) < \frac{1}{2}\rho(y, \tilde{y}).$$

Since $(y, \tilde{y}) \in U_n \backslash U_{n+1}$ for some $n \in Z\!\!\!Z_+$, we have that $2^{-(n+2)} \leq \rho(y, \tilde{y}) \leq \nu < 2^{-2}$ and thus $n \in I\!N$. Since $\tilde{y} \in W^s_{\epsilon/4}(y, d_Y, \pi) \cap yU_n = W^s_{\epsilon/4}(y, d_Y, \pi) \cap yV_{nM}(\epsilon/4)$, we obtain that

$$\begin{aligned}
T^{3M}(\tilde{y}) &\in W^s_{\epsilon/4}\Big(T^{3M}(y), d_Y, \pi\Big) \bigcap T^{3M}(y)V_{(n+3)M}(\epsilon/4) \\
&= W^s_{\epsilon/4}\Big(T^{3M}(y), d_Y, \pi\Big) \bigcap T^{3M}(y)U_{n+3} \\
&\subseteq T^{3M}(y)\Big\{(y_1, y_2) \in Y \times Y \Big| \rho(y_1, y_2) < 2^{-(n+3)}\Big\}
\end{aligned}$$

and so

$$\rho\Big(T^{3M}(\tilde{y}), T^{3M}(y)\Big) < 2^{-(n+3)} = \frac{1}{2}2^{-(n+2)} \leq \frac{1}{2}\rho(\tilde{y}, y).$$

It follows by induction that

$$\rho\Big(T^{k(3M)}(\tilde{y}), T^{k(3M)}(y)\Big) < \Big(\frac{1}{2}\Big)^k \rho(\tilde{y}, y)$$

for all $k \in I\!N$.

Define $\lambda = (1/2)^{1/3M}$. To conclude, we show that if $\tilde{y} \in W^s_\nu(y, \rho, \pi) \backslash \{y\}$, then

$$\rho\Big(T^m(y), T^m(\tilde{y})\Big) < 8\lambda^m \rho(y, \tilde{y})$$

for all $m \in Z\!\!\!Z_+$.

We write $m = k(3M) + j$ in such a way that $k \in \mathbb{Z}_+$ and $0 \leq j < 3M$. Since $(T^{k(3M)}(y), T^{k(3M)}(\tilde{y})) \in U_n \backslash U_{n+1}$ for some $n \in \mathbb{N}$, we have that

$$T^{k(3M)}(\tilde{y}) \in W^s_{\epsilon/4}\Big(T^{k(3M)}(y), d_Y, \pi\Big) \bigcap T^{k(3M)}(y) V_{nM}(\epsilon/4)$$

and thus

$$T^{k(3M)+j}(\tilde{y}) \in W^s_{\epsilon/4}\Big(T^{k(3M)+j}(y), d_Y, \pi\Big) \bigcap T^{k(3M)+j}(y) V_{nM+j}(\epsilon/4),$$

from which it appears that

$$\Big(T^{k(3M)+j}(\tilde{y}), T^{k(3M)+j}(y)\Big) \in V_{nM+j}(\epsilon/4) \subseteq V_{nM}(\epsilon/4) = U_n.$$

Since

$$\Big(T^{k(3M)}(\tilde{y}), T^{k(3M)}(y)\Big) \notin U_{n+1} \supseteq \Big\{(y_1, y_2) \in Y \times Y \,\Big|\, \rho(y_1, y_2) < 2^{-(n+2)}\Big\},$$

we know that

$$\rho\Big(T^{k(3M)}(\tilde{y}), T^{k(3M)}(y)\Big) \geq 2^{-(n+2)}$$

and thus

$$\rho\Big(T^{k(3M)+j}(\tilde{y}), T^{k(3M)+j}(y)\Big) < 2^{-n} \leq 4\rho\Big(T^{k(3M)}(\tilde{y}), T^{k(3M)}(y)\Big) \leq 4\Big(\frac{1}{2}\Big)^k \rho(\tilde{y}, y).$$

As $m = k(3M) + j$, we get that

$$\rho\Big(T^m(\tilde{y}), T^m(y)\Big) \leq 4\Big(\frac{1}{2}\Big)^k \rho(\tilde{y}, y) = 4\lambda^{k(3M)} \rho(\tilde{y}, y) < 8\lambda^{k(3M)+j} \rho(\tilde{y}, y) = 8\lambda^m \rho(\tilde{y}, y).$$

By replacing T by T^{-1}, one can establish the corresponding property for $\tilde{y} \in W^u_\nu(y, \rho, \pi)$.

Step 9. π is a contraction with respect to ρ.

The proof goes along the same lines as in the non-invertible case. □

The hypothesis of invertibility of the system means in particular that S is a homeomorphism. This requirement can be relaxed at the cost of a condition on π. Indeed, the critical point in the proof of Theorem 2.13 is that the fibres are both forward and backward preserved by T. Thanks to Lemma 2.2, we can show that it suffices for S to be uniformly locally injective whenever π is uniformly continuous. Note that it may happen that the fibres be not preserved under backward iteration, but the dynamics does respect, on sufficiently small local fibrewise ν-unstable sets (meaning that ν is small enough), the fibres under backward iteration.

Theorem 2.14 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a quasi invertible fibred system for which T is a uniform homeomorphism, S uniformly locally injective, π uniformly continuous and whose fibres are uniformly continuous in the Hausdorff metric. Suppose also that this system is fibrewise uniformly expansive. Then there exist a topologically equivalent metric ρ for Y and constants $\nu > 0$, $C > 0$ and $0 < \lambda < 1$ such that*

$$\tilde{y} \in W^s_\nu(y, \rho, \pi) \Rightarrow \rho\Big(T^n(\tilde{y}), T^n(y)\Big) \leq C\lambda^n \rho(\tilde{y}, y)$$

and

$$\tilde{y} \in W^u_\nu(y, \rho, \pi) \Rightarrow \rho\Big(T^{-n}(\tilde{y}), T^{-n}(y)\Big) \leq C\lambda^n \rho(\tilde{y}, y)$$

for all $n \in \mathbb{Z}_+$.

Furthermore, π is a strong contraction with respect to ρ, the identity map $Id : (Y, d_Y) \to (Y, \rho)$ is uniformly continuous and the fibres remain uniformly continuous in the Hausdorff metric induced by ρ. If, in addition, X is a discrete space, then ρ and d_Y are uniformly topologically equivalent (that is, Id is a uniform homeomorphism) and T retains its uniform continuity under ρ.

Proof. As π is assumed to be uniformly continuous, we can, according to Lemma 2.2, change the original metric d_Y on Y for a uniformly topologically equivalent one (also denoted by d_Y) with respect to which π is a strong contraction (and thereby uniformly continuous). This permits T to remain a uniform homeomorphism and the fibres to be uniformly continuous in the Hausdorff metric induced by the new metric. For the same reason, the system retains its fibrewise uniform expansiveness.

Moreover, since $S : X \to X$ is uniformly locally injective, the argument given in the proof of Lemma 2.10 shows that $\pi(T^{-n}(\tilde{y})) = \pi(T^{-n}(y))$ for all $n \in \mathbb{N}$ and $\tilde{y} \in W^u_\eta(y, d_Y, \pi)$, where $0 < \eta < \delta$.

From this fact and using the new metric, we can apply the proof of Theorem 2.13 with the additional constraint $\max(\epsilon, \nu) < \delta$. $\square$

For compact fibred systems, the foregoing theorem can be rephrased as follows.

Corollary 2.15 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a quasi invertible fibred system whose fibres are continuous in the Hausdorff metric. Suppose also that this system is fibrewise expansive. Then there exist a topologically equivalent metric ρ for Y and constants $\nu > 0$, $C > 0$ and $0 < \lambda < 1$ such that*

$$\tilde{y} \in W^s_\nu(y, \rho, \pi) \Rightarrow \rho\Big(T^n(\tilde{y}), T^n(y)\Big) \leq C\lambda^n \rho(\tilde{y}, y)$$

and

$$\tilde{y} \in W^u_\nu(y, \rho, \pi) \Rightarrow \rho\Big(T^{-n}(\tilde{y}), T^{-n}(y)\Big) \leq C\lambda^n \rho(\tilde{y}, y)$$

for all $n \in \mathbb{Z}_+$. Furthermore, π is a strong contraction with respect to ρ.

Proof. It is obvious that a uniformly locally injective map is locally injective. Though the converse is not true, a locally injective map on a compact metric space is uniformly locally injective. For instance, as S is assumed to be locally injective, given $x \in X$, there is $\delta_x > 0$ such that S is injective on $B_X(x, \delta_x)$. Since the family of balls $\{B_X(x, \delta_x)\}_{x \in X}$ is an open cover of the compact space X, we can extract a finite subcover $\{B_X(x_k, \delta_{x_k})\}_{k=1}^{l}$ for some $l \in I\!N$. Let $\delta > 0$ be a Lebesgue number for this cover. Then for any $x \in X$, we have $B_X(x, \delta) \subseteq B_X(x_k, \delta_{x_k})$ for some $k = k(x)$. Hence S is injective on $B_X(x, \delta)$ for all $x \in X$, for it is injective on each $B_X(x_k, \delta_{x_k})$. This precisely means that S is uniformly locally injective. The corollary follows from the simple observation that all the other assumptions of Theorem 2.14 reduce to the ones of the statement. □

2.3 Fibrewise Expanding vs. Fibrewise Ruelle Expanding Systems

It is well-known that an expanding compact dynamical system is Ruelle expanding if and only if it is open. This result could be restated by replacing the term *open* by the term *uniformly open* since the underlying space is compact.

The situation is somewhat different for fibred systems. Fibrewise openness and uniform fibrewise openness are two distinct properties, even for compact fibred systems.

Examples 1.19 and 1.20 also show that there are fibrewise open and fibrewise expanding compact systems that are not fibrewise Ruelle expanding.

This is because we must imperatively require in the theory of fibred systems that the system be uniformly fibrewise open in order to reach fibrewise Ruelle expandability from fibrewise expandability.

Theorem 2.16 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibrewise expanding system. Then $\mathcal{Y}$ is fibrewise Ruelle expanding if and only if it is uniformly fibrewise open.*

Proof. Assume that $\mathcal{Y}$ is uniformly fibrewise open and fibrewise (a, Λ)-expanding for some constants $a > 0$ and $\Lambda > 1$. If $y_1, y_2 \in Y$, $y_1 \neq y_2, \pi(y_1) = \pi(y_2)$ and $T(y_1) = T(y_2)$, then $d_Y(y_1, y_2) \geq 2a$. Hence for each $y \in Y$, the map $T : B_Y(y, a) \cap Y_{\pi(y)} \to Y_{S(\pi(y))}$ is injective. Now, let $0 < \delta < a$. As $\mathcal{Y}$ is uniformly fibrewise open, there exists $\epsilon > 0$ such that $T(B_Y(y, \delta) \cap Y_{\pi(y)}) \supseteq B_Y(T(y), 2\epsilon) \cap Y_{S(\pi(y))}$ for all $y \in Y$. Then for $y_1, y_2' \in Y$ such that $\pi(T(y_1)) = \pi(y_2')$ and $d_Y(T(y_1), y_2') < 2\epsilon$, there exists a unique $y_2 \in B_Y(y_1, \delta) \cap Y_{\pi(y_1)}$, with $T(y_2) = y_2'$, that satisfies

$$d_Y\Big(T(y_1), T(y_2)\Big) \geq \Lambda \, d_Y(y_1, y_2).$$

This implies that

$$d_Y(y_1, y_2) \leq \Lambda^{-1}\, d_Y\Big(T(y_1), y_2'\Big) < \Lambda^{-1}\, 2\epsilon < 2\epsilon.$$

Hence there exists a unique $y_2 \in Y$ such that $T(y_2) = y_2'$, $\pi(y_2) = \pi(y_1)$ and $d_Y(y_1, y_2) < 2\epsilon$. Furthermore, $d_Y(T(y_1), T(y_2)) \geq \Lambda\, d_Y(y_1, y_2)$. This precisely means that $\mathcal{Y}$ is fibrewise Ruelle (ϵ, Λ)-expanding.

Suppose now that $\mathcal{Y}$ is fibrewise Ruelle (a, Λ)-expanding, though not uniformly fibrewise open. Then there exists $\delta > 0$ such that for all $n \in I\!N$, we can find $y_n \in Y$ with

$$T\Big(B_Y(y_n, \delta) \cap Y_{\pi(y_n)}\Big) \not\supseteq B_Y\Big(T(y_n), 1/n\Big) \cap Y_{S(\pi(y_n))}.$$

For each $n \in I\!N$, choose

$$z_n' \in \Big[B_Y\Big(T(y_n), 1/n\Big) \cap Y_{S(\pi(y_n))}\Big] \Big\backslash T\Big(B_Y(y_n, \delta) \cap Y_{\pi(y_n)}\Big).$$

Let $N \in I\!N$ be such that $N \geq \max(1/2a, 1/\Lambda\delta)$ and fix $n \geq N$. Then y_n and z_n' are such that $\pi(T(y_n)) = \pi(z_n')$ and $d_Y(T(y_n), z_n') < 1/n \leq 2a$. Hence there exists a unique $z_n \in Y$ such that $T(z_n) = z_n'$, $\pi(z_n) = \pi(y_n)$ and $d_Y(z_n, y_n) < 2a$. Furthermore,

$$d_Y(z_n', T(y_n)) \geq \Lambda\, d_Y(z_n, y_n).$$

This implies that $d_Y(z_n, y_n) < 1/\Lambda n$ and therefore

$$z_n' \in T\Big(B_Y(y_n, 1/\Lambda n) \cap Y_{\pi(y_n)}\Big) \subseteq T\Big(B_Y(y_n, \delta) \cap Y_{\pi(y_n)}\Big).$$

This is a contradiction to the definition of the z_n''s. $\Box$

Let us now study some concrete systems.

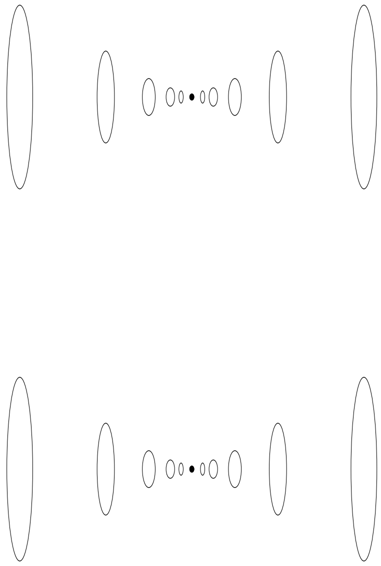
Example 2.17 *Set*

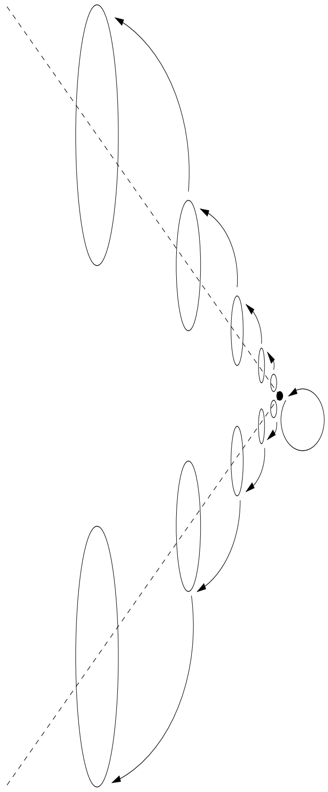
$$X = \Big\{\pm 1/2^s, 3 - 1/2^s\Big\}_{s \in I\!N \cup \{\infty\}} \cup \{2\}.$$

For each $x' \in X$ of the form $\pm 1/2^s$, the fibre $Y_{x'}$ consists of two circles of radius $|x'|$, centred at $(x', 0, 0)$ and $(x', 3, 0)$, that lie in the plane $x = x'$ of $I\!R^3$. For a x' of the form $3 - 1/2^s$, the fibre $Y_{x'}$ is composed of two circles of radius $1/2^s$, centred at $(x', \pm 1/2^{s-1}, 0)$, which lie in the plane $x = x'$. (Note that we interpret a circle of radius 0 as being a point.) Finally, the fibre $x' = 2$ is composed of two circles of radius 1, centred at $(2, \pm 2, 0)$, in the plane $x = 2$. The union of all these fibres constitutes the set $Y \subseteq I\!R^3$.

The map $\pi : Y \to X$ is here the projection on the x-axis, while the map S is defined by

$$S(x) = \begin{cases} 3 - 1/2^{s-1} & \text{if } \; x = \pm 1/2^s, 3 - 1/2^s \\ 2 & \text{if } \; x = 2. \end{cases}$$





Finally, the map $T : Y \to Y$ is defined by

$$T(x,y,z) = \begin{cases} (3-1/2^{s-1}, & 1/2^{s-2}+2(y-3), & 2z) & \text{if} & x = 1/2^s, & y > 1 \\ (3-1/2^{s-1}, & -1/2^{s-2}+2y, & 2z) & \text{if} & x = 1/2^s, & y < 1 \\ (3-1/2^{s-1}, & -1/2^{s-2}+2(y-3), & 2z) & \text{if} & x = -1/2^s, & y > 1 \\ (3-1/2^{s-1}, & 1/2^{s-2}+2y, & 2z) & \text{if} & x = -1/2^s, & y < 1 \\ (3-1/2^{s-1}, & 1/2^{s-2}+2(y-1/2^{s-1}), & 2z) & \text{if} & x = 3-1/2^s, & y > 0 \\ (3-1/2^{s-1}, & -1/2^{s-2}+2(y+1/2^{s-1}), & 2z) & \text{if} & x = 3-1/2^s, & y < 0 \\ \text{Doubling map} & & & \text{if} & x = 2. & \\ \text{on each circle} & & & & & \end{cases}$$

The dynamics associated to this fibred system is grossly depicted in the previous and next two figures.

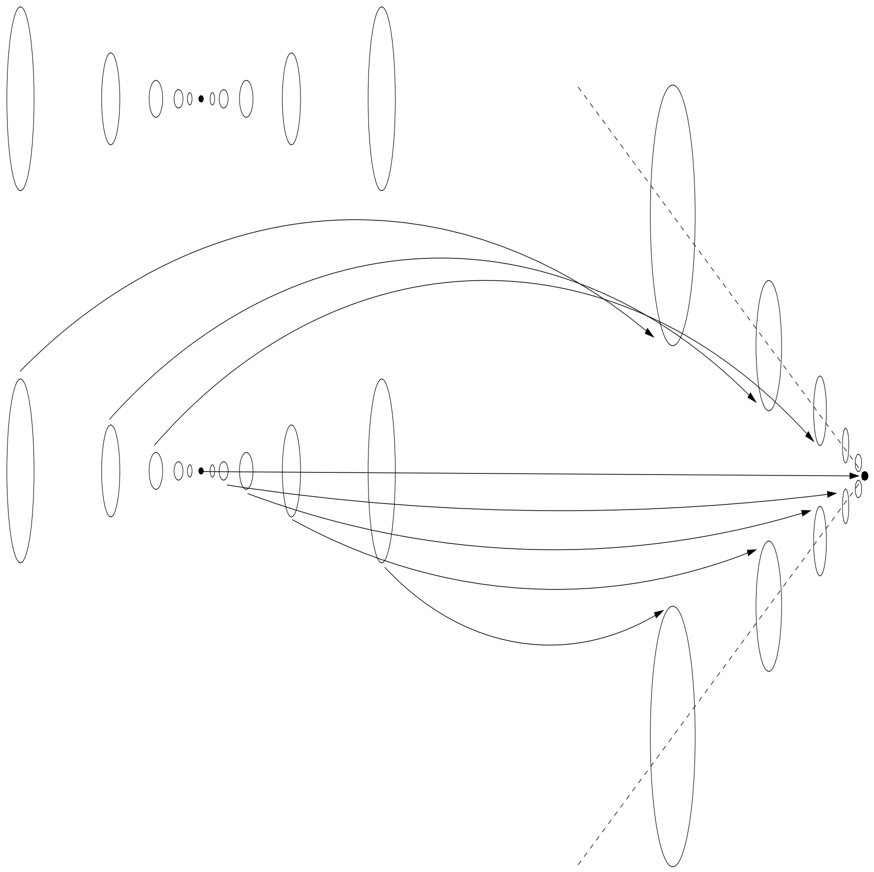
A similar, though simpler, system is the following.

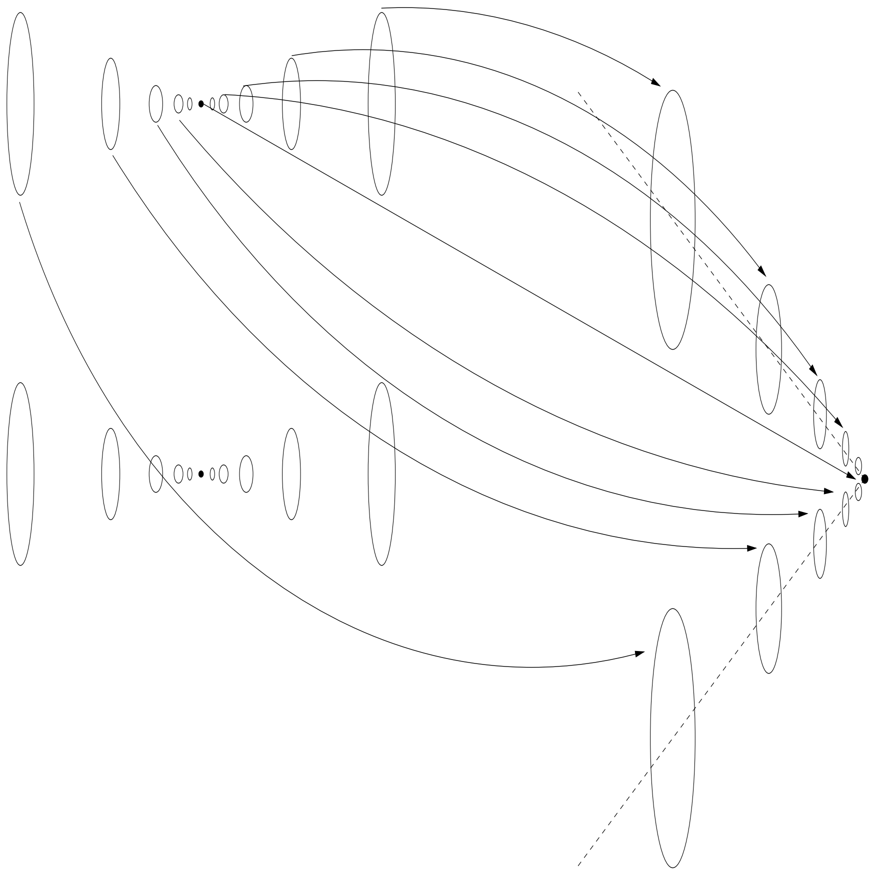
Example 2.18 *Set*

$$X = \left\{\pm 1/2^s, 3 - 1/2^s\right\}_{s \in I\!N \cup \{\infty\}} \cup \{2\}$$

and $Y = \cup_{x \in X} Y_x$, where

$$Y_x = \begin{cases} \{(x,0),(x,3)\} & \text{if} \quad x = \pm 1/2^s \\ \{(x, \pm 1/2^s)\} & \text{if} \quad x = 3 - 1/2^s \\ \{(2, \pm 1)\} & \text{if} \quad x = 2. \end{cases}$$





As usual, the map $\pi : Y \to X$ is the projection on the x-axis of $I\!R^2$, while the map $S : X \to X$ is defined by

$$S(x) = \begin{cases} 3 - 1/2^{s-1} & \text{if } \ x = \pm 1/2^s, 3 - 1/2^s \\ \hfill 2 & \text{if } \ x = 2. \end{cases}$$

Finally, the map $T : Y \to Y$ is defined by

$$T(x,y) = \begin{cases} (3 - 1/2^{s-1}, \quad 1/2^{s-1}) & \text{if } \quad x = 1/2^s, \ y = 3 \\ (3 - 1/2^{s-1}, \ -1/2^{s-1}) & \text{if } \quad x = 1/2^s, \ y = 0 \\ (3 - 1/2^{s-1}, \ -1/2^{s-1}) & \text{if } \quad x = -1/2^s, \ y = 3 \\ (3 - 1/2^{s-1}, \quad 1/2^{s-1}) & \text{if } \quad x = -1/2^s, \ y = 0 \\ (3 - 1/2^{s-1}, \quad 1/2^{s-1}) & \text{if } \ x = 3 - 1/2^s, \ y > 0 \\ (3 - 1/2^{s-1}, \ -1/2^{s-1}) & \text{if } \ x = 3 - 1/2^s, \ y < 0 \\ \hfill (x,y) & \text{if } \quad x = 2. \end{cases}$$

These two systems have interesting properties. The map T is a local homeomorphism and hence open, even uniformly open for Y is compact. Though it is fibrewise open, $\mathcal{Y}$ is not uniformly fibrewise open and therefore not fibrewise Ruelle expanding. Nevertheless, one can show that it is expanding and, thereby, fibrewise expanding. The fact that T is open and $\mathcal{Y}$ expanding implies that this latter is Ruelle expanding. Hence we see that a system may be Ruelle expanding without necessarily being fibrewise Ruelle expanding. This means that Ruelle expandability does not imply fibrewise Ruelle expandability.

Another property that the two systems just described have in common is that their base transformation S is finite-to-one and hence locally finite-to-one. In both cases, S fails to be locally injective at one and only one point (namely, $x = 0$). This may explain why these two systems are not uniformly fibrewise open and therefore not fibrewise Ruelle expanding. Indeed, we have the following result.

Theorem 2.19 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a compact fibred system with T open and S locally injective. Then $\mathcal{Y}$ is uniformly fibrewise open. Moreover, if $\mathcal{Y}$ is fibrewise expanding, then it is fibrewise Ruelle expanding.*

Proof. Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibred system with T open and S locally injective. Suppose though that it is not uniformly fibrewise open, that is, assume that there exist a $\delta > 0$ and sequences $(y_n)_{n \in I\!N}$ and $(z'_n)_{n \in I\!N}$ in Y such that

$$z'_n \in \Big[B_Y\Big(T(y_n), 1/n\Big) \cap Y_{S(\pi(y_n))}\Big] \Big\backslash T\Big(B_Y(y_n, \delta) \cap Y_{\pi(y_n)}\Big).$$

Without loss of generality, we may assume that the sequence $(y_n)_{n \in I\!N}$ converges to some point y_0. The continuity of T then ensures that the sequence $(T(y_n))_{n \in I\!N}$ converges to $T(y_0)$. It is evident that so does the sequence $(z'_n)_{n \in I\!N}$.

As S is locally injective, π continuous and T open, there exist $0 < \eta < \delta$ and $\beta > 0$ such that S is injective on $\pi(B_Y(y_0, \eta))$ and $B_Y(T(y_0), \beta) \subseteq T(B_Y(y_0, \eta))$. Considering a subsequence of the y_n's if necessary, we may suppose that all the z_n''s belong to $B_Y(T(y_0), \beta)$, so that $(T|_{B_Y(y_0,\eta)}^{-1}(z_n'))_{n\in I\!N}$ is a sequence of nonempty subsets of $B_Y(y_0, \eta)$. Choose $z_n \in T|_{B_Y(y_0,\eta)}^{-1}(z_n')$, which means that $z_n \in B_Y(y_0, \eta)$ and $T(z_n) = z_n'$. Therefore $d_Y(z_n, y_0) < \eta < \delta$. This implies that $d_Y(z_n, y_n) < \delta$ for all sufficiently large n's. Moreover, we know that $z_n' \in Y_{S(\pi(y_n))}$, that is, $\pi(z_n') = S(\pi(y_n))$, that $y_n, z_n \in B_Y(y_0, \eta)$ for big enough n's and that S is injective on $\pi(B_Y(y_0, \eta))$. Then $S(\pi(z_n)) = \pi(T(z_n)) = \pi(z_n') = S(\pi(y_n))$ implies that $\pi(z_n) = \pi(y_n)$.

We have hence shown that $z_n \in B_Y(y_n, \delta) \cap Y_{\pi(y_n)}$ for large n's. We correspondingly conclude that $z_n' = T(z_n) \in T(B_Y(y_n, \delta) \cap Y_{\pi(y_n)})$ for large enough n's. This contradicts the definition of the z_n''s. $\square$

So we now know that the openness of T is not sufficient to ensure that a compact fibred system be uniformly fibrewise open. One must additionally require that S be locally injective.

On the other hand, we remark that all the fibre maps T_x of the systems considered in Examples 2.17 and 2.18 are homeomorphisms, except for $x = 0$ and $x = 2$ (in the second case, only for $x = 0$). Moreover, T_0 and T_2 are surjective local homeomorphisms. This situation also prevails in Examples 1.19 and 1.20. In those examples, all the fibre maps are homeomorphisms, except T_1 which is a surjective local homeomorphism.

We now show that if T_1 (or T_0 and T_2 in the previous examples) had been homeomorphisms, then the systems would have been uniformly fibrewise open.

In the following, we will refer to a fibred system as a fibrewise homeomorphism if each fibre map T_x is a homeomorphism.

Theorem 2.20 *Any compact fibrewise homeomorphism $\mathcal{Y} = (Y, T, X, S, \pi)$ is uniformly fibrewise open. Moreover, if $\mathcal{Y}$ is fibrewise expanding, it is fibrewise Ruelle expanding.*

Proof. Suppose that $\mathcal{Y}$ is not uniformly fibrewise open, that is, there exist a $\delta > 0$ and sequences $(y_n)_{n\in I\!N}$ and $(z_n')_{n\in I\!N}$ in Y such that

$$z_n' \in \Big[B_Y\Big(T(y_n), 1/n\Big) \cap Y_{S(\pi(y_n))}\Big] \Big\backslash T\Big(B_Y(y_n, \delta) \cap Y_{\pi(y_n)}\Big).$$

As before, we may suppose that the sequence $(y_n)_{n\in I\!N}$ converges to some point y_0. The continuity of T then ensures that the sequence $(T(y_n))_{n\in I\!N}$ converges to $T(y_0)$, so that the sequence $(z_n')_{n\in I\!N}$ converges to $T(y_0)$ too.

But, since $\mathcal{Y}$ is a fibrewise homeomorphism, $T_{\pi(y_n)}$ is surjective and hence there exists $z_n \in Y_{\pi(y_n)} \backslash B_Y(y_n, \delta)$ such that $T(z_n) = z_n'$. Hence $d_Y(z_n, y_n) \geq \delta$. Without loss of generality, we may further assume that the z_n's converge to, say, z_0. Then

$$d_Y(z_0, y_0) = \lim_{n\to\infty} d_Y(z_n, y_n) \geq \delta.$$

Thus $z_0 \neq y_0$. Nevertheless,

$$T(z_0) = \lim_{n\to\infty} T(z_n) = \lim_{n\to\infty} z'_n = T(y_0)$$

and

$$\pi(z_0) = \lim_{n\to\infty} \pi(z_n) = \lim_{n\to\infty} \pi(y_n) = \pi(y_0).$$

Consequently, we deduce that $T_{\pi(y_0)}$ is not a homeomorphism. This is a contradiction. □

In fact, we can see from the previous proof that it is sufficient that all those fibre maps T_x, that are associated to accumulation points x of X, be homeomorphisms. All those fibre maps associated to isolated points of X may only be required to be open and surjective.

So far, our approach has consisted in determining conditions under which a system is uniformly fibrewise open. We could therewith deduce that if the system is, in addition, fibrewise expanding, then it is fibrewise Ruelle expanding.

Another possibility is to make that additional assumption right from the start. So we now assume that the system under scrutiny is fibrewise expanding and try to find topological conditions on the fibres that force the system into uniform fibrewise openness or, in other words, into fibrewise Ruelle expandability.

Theorem 2.21 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a compact fibred system which is fibrewise open, fibrewise expanding and uniformly fulc. Then $\mathcal{Y}$ is uniformly fibrewise open and hence fibrewise Ruelle expanding.*

Proof. Assume that $\mathcal{Y}$ is fibrewise (a, Λ)-expanding for some constants $a > 0$ and $\Lambda > 1$. However, suppose that $\mathcal{Y}$ is not uniformly fibrewise open. Then there exist $0 < \delta < a$ and sequences $(y_n)_{n\in I\!N}$ and $(z'_n)_{n\in I\!N}$ in Y such that

$$z'_n \in \Big[B_Y\big(T(y_n), 1/n\big) \cap Y_{S(\pi(y_n))}\Big] \Big\backslash T\Big(B_Y(y_n, \delta) \cap Y_{\pi(y_n)}\Big).$$

As earlier, we assume that the sequence $(y_n)_{n\in I\!N}$ converges to some point y_0, so that the sequences $(T(y_n))_{n\in I\!N}$ and $(z'_n)_{n\in I\!N}$ both converge to $T(y_0)$.

Since $\mathcal{Y}$ is uniformly fulpc, there exists $\zeta > 0$ such that any pair of points $y_1, y_2 \in Y$ with $\pi(y_1) = \pi(y_2)$ and $d_Y(y_1, y_2) < \zeta$ can be joined by a path in $Y_{\pi(y_i)}$ of diameter less than $\Lambda\,\delta$. Now, let $N \in I\!N$ be such that $1/N < \zeta/2$ and $d_Y(T(y_n), z'_n) < \zeta$ for all $n \geq N$. For each such n, the points $T(y_n)$ and z'_n can be joined by a path in

$Y_{S(\pi(y_n))}$ of diameter less than $\Lambda\,\delta$. Hence there exists a continuous map $\gamma_n : [0,1] \to B_Y(T(y_n), \Lambda\,\delta) \cap Y_{S(\pi(y_n))}$ with $\gamma_n(0) = T(y_n)$ and $\gamma_n(1) = z'_n$.

Let

$$t_n := \inf\Big\{t \in [0,1] \,\Big|\, \gamma_n(t) \notin T\Big(B_Y(y_n, \delta) \cap Y_{\pi(y_n)}\Big)\Big\}.$$

It is easy to see that $t_n > 0$ since $T(y_n) \in T(B_Y(y_n, \delta) \cap Y_{\pi(y_n)})$ and $\mathcal{Y}$ is fibrewise open (this ensures that $T(B_Y(y_n, \delta) \cap Y_{\pi(y_n)})$ is an open subset of $Y_{S(\pi(y_n))}$). Furthermore, t_n is well-defined since $z'_n \notin T(B_Y(y_n, \delta) \cap Y_{\pi(y_n)})$ and γ_n is continuous. One can also observe that $\gamma_n(t_n) \notin T(B_Y(y_n, \delta) \cap Y_{\pi(y_n)})$ since this latter set is open in $Y_{S(\pi(y_n))}$ and hence its complement is closed in that fibre. But, clearly, $\gamma_n(t_n) \in \overline{T(B_Y(y_n, \delta) \cap Y_{\pi(y_n)})}$. Hence $\gamma_n(t_n) \in \partial T(B_Y(y_n, \delta) \cap Y_{\pi(y_n)}) \subseteq T(\partial(B_Y(y_n, \delta) \cap Y_{\pi(y_n)}))$. So, let $z_n \in \partial(B_Y(y_n, \delta) \cap Y_{\pi(y_n)})$ be such that $T(z_n) = \gamma_n(t_n)$. Without loss of generality, we can restrict our attention to a subsequence $(z_{n_m})_{m \in I\!N}$ which converges to, say, z_0. Then

$$\lim_{m\to\infty} \gamma_{n_m}(t_{n_m}) = \lim_{m\to\infty} T(z_{n_m}) = T(z_0).$$

Therefore

$$d_Y\Big(T(z_0), T(y_0)\Big) = \lim_{m\to\infty} d_Y\Big(\gamma_{n_m}(t_{n_m}), T(y_{n_m})\Big) < \Lambda\,\delta.$$

However, since $z_n \in \partial(B_Y(y_n, \delta) \cap Y_{\pi(y_n)})$, we have $d_Y(z_n, y_n) = \delta$. This implies that $d_Y(z_0, y_0) = \delta < a$. Since $\pi(z_0) = \lim_{m\to\infty} \pi(z_{n_m}) = \lim_{m\to\infty} \pi(y_{n_m}) = \pi(y_0)$, it follows that

$$d_Y\Big(T(z_0), T(y_0)\Big) \geq \Lambda\, d_Y(z_0, y_0) = \Lambda\,\delta.$$

This is a contradiction. □

Nonetheless, remark that there are fibrewise open, fibrewise expanding ful(p)c, but not uniformly ful(p)c, compact fibred systems which are not uniformly fibrewise open or, in other words, not fibrewise Ruelle expanding. Examples 1.19, 1.20, 2.17 and 2.18 yield such systems.

2.4 Some Interesting Implications for the Families of Skew-products and Triangular Maps

The family of skew-products accounts for one of the most important subclasses of fibred systems. Its stronger topological structure restricts the dynamical behaviors that can be observed upon it.

The following result generalizes the fact that a positively expansive homeomorphism that lives on a compact space can only survive on a finite space (see [1]).

Theorem 2.22 *Let $\mathcal{Y} = (Y = X \times Z, T, X, S, \pi)$ be a fibrewise uniformly expansive uniformly fibrewise open skew-product, with Z compact and T uniformly continuous* *with respect to $d_Y = d_X + d_Z$, and assume that there exists $x \in X$ such that $T_{S^i(x)}$ is a homeomorphism for all $i \in \mathbb{Z}_+$. Then Z is a finite set.*

Proof. Since $\mathcal{Y}$ is a skew-product with $d_Y = d_X + d_Z$, it is clear that π is a contraction and hence uniformly continuous. It is also easy to see that $d_Y^H(Y_{x_1}, Y_{x_2}) = d_X(x_1, x_2)$ for all $x_1, x_2 \in X$, from which it follows that the fibres of $\mathcal{Y}$ are uniformly continuous. (In fact, we observe from the latter relation that $\overline{\pi}$ is an isometry.)

Now we change the metric on X and, correspondingly, the metric on Y, replacing the original metric d_X by the discrete metric. So, for now on, d_X will denote the discrete metric. Note that this change of metric does not alter the properties the system has. Indeed, the fibrewise uniform expansiveness and uniform fibrewise openness depend only on the metric d_Z. So does the compactness of Z. Moreover, π remains a contraction (and hence is uniformly continuous), whatever the metric on X is. It is also easy to see that S and T are uniformly continuous and the $T_{S^i(x)}$'s homeomorphisms under the sovereignty of the discrete metric. Finally, since $\overline{\pi}$ is an isometry whichever metric X is equipped with, the fibres retain their uniform continuity under the proposed change of metric.

Note that the topology on Y may have changed. But this is of no importance here since we are only concerned with the topology on Z.

Hence $\mathcal{Y}$ (with the discrete metric on X) satisfies all the assumptions of Theorems 2.4 and 2.16. Therefore there exists a uniformly topologically equivalent metric ρ for Y under which the system is fibrewise Ruelle expanding. This means that there are some constants $a > 0$ and $\Lambda = 1/\lambda > 1$ such that if $y_1, y_2' \in Y, \pi(T(y_1)) = \pi(y_2')$ and $\rho(T(y_1), y_2') < 2a$, then there is a unique $y_2 \in Y$ with $T(y_2) = y_2'$, $\pi(y_2) = \pi(y_1)$, $d_Y(y_2, y_1) < 2a$ and

$$\rho\Big(T(y_1), T(y_2)\Big) \geq \Lambda\, \rho(y_1, y_2).$$

As $\mathcal{Y}$ is a skew-product, the homeomorphism $T_x : Y_x = \{x\} \times Z \to Y_{S(x)}$ induces a homeomorphism (also denoted by T_x)

$$\begin{array}{rccc} T_x : & Z & \to & Z \\ & z & \mapsto & T(x, z), \end{array}$$

where the domain Z of T_x is endowed with the relative metric ρ_x (defined as $\rho_x(z_1, z_2) = \rho((x, z_1), (x, z_2)))$, whereas its range Z is equipped with the relative metric $\rho_{S(x)}$ (which is defined in a similar manner). The inverse $(T_x)^{-1}$ of T_x is such that if $z_1, z_2 \in Z$ and $\rho_{S(x)}(z_1, z_2) < a$, then

$$\rho_x\Big((T_x)^{-1}(z_1), (T_x)^{-1}(z_2)\Big) \leq \lambda\, \rho_{S(x)}(z_1, z_2).$$

By induction on i, we have that if $z_1, z_2 \in Z, \rho_{S^i(x)}(z_1, z_2) < a$, then

$$\rho_x\Big((T_x^i)^{-1}(z_1), (T_x^i)^{-1}(z_2)\Big) \leq \lambda^i \, \rho_{S^i(x)}(z_1, z_2)$$

for all $i \in I\!N$.

The homeomorphisms of the family $\{T_x^i\}_{i \in I\!N}$ share the same domain but have different ranges (the underlying set is the same but the metrics vary). We would hence be delighted if we could replace all these metrics by a common one, namely, ρ_x. This is possible because of the uniform compatibility of the metrics d_Y and ρ. For this, let $0 < \alpha < a$.

On the one hand, there exists $\beta > 0$ such that

$$d_Y(y_1, y_2) < \beta \Rightarrow \rho(y_1, y_2) < \alpha.$$

This implies that

$$d_Z(z_1, z_2) < \beta \Rightarrow \sup_{x' \in X} \rho_{x'}(z_1, z_2) < \alpha.$$

On the other hand, there is $\gamma > 0$ such that

$$\rho(y_1, y_2) < \gamma \Rightarrow d_Y(y_1, y_2) < \beta,$$

from which it follows that

$$\rho_x(z_1, z_2) < \gamma \Rightarrow d_Z(z_1, z_2) < \beta.$$

Hence

$$\rho_x(z_1, z_2) < \gamma \Rightarrow \sup_{x' \in X} \rho_{x'}(z_1, z_2) < \alpha.$$

We accordingly conclude that

$$\rho_x\Big((T_x^i)^{-1}(z_1), (T_x^i)^{-1}(z_2)\Big) < \alpha$$

whenever $z_1, z_2 \in Z$ and $\rho_x(z_1, z_2) < \gamma$.

This means that the family of homeomorphisms $\{(T_x^i)^{-1}\}_{i \in I\!N}$ is uniformly equicontinuous on Z with respect to the metric ρ_x (put on both the domain and the range). As Z is compact, it follows from Arzela-Ascoli's theorem that this family is totally bounded with respect to the uniform metric generated by ρ_x. Now, we can show that the map

$$\begin{array}{rccc} \Phi : & \Big\{(T_x^i)^{-1} \,\Big|\, i \in I\!N\Big\} & \to & \Big\{T_x^i \,\Big|\, i \in I\!N\Big\} \\ & (T_x^i)^{-1} & \mapsto & T_x^i \end{array}$$

is a uniform isometry.

Let $i, j \in I\!N, i < j$. Then

$$\begin{aligned}
d\Big(\Phi((T_x^i)^{-1}), \Phi((T_x^j)^{-1})\Big) &= d\Big(T_x^i, T_x^j\Big) \\
&:= \sup_{z \in Z} \rho_x\Big(T_{S^{i-1}(x)} \circ \ldots \circ T_x(z), T_{S^{j-1}(x)} \circ \ldots \circ T_x(z)\Big) \\
&= \sup_{z' \in Z} \rho_x\Big(z', T_{S^{j-1}(x)} \circ \ldots \circ T_{S^i(x)}(z')\Big) \\
&= \sup_{w' \in Z} \rho_x\Big(T^{-1}_{S^i(x)} \circ \ldots \circ T^{-1}_{S^{j-1}(x)}(w'), w'\Big) \\
&= \sup_{w \in Z} \rho_x\Big(T_x^{-1} \circ \ldots \circ T^{-1}_{S^{j-1}(x)}(w), T_x^{-1} \circ \ldots \circ T^{-1}_{S^{i-1}(x)}(w)\Big) \\
&= \sup_{w \in Z} \rho_x\Big((T_{S^{j-1}(x)} \circ \ldots \circ T_x)^{-1}(w), (T_{S^{i-1}(x)} \circ \ldots \circ T_x)^{-1}(w)\Big) \\
&=: d\Big((T_x^i)^{-1}, (T_x^j)^{-1}\Big).
\end{aligned}$$

It follows that the family of homeomorphisms $\{T_x^i\}_{i \in I\!N}$ is totally bounded and hence uniformly equicontinuous. But this is clearly impossible. Indeed, let $x \in X$ and $z_1, z_2 \in Z$ be such that $\rho_x(z_1, z_2) < a$. Because of the fibrewise expandability of the system, we have that

$$\sup_{i \in Z\!\!Z_+} \rho_{S^i(x)}\Big(T_x^i(z_1), T_x^i(z_2)\Big) \geq a,$$

from which it turns out that

$$\sup_{i \in Z\!\!Z_+} \rho_x\Big(T_x^i(z_1), T_x^i(z_2)\Big) \geq \gamma.$$

This contradicts the uniform equicontinuity of the family $\{T_x^i\}_{i \in I\!N}$, unless Z is finite. □

In the compact case, we obtain the following result.

Corollary 2.23 *Let $\mathcal{Y} = (Y = X \times Z, T, X, S, \pi)$ be a fibrewise expansive, uniformly fibrewise open compact skew-product and assume that there exists $x \in X$ such that $T_{S^i(x)}$ is a homeomorphism for all $i \in Z\!\!Z_+$. Then Z is a finite set.*

From one of the results we have developed before, we deduce the following corollary.

Corollary 2.24 *Let $\mathcal{Y} = (Y = X \times Z, T, X, S, \pi)$ be a fibrewise expansive compact skew-product which forms a fibrewise homeomorphism. Then Z is a finite set.*

Proof. Theorem 2.20 ensures that $\mathcal{Y}$ is uniformly fibrewise open and hence the result follows immediately from the previous corollary. □

It is also worth noticing that we obtain a generalization of the fact that the unit interval cannot support any expansive map.

Corollary 2.25 *Let $\mathcal{Y} = (Y = X \times [0,1], T, X, S, \pi)$ be a fibrewise surjective compact skew-product. Then $\mathcal{Y}$ cannot be fibrewise expansive.*

Proof. Let $0 < \epsilon < 1$, $x \in X$ and $y \in [0, 1-\epsilon]$. Denote as $I_{x,y}(\epsilon)$ the closed subinterval $[y, y+\epsilon]$ in the fibre Y_x. If $\mathcal{Y}$ were fibrewise expansive, there would exist $0 < \epsilon < 1$ such that $T_x : I_{x,y}(\epsilon) \subseteq Y_x \to Y_{S(x)}$ is injective for all $y \in [0, 1-\epsilon]$. The point $a_x \in I_{x,0}(\epsilon/2)$ which gives the greatest value of $T_x|_{I_{x,0}(\epsilon/2)}$ is one of the endpoints of $I_{x,0}(\epsilon/2)$, that is, 0 or $\epsilon/2$ (notice that the point hence identified depends continuously on x). If $a_x = \epsilon/2$ gives the greatest value of T_x on $I_{x,0}(\epsilon/2)$, then the point ϵ must be the point of greatest value on $I_{x,\epsilon/2}(\epsilon/2)$. Continuing this argument, we conclude that T_x is a homeomorphism for every $x \in X$ (we use here the assumption that $\mathcal{Y}$ is fibrewise surjective) and the result follows immediately from Corollary 2.24. □

Corollary 2.26 *No fibrewise surjective triangular map is fibrewise expansive, less expansive.*

Proof. Recall that a triangular map is a very special kind of skew-product. A triangular map is defined as a skew-product with $X = [0,1] = Z$ and $\pi : [0,1] \times [0,1]$ by $\pi(x, z) = x$, that is, the natural projection on X. This is therefore a special case of Corollary 2.25. □

Chapitre 3

On the Existence of a Gibbs Family for Fibrewise Expansive Systems

Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a compact fibred system. Throughout this chapter we make the assumption that T is bounded-to-one on fibres, an assumption originally due to Denker and Gordin [8]. This means that there exists a constant $M \in I\!N$ such that for all $y \in Y$

$$\operatorname{card}\Big\{y' \in Y \,\Big|\, T(y) = T(y'),\ \pi(y) = \pi(y')\Big\} \leq M,$$

or, in other words, the number of preimages in each fibre is bounded by M.

We also assume, like Denker and Gordin, that each fibre contains at least two points, that S, like T, is bounded-to-one on fibres and that $\mathcal{Y}$ is fibrewise surjective, that is, each fibre map $T_x : Y_x \to Y_{S(x)}$ is surjective.

Recall that a family $\{\mu_x\}_{x\in X}$ of probability measures on Y is called a (measurable) system of conditional probabilities for $\mathcal{Y}$ if $\mu_x(Y_x) = 1$ for all $x \in X$ and the integral $\int f(y)\, \mu_x(dy)$ is a Borel-measurable function of x for every bounded Borel-measurable function f on Y.

Denker and Gordin [8] introduced the notion of Gibbs family for fibred systems. They defined it as follows.

Definition 3.1 *A system $\{\mu_x^\varphi\}_{x\in X}$ of conditional probabilities for $\mathcal{Y}$ is called a Gibbs family for a measurable function $\varphi : Y \to I\!R$ if there exists a positive Borel-measurable function $A_\varphi : X \to I\!R$ with the following property:*
For all $x \in X$ the Jacobian of μ_x^φ with respect to the map T is given by

$$\frac{d(\mu_{S(x)}^\varphi \circ T_x)}{d\mu_x^\varphi} = A_\varphi(x) \exp(-\varphi) \quad \mu_x^\varphi\text{-}a.e.$$

They also remarked that the Gibbs property is equivalent to the following statement: For every bounded Borel-measurable function $f : Y \to I\!R$ and all $x \in X$, one has that

$$\int (V_x^\varphi f)(y)\, \mu_{S(x)}^\varphi(dy) = A_\varphi(x) \int f(y)\, \mu_x^\varphi(dy),$$

where V_x^φ is a conditional transfer operator between the spaces of bounded Borel-measurable functions B_x on the fibres Y_x. These operators $V_x^\varphi : B_x \to B_{S(x)}$ are defined by

$$(V_x^\varphi f)(y) = \sum_{y' \in T^{-1}(y) \cap Y_x} f(y') \exp(\phi(y')),$$

where $y \in Y_{S(x)}$.

Denker and Gordin derived the existence and uniqueness of a Gibbs family for Hölder continuous potentials and fibrewise Ruelle expanding and topologically exact systems (see [8]). It was later remarked (see [9], page 13) that their results hold for potentials that are globally continuous and uniformly Hölder continuous on the fibres.

Recall that a potential $\varphi : Y \to I\!R$ is uniformly fibrewise Hölder continuous if there exists $\alpha \in (0,1]$ and $C > 0$ such that if $y_1, y_2 \in Y$ are such that $\pi(y_1) = \pi(y_2)$, then $|\varphi(y_1) - \varphi(y_2)| \leq C d_Y^\alpha(y_1, y_2)$.

For the sake of completeness, we recall Denker and Gordin's results (see [8], Theorems 2.6, 2.7 and 2.10, and [9], page 13).

Existence Theorem 3.2 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a compact fibred system. Assume that $\mathcal{Y}$ is fibrewise Ruelle expanding and topologically exact along fibres. Then for every globally continuous, uniformly fibrewise Hölder continuous function $\varphi : Y \to I\!R$, there exists a unique Gibbs family $\{\mu_x^\varphi\}_{x \in X}$ for φ. The function A_φ associated with φ by Definition 3.1 is also unique and supp $\mu_x^\varphi = Y_x$.*

In order to provide sufficient conditions for the continuity of the Gibbs family (see also [7]) one needs to consider the spaces

$$Y_n = \Big\{(x, y) \in X \times Y \,\Big|\, S^n(x) = \pi(y)\Big\},$$

where $n \in Z\!\!\!Z_+$. Y_n can be considered as a subspace of $X \times Y$ and thus inherits the induced topology. The continuity of the Gibbs family depends on properties of the maps $i_n : Y \to Y_n$ defined by $i_n(y) = (\pi(y), T^n(y))$.

Continuity Theorem 3.3 *Let $\{\mu_x^\varphi\}_{x \in X}$ and (φ, A_φ) be as in the Existence Theorem 3.2. Assume that S and π are open maps and that i_1 is a local homeomorphism. Then the Gibbs family $\{\mu_x^\varphi\}_{x \in X}$ and the function $A_\varphi : X \to I\!R$ are continuous.*

In order to obtain Hölder continuity of the Gibbs family, Denker and Gordin required that the following condition be fulfilled:

(RH) A fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is called relatively Hölder continuous if for some $\alpha \in (0,1]$ and $b, c > 0$ the following holds:
If $y_1, y_0' \in Y$, $x_1' \in X$ are such that $S(x_1') = \pi(y_0')$, $d_X\big(\pi(y_1), x_1'\big) < 2b$ and $d_Y(T(y_1), y_0')$

$< 2c$, then there exists a unique $y_1' \in Y$ such that $\pi(y_1') = x_1'$, $T(y_1') = y_0'$ and $d_Y(y_1, y_1') < 2c$. Moreover, it is assumed (letting $y_0 = T(y_1)$) that

$$d_Y(y_1, y_1')\, d_X^\alpha\Big(\pi(y_0), \pi(y_0')\Big) \leq d_Y(y_0, y_0')\, d_X^\alpha\Big(\pi(y_1), \pi(y_1')\Big).$$

In such a situation, α is called the exponent of Hölder continuity.

As Denker and Gordin remarked (see [8], Remark 2.9), condition (RH) and continuity of T and π imply that i_1 is a local homeomorphism. Furthermore, condition (RH) holds for expanding maps T and differentiable maps S.

Hölder Continuity Theorem 3.4 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$, $\{\mu_x^\varphi\}_{x\in X}$ and (φ, A_φ) be as in the Continuity Theorem 3.3. Let T be an open map, π be a contraction, and let condition (RH) be satisfied. Moreover, assume that S is an expansive, Lipschitz continuous and locally non-contracting map. Then for any $\gamma \in (0,1]$, there exist $\eta(\gamma) \in (0,1]$ and $C_\gamma > 0$ with the following property:*
If f is (D,γ)-Hölder continuous, then

$$x \mapsto \int_Y f(y)\, \mu_x^\varphi(dy)$$

is $(C_\gamma D, \eta(\gamma))$-Hölder continuous. In particular, the function $A_\varphi : X \to I\!R$ is Hölder continuous.

Now we state new results. First of all, let us consider X to be a countable set.

Existence Theorem 3.5 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a compact fibred system. Suppose that it is fibrewise expansive and uniformly fibrewise open, that its fibres are continuous in the Hausdorff metric and that it is topologically exact along those latter. Assume also that X is countable. Then for every continuous potential $\varphi : Y \to I\!R$, there exist a (non necessarily unique) Gibbs family $\{\mu_x^\varphi\}_{x\in X}$ for φ and a (non necessarily unique) function A_φ associated with φ that satisfy Definition 3.1. Moreover, supp $\mu_x^\varphi = Y_x$.*

Proof. According to Corollary 2.6, there exists a topologically equivalent metric ρ under which the system is fibrewise expanding. As the system is also uniformly fibrewise open under ρ (see Theorem 1.22), it follows from Theorem 2.16 that it is fibrewise Ruelle expanding under this metric. It is furthermore topologically exact along fibres under ρ, since this notion was shown to be independent of the compatible metric considered (see Theorem 1.34). Now, let $\varphi : Y \to I\!R$ be a continuous potential under the original metric. Remark that φ is also continuous with respect to the metric ρ, for ρ is compatible with the topology on Y. Since the Lipschitz potentials (with respect to ρ) form a dense subset of the collection of all continuous potentials in the uniform metric induced by ρ (this follows from a simple application of Stone-Weierstrass theorem), there exists a sequence $(\psi_n)_{n\in I\!N}$ of Lipschitz potentials that converge uniformly to φ.

We can therefore apply the Existence Theorem 3.2 to each potential ψ_n and hence obtain the existence of a unique Gibbs family $\{\mu_x^{\psi_n}\}_{x\in X}$ for ψ_n such that supp $\mu_x^{\psi_n} = Y_x$. Furthermore, there exists a unique function A_{ψ_n} satisfying Definition 3.1. This means that the following relations hold:

1. supp $\mu_x^{\psi_n} = Y_x$ for all $x \in X$;

2. The function $x \mapsto \int f(y)\, \mu_x^{\psi_n}(dy)$ is Borel-measurable for every bounded Borel-measurable function $f : Y \to I\!R$;

3. $A_{\psi_n} : X \to I\!R$ is a positive Borel-measurable function such that for every bounded Borel-measurable function $f : Y \to I\!R$, one has
$$\int (V_x^{\psi_n} f)(y)\, \mu_{S(x)}^{\psi_n}(dy) = A_{\psi_n}(x) \int f(y)\, \mu_x^{\psi_n}(dy).$$

Since X is countable, one can find, via a Cantor's diagonal type of argument, a subsequence $(\psi_{n_m})_{m\in I\!N}$ with the following properties:

(i) $\psi_{n_m} \to \varphi$ in the uniform metric induced by ρ;

(ii) For each $x \in X$, one has that $\mu_x^{\psi_{n_m}} \to \mu_x$ for some $\mu_x \in M(Y)$, the convergence taking place in the weak* topology on $M(Y)$, the set of all Borel probability measures on Y. Henceforth, we define $\mu_x^{\varphi} := \mu_x$;

(iii) For every $x \in X$, one has that $A_{\psi_{n_m}}(x) \to A(x)$ for some $A(x) > 0$ (this is possible since $A_{\psi_n}(x) = \int \sum_{y'\in T^{-1}(y)\cap Y_x} \exp(\psi(y'))\, \mu_{S(x)}^{\psi_n}(dy)$ and hence $\exp(-\|\psi_n\|) \leq A_{\psi_n}(x) \leq M \exp \|\psi_n\|)$. We also define $A_{\varphi}(x) := A(x)$.

It follows from this that:

1. supp $\mu_x^{\varphi} = Y_x$ for all $x \in X$ (for μ_x^{φ} is the weak* limit of the sequence $(\mu_x^{\psi_{n_m}})_{m\in I\!N}$, which satisfies this condition);

2. The function $x \mapsto \int f(y)\, \mu_x^{\varphi}(dy)$ is Borel-measurable for every bounded Borel-measurable function $f : Y \to I\!R$ (for it is the pointwise limit of the sequence of Borel-measurable functions $x \mapsto \int f(y)\, \mu_x^{\psi_{n_m}}(dy)$);

3. $A_{\varphi} : X \to I\!R$ is positive Borel-measurable (for it is the pointwise limit of the sequence of Borel-measurable functions $A_{\psi_{n_m}}$) and for each bounded Borel-measurable function $f : Y \to I\!R$, one has that
$$\int (V_x^{\varphi} f)(y)\, \mu_{S(x)}^{\varphi}(dy) = A_{\varphi}(x) \int f(y)\, \mu_x^{\varphi}(dy).$$

We have hence found a Gibbs family for φ. $\square$

We now take a look at the case X uncountable. This is a more difficult situation since we cannot apply Cantor's diagonal argument to obtain a 'nice' subsequence of Lipschitz potentials that converges to the potential considered.

We rather want to find a family that keeps its uniform fibrewise Hölder continuity property under any compatible metric. We would evidently like this family to be as large as possible.

Definition 3.6 *Let (Y, X, π) be a fibred space. A potential $\varphi : Y \to I\!R$ is said to be uniformly fibrewise locally constant if*

$$d_{\text{crit}} := \inf_{x \in X} \inf_{\substack{r,\, r' \,\in\, I\!R \\ r \neq r'}} d_Y\Big(Y_x(r), Y_x(r')\Big) > 0,$$

where

$$Y_x(r) = \Big\{ y \in Y_x \,\Big|\, \varphi(y) = r \Big\}$$

is the r-level set of φ in the fibre Y_x, and the inmost infimum is taken over all pairs of reals $r \neq r'$ for which $Y_x(r) \neq \emptyset$ and $Y_x(r') \neq \emptyset$.

Evidently, a uniformly fibrewise locally constant potential is fibrewise locally constant (that is, locally constant on each fibre). The converse is not true though. Indeed, consider the following fibred space, which is somewhat similar to the one described in Example 1.20 and should be regarded as being imbedded into $I\!R^2$.

Example 3.7 *Let (Y, X, π) be defined by setting $X = \{1/2^s\}_{s \in I\!N \cup \{\infty\}}$ and $Y = \cup_{x \in X} Y_x$, where $Y_x = \{(x, 0), (x, x)\}$. Define on this space a potential φ by $\varphi(x, 0) = 0$ and $\varphi(1/2^s, 1/2^s) = 1/s$ for all s.*

The potential φ is fibrewise locally constant since the fibres are discrete spaces. Moreover, in the fibre $Y_{1/2^s}$, we have that $Y_{1/2^s}(0) = \{(1/2^s, 0)\}$ and $Y_{1/2^s}(1/s) = \{(1/2^s, 1/2^s)\}$, whereas $Y_{1/2^s}(r) = \emptyset$ for all $r \in I\!R \backslash \{0, 1/s\}$. Hence we have that $d_Y(Y_{1/2^s}(0), Y_{1/2^s}(1/s)) = 1/2^s$, from which it follows that $d_{\text{crit}} = 0$. Therefore φ is a globally continuous, fibrewise locally constant potential which is not uniformly fibrewise locally constant.

Remark also that φ is fibrewise Hölder continuous (in fact, fibrewise Lipschitz continuous), though not uniformly fibrewise Hölder continuous with respect to the usual metric of $I\!R^2$.

On the other hand, note that a globally continuous, uniformly fibrewise locally constant potential on a compact fibred space is uniformly fibrewise Hölder continuous with respect to any compatible metric. We will come back to this fact in the proof of the forthcoming theorem.

A uniformly fibrewise locally constant, globally continuous potential on a compact fibred space only admits a finite number of values on each fibre. Indeed, if it were assuming infinitely many distinct values on a fibre Y_x, there would exist sequences $(y_n)_{n\in I\!N}$ of distinct points in Y_x and $(r_n)_{n\in I\!N}$ of distinct values in $I\!R$ such that $\varphi(y_n) = r_n$, $y_n \to y$ for some $y \in Y_x$ and $r_n \to r$ for some $r \in I\!R$ when $n \to \infty$. This would then imply that φ is not constant on any neighbourhood of y. Hence, the inmost infimum in the definition of d_{crit} can be replaced, in the compact case, by a minimum. Moreover, remark that the fibre level sets $Y_x(r)$ are closed.

Note also that the uniform fibrewise local constancy is invariant under a uniformly topologically equivalent change of metric, even though the value of d_{crit} generally depends on the metric considered.

Existence Theorem 3.8 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a compact fibred system. Suppose that it is fibrewise expansive and uniformly fibrewise open, that its fibres are continuous in the Hausdorff metric and that it is topologically exact along those latter. Then for every globally continuous, uniformly fibrewise locally constant potential φ, there exist a unique Gibbs family $\{\mu_x^\varphi\}_{x\in X}$ for φ and a unique function A_φ that satisfy Definition 3.1. Moreover, supp $\mu_x^\varphi = Y_x$.*

Proof. As before, there exists a topologically equivalent metric ρ under which the system is fibrewise Ruelle expanding and topologically exact along fibres. Now, let $\varphi : Y \to I\!R$ be a globally continuous, uniformly fibrewise locally constant potential and let us show that φ is a uniformly fibrewise Hölder continuous function with respect to ρ (in fact, under any compatible metric, following the observation that the uniform fibrewise local constancy of the function is independent of the metric imposed on the space as long as it is compatible with the topology). For this, let $x \in X$ and $y, y' \in Y_x$. If, on the one hand, $y, y' \in Y_x(r)$ for some $r \in I\!R$, then $\varphi(y) = \varphi(y')$ and the Hölder property is trivially fulfilled. If, on the other hand, $y \in Y_x(r)$ and $y' \in Y_x(r')$ for some $r, r' \in I\!R, r \neq r'$, then

$$|\varphi(y) - \varphi(y')| \leq 2\|\varphi\| = \frac{2\|\varphi\|}{d_{\text{crit}}} d_{\text{crit}} \leq \frac{2\|\varphi\|}{d_{\text{crit}}} \rho\big(Y_x(r), Y_x(r')\big) \leq \frac{2\|\varphi\|}{d_{\text{crit}}} \rho(y, y'),$$

where the value of d_{crit} was calculated using the metric ρ. Hence φ is a uniform fibrewise Lipschitz function under the metric ρ. Therefore we can apply Theorem 3.2 to obtain the result. □

Continuity Theorem 3.9 *Let $\{\mu_x^\varphi\}_{x\in X}$ and (φ, A_φ) be as in the Existence Theorem 3.8. Assume that S and π are open maps and that i_1 is a local homeomorphism. Then the Gibbs family $\{\mu_x^\varphi\}_{x\in X}$ and the function $A_\varphi : X \to I\!R$ are continuous.*

Proof. The result follows directly from our Existence Theorem 3.8 and Denker and Gordin's Continuity Theorem 3.3. □

Finally, we have:

Hölder Continuity Theorem 3.10 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$, $\{\mu_x^\varphi\}_{x \in X}$ and (φ, A_φ) be as in the Continuity Theorem 3.9. Let T be a locally non-contracting open map and let condition (RH) be satisfied. Moreover, assume that S is an expansive, Lipschitz continuous and locally non-contracting map. Then for any $\gamma \in (0, 1]$, there exist $\eta(\gamma) \in (0, 1]$ and $C_\gamma > 0$ with the following property:*
If f is (D, γ)-Hölder continuous, then

$$x \to \int_Y f(y) d\mu_x^\varphi(y)$$

is $(C_\gamma D, \eta(\gamma))$-Hölder continuous. In particular, the function $A_\varphi : X \to \mathbb{R}$ is Hölder continuous.

Proof. The map T is by assumption open. Moreover, since S is expansive, S is locally injective. It therefore follows from Theorem 2.19 that the system is uniformly fibrewise open. Remark also that we do not, contrarily to Denker and Gordin, require that the map π be a contraction since this is ensured by the change of metric. The result is a direct consequence of our Continuity Theorem 3.9 and Denker and Gordin's Hölder Continuity Theorem 3.4. □

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