The Bowen - Series Map for some free groups

Dissertation
zur Erlangung des Doktorgrades
der Mathematisch - Naturwissenschaftlichen Fakultäten
der Georg - August - Universität zu Göttingen

vorgelegt von
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aus Karlsruhe

Göttingen 2002

Acknowledgements

First of all, I want to express my gratitude to Prof. Manfred Denker for help and fruitful mathematical discussion. I would further like to thank Prof. Samuel J. Patterson, Dr. Bernd O. Stratmann and Dr. Kurt Falk for drawing my attention to many concepts of hyperbolic geometry. There are many others to be mentioned: I would like to express my thanks to my colleagues for a nice atmosphere in the institute. Last but not least, I am obliged to my family and my friends for all the support they gave me during the last years. Finally, I would like to mention, that I was sponsored by a scholarship of the graduate school "Gruppen und Geometrie" while I was doing my doctorate.

Contents

1 Introduction

The coding of the geodesics, i.e. the representation as infinite words in some alphabet, of a surface M of constant negative curvature was studied over a long time. Several authors (Morse 1921, Artin 1965, Hedlund 1934) developed methods to code the geodesics for some special cases. Morse defined so called cutting sequences. This means that a geodesic is coded by the sequence in which it cuts a fixed set of curves on the surface where these curves are chosen to be projections of the sides of some fundamental polygon in the universal cover. This approach is closely related to the dynamics of the geodesic flow on M. It presupposes the representation of the nonwandering set of the geodesic flow as special (or suspension) flow. Exactly this was done by Adler & Flatto for M compact (cf. [AF]) and by the author in a more general setup in this work (cf. (3.8)). This gives an explicit construction (which will be called canonical in the sequel) with the aim to find dynamical properties of the flow. Hence, this is in some sense the opposite of well known results of Bowen for hyperbolic flows (cf. [Bo]) or the result of Ambrose & Kakutani (cf. [AK]), that any conservative flow admits such a representation.
The other approach, which was done by Artin, is the so called boundary expansion. Here the coding sequence of a geodesic is given by coding the endpoints at infinity of some lift of this geodesic. For example, if M is the modular surface, the boundary expansion reduces to the continued fraction expansion on $\mathbb{R} \cup \{\infty\}$. An overview of those methods can be found in [Se]. In addition, Series proved that there is a correspondence between these two methods: there is a bijection between the different codings commuting with the shift.
The main motivation for this work is the paper of Bowen & Series "Markov maps associated to Fuchsian group" ([BS]). In contrast to the two methods described before, they associated a Markov map to the group G of the Fuchsian model $\mathbb{B}/_G$ of M, where M is assumed to be of finite hyperbolic area. This Markov map T is defined as a transformation of $\partial\mathbb{B} = S^1$ (here $\mathbb{B}$ denotes the disc model of the hyperbolic plane). They used the local differentiable structure of T to show:

- If G is cocompact, T is Markov with respect to a finite partition α of $\partial\mathbb{B}$. In addition, $|DT^n(x)| > \delta > 1$ for all $n \geq 2$ and $\sup_{x\in\partial\mathbb{B}} |D^2Tx|/|DTx|^2 < \infty$ for Lebesgue a.e. x. By using these two estimates they could show that T is ergodic. Nowadays, this situation is called eventually expanding and C^2 - Markov. By these properties, it is standard to derive the Gibbs - Markov property and the existence of a finite invariant measure which is equivalent to Lebesgue measure.

- If G is not cocompact, Bowen & Series showed, that T is Markov with respect to a infinite partition α. In this case, there is some K being the union of finitely many atoms of α, such that the induced transformation T_K has the properties described above. By the ergodicity of T_K, the ergodicity of T follows.

But this approach has a priori no connection with the dynamics of the flow resp. the geometry of $\mathbb{H}/_G$. The connection mentioned in this paper is the so called orbit equivalence: $gx = y$ for some $g \in G \iff \exists\, n, m > 0$ with $T^n x = T^m y$. From this property, Bowen

& Series derive the ergodicity of the geodesic flow. It has to be pointed out that they did not prove these assertions for arbitrary cofinite G. By quasiconformal deformation they achieved a group with a ford domain with the extra property, that $G(\partial P)$ is the union of complete geodesics. By the ford property the eventual expandingness follows and this extra property gives the Markov property. Using the same method of quasiconformal conjugation, Adler & Flatto gave a geometrical interpretation of the Bowen - Series map T. They showed for G cocompact that T is a factor of some S which is measuretheoretical isomorphic to the canonical section for the flow (i.e. the geodesic flow is representable as a special flow over S). In addition, S is shown to be the natural extension of T. Another kind of geometrical interpretation of T is given by Series ([Se]). She identified T with the one sided shift given by the canonical factor of the two sided shift defined on the boundary expansion.
It has to be mentioned that a quasiconformal deformation gives a homeomorphism between $\mathbb{H}/_{G}$ for given G and $\mathbb{H}/_{G'}$ where G' is the special model used in [BS] and [AF]. But there is no measuretheoretical equivalence between the Liouiville measures on the corresponding surfaces in general (this is only the case if the deformation is given by the conjugation with an isometry). Here the papers of Rees ([Re1], [Re2]) have to be mentioned. She developed criteria for the ergodicity of the flow if G is a normal subgroup of a Fuchsian group, which is either cocompact ([Re1]) or cofinite and not cocompact ([Re2])).
This is the context in which this work has to be put in. As in the papers of Rees, no quasiconformal deformation is used here. First of all, assume in the sequel that G is of first kind but not necessarily finitely generated. Then the Liouiville measure is the natural measure on the sphere bundle[1]. By arguments similar to [AF], it is shown (cf. proposition (3.8)) that the geodesic flow is representable Liouiville a.e. as special flow over the canonical section. Therefore, a condition called coding assumption resp. (CA) is necessary to ensure that the set corresponding to the vertices of P has measure zero. As it was shown in proposition (3.14), this is in the geometrical finite case equivalent to G being of first kind resp. equivalent to G being cofinite. For the the geometrical infinite case, it is shown that this condition is stronger than first kindness but weaker than ergodicity of the flow.
In order to define the Bowen - Series map, an additional property is introduced. This property (GC) states that there is a fundamental polygon for G whose sides are complete geodesics. While writing this thesis, there was some discussion about this condition (GC) in the cofinite case. It was claimed e.g. in [Re2] that any hyperbolic surface of finite type with cusps has this property. If this would be true, the geometrical meaning of the Bowen - Series map would be understood for all cofinite groups (which was the motivation for this definition). But as the author could not find a reference, this is left open.
Under (CA) and (GC) the author was able to prove a result corresponding to [AF]: Proposition (4.2) states that under these assumptions the Bowen - Series map T is a factor of the canonical section S. But in contrast to the compact case both maps S and T are shown to be infinite measure preserving. By theorem (4.7), T is a topologically mixing, infinite

[1] In the case that G is not necessarily of first kind but finitely generated, this would be the flow invariant measure given by the unique G - invariant conformal density on the limit set introduced by Patterson.

measure preserving Markov map.
If G is cofinite with (GC), then theorem (5.14) states that the inducing on some set of finite measure A gives the Gibbs - Markov property which is the analog for the result obtained by [BS]. In addition, ergodic properties of T itself are described. Theorem (5.19) states that T is pointwise dual ergodic with return sequence proportional to $n/\log(n)$. Hence, the canonical section is shown to be rationally ergodic with the same return sequence as T. Via T_A it is possible to define another section which is a finite measure preserving, conservative and ergodic section and has T_A as factor. In comparison to the result in [Bo] for the compact case, where the section is shown to be a finite measure preserving Markov map with respect to a finite partition, these results can be interpreted as follows: either the section is an infinite measure preserving Markov map with respect to a finite partition or a finite measure preserving Markov map with respect to an infinite partition. Hence, in contrast to the compact case, where it is possible to represent the geodesic flow as the suspension over a Markov shift in finitely many states, here this is possible only for a shift in countably many states.
As an immediate application of these results, it should be possible to determine the Poincaré series of an abelian cover of a cofinite subgroup using the methods developed by Aaronson & Denker in [AD].

2 Hyperbolic Geometry

In this section, basic properties of hyperbolic geometry and the geodesic flow will be described. For a reference of the described results, see e.g. [Ra] and [Kat]. One of the standard models of the hyperbolic plane is the ball model $\mathbb{B} := \{z \in \mathbb{C} \ \ |z| < 1\}$ with arc length and area given by

$$ds(z) = \frac{2|dz|}{1-|z|^2} \text{ and } dA(z) = \frac{4dz}{(1-|z|^2)^2},$$

where $|z|$ denotes the euclidean norm of $z \in \mathbb{C}$. As a fact, any two points $z_1, z_2 \in \mathbb{B}$ can be joined by a curve $\gamma : [a,b] \to \mathbb{B}$ such that γ is an isometry, i.e. $|x-y| = d_{\mathbb{B}}(\gamma(x), \gamma(y)) \ \forall x, y \in [a,b]$. This curve can be uniquely extended to a curve $\gamma' : \mathbb{R} \to \mathbb{B}$, which is again an isometry. In the following, these curves joining two points as described will be called *geodesic arcs* and their extensions *geodesic lines* or just *geodesics*. To make notation easier one can generalize the notion of an endpoint of a geodesic arc resp. line, if one does not distinguish if this point is in $\mathbb{B}$ or $\partial\mathbb{B}$. This means: The set of endpoints of $\gamma : [a,b] \to \mathbb{B}$ is the set $\{\gamma(a), \gamma(b)\}$, the set of endpoints of $\gamma : \mathbb{R} \to \mathbb{B}$ is the set $\{\lim_{t\to-\infty}\gamma(t), \lim_{t\to\infty}\gamma(t)\}$. Conversely define for two points $a, b \in \mathrm{Clos}(\mathbb{B})$ the directed geodesic from a to b by $\gamma_{a,b}$.

2.1 The Geodesic Flow on $\mathbb{B}$

The geodesic flow is a flow acting on the sphere bundle $T^1\mathbb{B}$. To define the flow, it is useful to use the following representation of $T^1\mathbb{B}$: Let $\eta, \xi \in S^1 = \partial\mathbb{B}, \eta \neq \xi$. Let $\gamma_{\eta,\xi}$ be the directed geodesic from ξ to η (i.e. $\xi = \lim_{t\to-\infty}\gamma(t)$, $\eta = \lim_{t\to\infty}\gamma(t)$) with the additional property that $\gamma(0)$ is the unique point in $\mathbb{B}$ where the euclidean distances $d_E(\eta, \gamma(0))$ and $d_E(\xi, \gamma(0))$ are equal.
Define $\mathcal{X}_{\mathbb{B}} := ((S^1)^2 \backslash \Delta) \times \mathbb{R}$ with $\Delta = \{(\eta, \xi) \in (S^1)^2 \mid \eta \neq \xi, \}$. Then the mapping Φ from $\mathcal{X}_{\mathbb{B}}$ to $\mathbb{B} \times S^1$, which is by definition $T^1\mathbb{B}$, given by

$$\begin{array}{rcl} \Phi : \mathcal{X}_{\mathbb{B}} & \to & \mathbb{B} \times S^1 \\ (\eta, \xi, t) & \mapsto & \gamma_{\eta,\xi}(t), \arg\gamma'_{\eta,\xi}(t)) \end{array}$$

is a diffeomorphism. Now, the flow is defined as :

Definition 2.1

$$\begin{array}{rcl} \varphi_t : \mathcal{X}_{\mathbb{B}} & \to & \mathcal{X}_{\mathbb{B}} \\ (\eta, \xi, s) & \mapsto & (\eta, \xi, s+t) \end{array}$$

is the geodesic flow on $T^1\mathbb{B}$.

Theorem 2.2 (Liouiville) The Liouiville - measure $dm_L = dAd\theta$ on $T^1\mathbb{B}$ is flow invariant. With respect to the representation $\mathcal{X}_{\mathbb{B}}$ the measure is given by

$$dm_L(\eta, \xi, t) = \frac{2|d\eta||d\xi|dt}{|\eta - \xi|^2}$$

Proof: cf. [Ho] for the invariance, cf. [Aa], [AF] for the representation on $\mathcal{X}_{\mathbb{B}}$.

2.2 Geometry of Discrete Groups

As it is well known, the full group of orientation preserving isometries of $\mathbb{B}$ is the Moebius group $\mathrm{Moeb}^+(\mathbb{B})$ on $\mathbb{B}$ (cf. [Ra]). A discrete subgroup G of $\mathrm{Iso}^+(\mathbb{B})$ is called *Fuchsian group.* If G is in addition torsionfree, then the quotient map

$$p : \mathbb{B} \to \mathbb{B}/_{G}$$

is a local diffeomorphism and $\mathbb{B}/_{G}$ is a hyperbolic manifold with respect to the metric induced by p. Now, by relating properties of G (for G torsionfree) with properties of $\mathbb{B}/_{G}$, one gets the following definitions:

Definition 2.3

- G is called *cocompact* if $\mathbb{B}/_{G}$ is compact.
- G is called *cofinite* if $\mathrm{Area}(\mathbb{B}/_{G})$ is finite.
- G is called of *first kind* if the limit set

  $$\Omega(G) := \{z \in \partial B \mid z \text{ is an accumulation point of } G(0)\}$$

  is dense in S^1, otherwise, G is called of second kind.

For the further understanding of the action of a Fuchsian Group on B and the geometry of $\mathbb{B}/_{G}$, the approach via an exact fundamental polygon is standard. As there are many ways to define polygons, the definitions used here are mentioned (cf. [Ra], section 6.2 and 6.3):

Definition 2.4

- $C \subset \mathbb{B}$ is *(hyperbolically) convex* if and only if for each pair of distinct points x, y the geodesic arc from x to y is contained in C.
- A *side* of a convex set $C \subset \mathbb{B}$ is a nonempty, maximal, convex subset of ∂C
- A *convex polygon* is a nonempty, closed, convex subset of $\mathbb{B}$, such that the collection of sides is locally finite[2].

[2]Locally finiteness of a collection of sets $\mathcal{S}$ is defined as follows: every point in $z \in \mathbb{B}$ has a neighbourhood, which has nonempty intersection only with finitely many members of $\mathcal{S}$

Looking at the G - images of a convex polygon P, it is sometimes useful to represent P as intersection of half - planes, where a half - plane H_γ is defined as the closure of one of the components of $\mathbb{B}\backslash\gamma$ for some geodesic line γ. This is given by:

Proposition 2.5 Let P be a two dimensional convex polygon unequal to $\mathbb{B}$ with set of sides $\mathcal{S}$. For each side s of $\mathbb{B}$, let H_s be the closed half - space such that $\partial H_s \supset s$ and $P \subset H_s$. Then:

$$P = \bigcap_{s\in\mathcal{S}} H_s$$

Proof: see [Ra], theorem 6.3.2

Definition 2.6 Assume G is a discrete, torsionfree subgroup of $\mathrm{Iso}^+(\mathbb{B})$. Then

- A *fundamental region* R for G is an open set with $G\overline{R} = \mathbb{B}$ and $g(R) \cap h(R) = \emptyset \iff g \neq h$.
- A *(convex) fundamental polygon* P for G is a convex polygon P, whose interior is fundamental domain and the collection $\{g(\mathrm{Int}(P)) \mid g \in G\}$ is locally finite.
- A fundamental polygon P is *exact* if for each side s, there is an element $g_s \in G$ with $s = P \cap g_s(P)$.

As the Dirichlet region is an exact fundamental polygon [Ra], there exists for any G Fuchsian and torsionfree a fundamental polygon, which is exact. By exactness, each side s of the set of sides $\mathcal{S}$ is mapped via g_s to another side s' and $g_s^{-1} = g_{s'}$. This gives an involution on the set of sides $' : \mathcal{S} \to \mathcal{S}, s \mapsto s'$, called *side pairing*. Now, by Poincaré's theorem, the set $\{g_s \mid s \in \mathcal{S}\}$ is generating G and all relations between the generators are given by the so called edge cycles and the side pairing relation $g_s^{-1} = g_{s'}$.

2.3 The Geodesic Flow on $\mathbb{B}/_G$

To define the geodesic flow on $\mathbb{B}/_G$, one has to define the action[3] of an element g of $\mathrm{Iso}^+(\mathbb{B})$ on $T^1\mathbb{B}$:

$$g(x,\theta) = (gx, \theta + \arg(g'(x)))$$

Calculation shows that this action and the flow are commuting, i.e. $g \circ \varphi_t = \varphi_t \circ g$ $\forall g \in \mathrm{Iso}^+(\mathbb{B}), t \in \mathbb{R}$. Thus, the flow on $\mathbb{B}/_G$ for G torsionfree is given by the commuting diagram

$$\begin{tikzcd}
\mathbb{B} \times S^1 \arrow[r, "\varphi_t"] \arrow[d, "\tilde{p}"] & \mathbb{B} \times S^1 \arrow[d, "\tilde{p}"] \\
\mathbb{B} \times S^1/_G \arrow[r, "\tilde{\varphi}_t"] & \mathbb{B} \times S^1/_G
\end{tikzcd}$$

[3] To be more sophisticated: one can define a group structure on $T^1\mathbb{B}$, such that the mapping $\mathrm{Iso}^+(\mathbb{B}) \to T^1\mathbb{B}, g \mapsto (g(0), \arg(g'(0))$ is a group isomorphism. Now, the flow can be defined as left multiplication by a one parameter subgroup.

Now, if P is a fundamental polygon for G, it is well known that the set

$$P_\varphi := \{(x,\theta) \in P \times S^1 \mid \mathrm{pr}_1 \circ \varphi_t(x,\theta) \in P \text{ for } t \text{ sufficiently small } \}$$

is a fundamental domain for the action of G on $T^1\mathbb{B}$ and that

$$(T^1\mathbb{B}/_G, \mathcal{B}, m_L, \varphi_t) \cong (P_\varphi/\sim, \mathcal{B}, m_L|_{P_\varphi}, \varphi_t^*)$$

is an isomorphism, where $\sim$ is induced by the action of the group, $m_L|_{P_\varphi}$ is the to P_φ restricted Liouiville measure and φ_t^* is given by $\varphi_t^*(z,\theta) = g_{t,z,\theta}(\varphi_t(z,\theta))$ with $g_{t,z,\theta} \in G$ is unique (mod m_L) by $g_{t,z,\theta} \in P_\varphi$.

2.4 The Upper Half Space $\mathbb{U}$

There is another standard model in hyperbolic 2 – geometry, the so called *Upper Half Space model*

$$\mathbb{U} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

equipped with arc length and area given by

$$ds(z) = \frac{|dz|}{\mathrm{Im}(z)} \quad \text{and} \quad dA(z) = \frac{dz}{(\,\mathrm{Im}(z))^2}.$$

As the map $\vartheta : \mathbb{B} \to \mathbb{U}$, $z \mapsto \frac{(1+i)z+1+i}{-(1-i)z+1-i}$ is with regard to these definitions distance and area preserving, one does not have to distinguish between those models. If there is no confusion, $\mathbb{H}$ will be used as notation for the hyperbolic 2 – plane. Translating the statements of this section to the new model, one gets:

$$\begin{aligned} dm_L &= dAd\theta \text{ as Liouiville - measure on } T^1\mathbb{U} \\ \mathcal{X}_{\mathbb{U}} &= (\mathbb{R} \cup \{\infty\})^2 \backslash \Delta \times \mathbb{R} \text{ as alternative representation of } T^1\mathbb{U} \\ dm_L &= \frac{2d\xi d\eta ds}{(\xi - \eta)^2} \text{ as Liouiville measure on } \mathcal{X}_{\mathbb{U}} \end{aligned}$$

3 The Special Flow Representation of the Flow

The aim of this section is to use the product structure of Liouiville measure to represent the geodesic flow on $\mathbb{H}/_G$ as special (or suspension) flow over some invertible transformation $S : Y \to Y$. This approach is closely related to code the flow via "cutting sequences" (cf. [Se]) and uses ideas of [AF]. It will be discussed whether this is possible by this special method: if $\mathbb{H}/_G$ is geometrically finite, this is equivalent to G being of first kind. But in the infinite case, this is not true at all. Hence, the possibility of such a representation gives another characteristic for these groups.

3.1 Special Flows

Definition 3.1 Suppose T is a nonsingular, invertible, measure – preserving transformation of the standard, σ – finite measure space $(X, \mathcal{B}, m)$ and that $h : X \to \mathbb{R}_+$ is measurable. Then the *special flow* over T with height function h is defined by:

$$
\begin{array}{rcl}
X_h & := & \{(x,y) \mid x \in X, 0 \leq y < h(x)\} \\
 & \text{by} & \\
\varphi_t^{X_h}(x,y) & := & (T^n x, y + t - h_n(x)) \text{ with n } \in \mathbb{Z} \text{ given by} \\
h_n(x) & \leq & y + z < h_{n+1}(x) \text{ where} \\
h_n(x) & := & \left\{ \begin{array}{lcl} 0 & : & n = 0 \\ \sum_{k=0}^{n-1} h(T^k(x)) & : & n \geq 1 \\ -h_n(T^n(x)) & : & n < 0 \end{array} \right.
\end{array}
$$

with the product measure $\mu := m \times \lambda$ and corresponding σ - algebra $\mathcal{B}_h$, where λ is the Lebesgue measure. In this context, T resp. $(X, \mathcal{B}, m, T)$ is called a *section* for $\varphi_t^{X_h}$.

There is a close connection between the section map and the flow (cf. [HIK]):

Proposition 3.2

1. $\varphi_t^{X_h}$ is ergodic $\iff$ T is ergodic.
2. $\varphi_t^{X_h}$ is measure preserving $\iff$ T is measure preserving.

In the case that the flow is measure preserving and there is a set $A \in \mathcal{B}$ in the section so that the induced transformation is well defined, it is sometimes possible to get a further section for the flow:

Definition 3.3 Let T be a nonsingular transformation of the σ – finite measure space $(X, \mathcal{B}, m)$. The *return time* of T on $A \in \mathcal{B}$ is

$$
\phi_A : X \to \{1, 2 \ldots\} \cup \{\infty\} \ , \ x \mapsto \left\{ \begin{array}{lcl} \min\{n : T^n(x) \in A\} & : & \text{if it exists} \\ \infty & : & \text{else} \end{array} \right.
$$

If $\phi_A(x) < \infty$ a.e. on A (e.g. if T is conservative), then

$$T_A : (A, \mathcal{B} \cap A, m|_A) \to (A, \mathcal{B} \cap A, m|_A)\ ,\ x \mapsto T^{\phi_A(x)}(x)$$

is the *induced transformation* of T on A.

The next theorem is folklore:

Proposition 3.4 Let $(X, \mathcal{B}, m, T)$ be a measure preserving section of the special flow on X_h. Let $A \in \mathcal{B}$ be a set with $\phi_A(x) < \infty$ a.e. and $\bigcup_{n\in\mathbb{N}} T^n(A) = X \bmod m$. Then the induced transformation $(A, \mathcal{B}\cap A, m|_A.T_A)$ is also a section for the flow with height function

$$h_A(x) := \sum_{k=0}^{\phi_A(x)-1} h \circ T^k(x)$$

3.2 The Special Flow Representation

As it was mentioned before, the next step is to decide whether it is possible to find a special flow representation for the geodesic flow on $\mathbb{H}/_G$ with G Fuchsian and torsionfree. The outline for that is to construct a section Y via the boundary of some fundamental polygon and the corresponding identifications by G. In contrast to the result of Ambrose and Kakutani, that any measure preserving conservative flow admits a section, such a section will be defined explicitly and will be used to decide whether the flow is conservative and ergodic or not.
As some elements of $T^1\mathbb{H}/_G$ may not be covered by this method one has to look if the corresponding set is of zero measure which leads to the following definition:

Definition 3.5 (CA) Let G be a torsionfree Fuchsian group. If there exists an exact fundamental polygon P, such that

$$A_P := \left\{(z, \theta) \in P \times S^1 \mid \exists\, t \in \mathbb{R} \cup \{\pm\infty\} \text{ such that } \gamma_{z,\theta}(t) \in GV\right\}$$

is a set of Liouiville - measure zero, where

$$V_P := \{ \text{ set of vertices of } P \text{ in } \mathbb{H} \ \} \cup (\mathrm{Clos}_{\overline{\mathbb{H}}}(P) \cap \partial\mathbb{H}))\,,$$

then G resp. the pair (G, P) is said to fulfill the *coding assumption*, abbreviated by **(CA)**.

Remark 3.6 By definition, the set A_P is invariant with respect to the flow and to the action of G. In addition, $A_P \neq \mathbb{H} \bmod m_L$. Hence, $m_L(A_P) > 0$ forces the flow to be not ergodic. A further description of this condition will be done in the next subsection.

If G is **(CA)**, then as noted before:

$$P_\phi := \{(x,\theta) \in P \times S^1 \mid \mathrm{pr}_1 \circ \phi_t(x,\theta) \in P \text{ for } t \text{ sufficiently small } \}$$

is a fundamental domain for the action of G on $T^1\mathbb{H}$ and $T^1\mathbb{H}/_G \cong P_\phi/\sim$ in the category of measure preserving flows (cf. section 2.3). Hence, by **(CA)** and the fact, that A_P is invariant with respect to the G - action, there is the following (measure theoretical) equivalence:

$$T^1\mathbb{H}/_G \cong (P_\phi \backslash A_P)/\sim$$

Let $\tilde{P}_\phi$ resp. $\tilde{A}_P$ be the sets P_ϕ resp. A_P in the $\mathcal{X}_\mathbb{H}$ - representation and define:

Definition 3.7 (ξ,η) is P - *admissible* iff $\exists\, t \in \mathbb{R}$, such that $\gamma_{\xi,\eta}(t) \in (P)$. If (ξ,η) is P – admissible, then

$$\begin{aligned} t^+_{\xi,\eta} &:= \sup\{t \mid \gamma_{\xi,\eta}(t) \in P\} \leq \infty \\ t^-_{\xi,\eta} &:= \inf\{t \mid \gamma_{\xi,\eta}(t) \in P\} \geq -\infty \\ h(\xi,\eta) &:= t^+_{\xi,\eta} - t^-_{\xi,\eta} \end{aligned}$$

As P is convex, $[t^-_{\xi,\eta}, t^+_{\xi,\eta}]$ and $\{t \mid \gamma_{\xi,\eta}(t) \in P\}$ coincide. Now the coding assumption allows to find an explicit special flow representation (cf. [AF]):

Proposition 3.8 Assume G is **(CA)**. Then $\left(T^1\mathbb{H}/_G, \mathcal{B}, m_L, \phi_t\right)$ and the special flow over $(Y,\mathcal{B},m,S)$ with height function h are isomorphic in the category of measure preserving flows, where $(Y,\mathcal{B},m,S)$ is defined as follows:

- $Y = \{(\xi,\eta) \in (\partial\mathbb{H})^2 \backslash \Delta \mid (\xi,\eta) \text{ admissible}\}$
- $dm = 2|d\xi||d\eta|/|\xi-\eta|^2$ with respect to the disc model $\mathbb{B}$ and $dm = 2d\xi d\eta/(\xi-\eta)^2$ with respect to $\mathbb{U}$, defined on the Borel σ - field $\mathcal{B}$
- S is defined piecewise: by **(CA)**, for a.e. $(\xi,\eta) \in Y$, $\gamma_{\xi,\eta}(t^+_{\xi,\eta})$ is element of some side s of P. Let g_s be the corresponding side - pairing. Then $S(\xi,\eta) = (g_s(\xi), g_s(\eta))$.

Proof: First of all, the conditions on S have to be checked:
S is defined a.e., as by **(CA)** m - a.e. $(\xi,\eta) \in Y$, $\gamma_{\xi,\eta}(t^+_{\xi,\eta})$ is an element of a side s of P. Assume $\gamma_{\xi,\eta}(t^+_{\xi,\eta}) \in s$. Then $g_s(\gamma_{\xi,\eta}(t^+_{\xi,\eta}))$ is an element of the side $g_s(s) = s'$ and is equal to $\gamma_{g_s\xi,g_s\eta}(t^-_{g_s\xi,g_s\eta})$ (cf. figure (1)). Hence, $S(\xi,\eta) = (g_s\xi, g_s\eta)$ is m - a.e. admissible . In addition, as $\gamma_{g_s\xi,g_s\eta}(t^-_{g_s\xi,g_s\eta}) = \gamma_{g_s\eta,g_s\xi}(t^+_{g_s\eta,g_s\xi})$ and $g_{s'} = g_s^{-1}$, S is invertible.
Now fix the disc model $\mathbb{B}$ and define $\tilde{A}$ to be the corresponding subset of $\mathcal{X}_\mathbb{B}$ for any set $A \subset \mathbb{B} \times S^1$. Then

$$\begin{aligned} \tilde{P}_\phi &= \left\{(\xi,\eta,t) \mid (\xi,\eta) \text{ is } P \text{ - admissible}, t^-_{\xi,\eta} \leq t < t^+_{\xi,\eta}\right\} \\ &\overset{m_L}{=} \left\{(\xi,\eta,t) \mid (\xi,\eta) \text{ is } P \text{ - admissible}, t^-_{\xi,\eta} \leq t < t^+_{\xi,\eta}\right\} \backslash \tilde{A}_P \\ &\overset{m_L}{=} \left\{(\xi,\eta,t) \mid (\xi,\eta) \text{ is } P \text{ - admissible}, -\infty < t^-_{\xi,\eta} \leq t < t^+_{\xi,\eta} < \infty\right\} \\ &\overset{m_L}{=} \left\{(\xi,\eta,t) \mid (\xi,\eta) \text{ is } P \text{ - admissible}, 0 \leq t - t^-_{\xi,\eta} < h(\xi,\eta) < \infty\right\} \end{aligned}$$

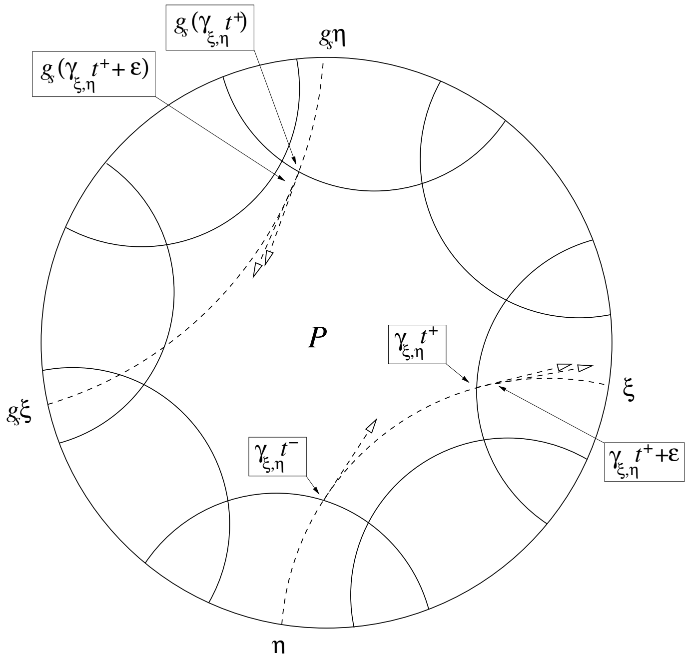

Figure 1: The section map S

Hence the mapping $(\xi, \eta, t) \mapsto (\xi, \eta, t - t^-_{\xi,\eta})$ is a measure theoretical isomorphism[4]

$$\left(\tilde{P}_\phi, \mathcal{B}, 2\frac{|d\xi||d\eta|dt}{|\xi-\eta|^2}\right) \to \left(Y_h, \mathcal{B}, 2\frac{|d\xi||d\eta|}{|\xi-\eta|^2}dt\right)$$

What is left to show is that the flows are isomorphic. This boils down to show that for $(\xi, \eta, t_0) \in Y_h$, $t \in \mathbb{R} \,:\, t + t_0 > h(\xi, \eta)$, there exists $n \in \mathbb{N}\backslash\{0\}$ and $g \in G$, such that

$$g(\gamma_{\xi,\eta}(t^-_{\xi,\eta} + t + t_0)) = \gamma_{S^n(\xi,\eta)}(t^-_{S^n(\xi,\eta)} + t + t_0 - h_n(\xi, \eta)).$$

As the tesselation GP is locally finite, it suffices to show this for $t \;:\; h(\xi,\eta) < t + t_0 < h_2(\xi, \eta)$. But this is a consequence of the side - pairing property: any side - pairing g_s is a hyperbolic isometry and

$$g_s(\gamma_{\xi,\eta}(t^+_{\xi,\eta})) = \gamma_{g_s(\xi),g_s(\eta)}(t^-_{g_s(\xi),g_s(\eta)}),$$

where s is given by $\gamma_{\xi,\eta}(t^+_{\xi,\eta}) \in s$. $\square$

Now by proposition (3.2):

[4]The measurability of this map is a consequence of the implicit function theorem: let H_s be a hyperbolic half - space with boundary s. Then, $(\xi, \eta) \mapsto \inf\{t \mid \gamma_{\xi,\eta}(t) \in H_s\}$ is a differentiable map in its domain of definition. As P is representable as a countable intersection of half - spaces (cf. proposition (2.5)), the measurability follows.

Corollary 3.9 $S : Y \to Y$ is a measure preserving transformation with respect to the measure given in proposition (3.8). In addition, S is ergodic if and only if the flow is ergodic.

3.3 Condition (CA)

The aim of this section is to characterize the groups with respect to **(CA)**. As noted before, new phenomena arise if one passes from the geometrically finite to the infinite case. But first of all, independent of the geometrical finiteness, the limit set of a group G gives a criteria for the coding assumption: a group of second kind is not of type **(CA)** which is a consequence of the following proposition ([Ra], Theorem 12.1.14):

Proposition 3.10 G is of second kind if and only if any convex fundamental polygon of P contains a closed hyperbolic half - space.

Corollary 3.11 Assume G is of second kind. Then G is not of type **(CA)** and the flow is not ergodic.

Proof: Let H be the hyperbolic half - space given by the last proposition. Define $W := \mathrm{Clos}_{\overline{\mathbb{H}}}(H) \cap \partial\mathbb{H}$. Then $(W \times W \backslash \Delta) \times \mathbb{R}$ is a flow - invariant subset of $\tilde{A}_P$ of positive measure. $\square$

Now, if $\mathbb{H}/_G$ is of finite volume, the next proposition (cf. [Ra], Theorem 9.8.1) gives the structure of some fundamental polygon:

Proposition 3.12 If G is cofinite, then there is an exact fundamental polygon P with finitely many sides $\mathcal{S}$. The set of sides can be cyclically ordered such that any two consecutive sides meet in a vertex in $\mathbb{H}$ or an ideal vertex in $\partial\mathbb{H}$.

Remark: If the set of sides is finite and any two consecutive sides meet in some vertex, the set V_P from the definition of **(CA)** is finite. This implies that A_P is the countable union $\bigcup_{z \in GV} A_z$, where

$$A_z := \{(x, \theta) \in T^1\mathbb{H} \mid \exists t \in \mathbb{R} \cup \{\pm\infty\} \text{ such that } \gamma_{x,\theta}(t) = z\}.$$

As the A_z are of Liouiville - measure zero, **(CA)** follows in the case that G is cofinite. To obtain a complete characterization, geometrically finiteness is introduced:

Definition 3.13 G is geometrically finite if and only if there is an exact fundamental polygon for G with finitely many sides.

Proposition 3.14 Assume G is geometrically finite. Then the following are equivalent:

(1) G is cofinite.

(2) G is of first kind.

(3) G has property **(CA)**.

(4) The flow on $\mathbb{H}/_G$ is ergodic.

Proof: '(1) $\Rightarrow$ (3)' is the last remark, '(1) $\Rightarrow$ (2)' is a consequence of propositions (3.12) and (3.10), '(3) $\Rightarrow$ (2)' is corollary (3.11) and '(2) $\Rightarrow$ (1)' uses the fact, that G is geometrically finite (cf. [Ra], Theorem 12.3.8).
By [Ho], the flow on $\mathbb{H}/_G$ for G cofinite is ergodic, which gives '(1) $\Rightarrow$ (4)'. And by (3.11), ergodicity is implying (2). $\square$

Now, if G is not geometrically finite, things are different. It will be shown via three examples that the set GV from definition of **(CA)** for G of first kind can be either countable, uncountable of zero Lebesgue - measure or uncountable of positive Lebesgue measure (with respect to the disc model). They will give counterexamples for a statement similar to the last one for the geometrically infinite case.

Example 3.15 In the upper half - space model, a convex polygon P is constructed such that the corresponding group is a subgroup of $\mathrm{PSL}_2\mathbb{Z}$. Let P be the polygon, whose sides are geodesics joining n with $n+1$ for $n \in \mathbb{Z}$. Then the side - pairings are defined as follows:

$$\sigma := \left(z \mapsto \frac{2-3z}{1-2z} \right) \in \mathrm{PSL}_2\mathbb{Z}$$

is sending $\gamma_{0,1}$ to $\gamma_{1,2}$ with parabolic fixed point 1. Let τ be given by $z \mapsto z+2$. Hence, $\tau^k \sigma \tau^{-k}$ maps $\gamma_{2k,2k+1}$ to $\gamma_{2k+1,2k+2}$, fixing the point $2k+1$. So the set

$$\{(s, g_s) \mid s = \gamma_{2k,2k+1}, g_s = \tau^k \sigma \tau^{-k} \text{ or } s = \gamma_{2k+1,2k+2}, g_s = \tau^{-k} \sigma^{-1} \tau^k\}$$

gives a complete system of sides of P and corresponding side - pairings. To apply Poincaré's Theorem, one has to check the vertex cycles given by the side - pairings. It turns out that there are only two possibilities: Either the cycle has infinite length or length one. In the second case, the cycle is the fixed point of some g_s (and hence fulfills the parabolic cycle condition). By Poincaré: P is the fundamental polygon of the group $G_P =< \tau^k \sigma \tau^{-k} \mid k \in \mathbb{Z} >$. Obviously $V_P = \mathbb{Z} \cup \{\infty\}$. In this case, by proposition (3.10), G_P is of first kind and by the same argument as in the remark to (3.12), property **(CA)** follows.

The next two examples rely one some Cantor set like construction. Here, the turning point is the possibility of defining a set, which is totally disconnected, uncountable and sometimes of positive Lebesgue - measure. This will lead to examples, which do not occur in the case, where the corresponding group is geometrically finite: a group G_P of first kind will be defined, where the corresponding sets V_P are uncountable resp. of positive Lebesgue measure.

Assume I_0 is the unit interval with Lebesgue - measure λ. Fix $\alpha\ :\ 0<\alpha\leq 1/3$ and define inductively:

$$\begin{aligned}
I_n &= \{\text{disjoint union of } 2^n \text{ closed intervals } I_n^1,\dots I_n^{2^n} \text{ of the same length}\}\\
B_n &= \{\text{disjoint union of } 2^n \text{ open intervals } B_n^1,\dots B_n^{2^n},\ \text{each of length } \alpha^{n+1},\\
&\quad\ \text{where each of the } B_n^i \text{ is placed in the middle of } I_n^i,\ \text{i.e. if}\\
&\quad\ I_n^i=[a,b],\ B_n^i=(\tfrac{a+b}{2}-\tfrac{\alpha^{n+1}}{2},\tfrac{a+b}{2}+\tfrac{\alpha^{n+1}}{2})\}\\
I_{n+1} &:= I_n\backslash B_n
\end{aligned}$$

To show that this is well defined, one has to check that the following holds: $\forall\ n\in\mathbb{N},1\leq i\leq 2^n$, $B_n^i\subset I_n^i$. So assume that this is true for $n-1$. Then:

$$\begin{aligned}
\lambda(I_n) &= 1-\sum_{i=0}^{n-1}2^n\alpha^{i+1}=1-\alpha\sum_{i=0}^{n-1}2^n\alpha^i\\
&= 1-\alpha\frac{1-(2\alpha)^n}{1-2\alpha}=\frac{1-3\alpha+\alpha(2\alpha)^n}{1-2\alpha}
\end{aligned}$$

As I_n resp.B_n is the disjoint union of 2^n intervals of the same length and $\lambda(B_n)=2^n\alpha^{n+1}$, it remains to show that $\lambda(I_n)/\lambda(B_n)>1$:

$$\begin{aligned}
\frac{\lambda(I_n)}{\lambda(B_n)} &= \frac{1-3\alpha+\alpha(2\alpha)^n}{(1-2\alpha)2^n\alpha^{n+1}}\\
&= \frac{1-3\alpha}{(1-2\alpha)(2\alpha)^n\alpha}+\frac{1}{1-2\alpha}
\end{aligned}$$

Hence, as for $0<\alpha\leq\frac{1}{3}$, $\frac{1-3\alpha}{1-2\alpha}\geq 0$ and $\frac{1}{1-2\alpha}>1$, the construction is well defined.

In addition, it follows that the set $I_\infty(\alpha)\equiv I_\infty:=\bigcap_{n\in\mathbb{N}}I_n$ has Lebesgue measure $\lambda(I_\infty)=\frac{1-3\alpha}{1-2\alpha}\geq 0$ as $I_n\supset I_{n+1}$. To be more precise, for $\alpha=\frac{1}{3}$, I_∞ is the normal $\frac{1}{3}$ - Cantor set with Lebesgue measure zero. Otherwise, for $0<\alpha<\frac{1}{3}$, $\lambda(I_\infty(\alpha))>0$.

Example 3.16 Use the disc model and let $\alpha=\frac{1}{3}$. If B_n^i is the interval (a,b), define for $n\in\mathbb{N},0<i\leq 2^n$:

s_n^i	is the geodesic with end points $e^{2\pi ia}$ and $e^{2\pi i\frac{a+b}{2}}$.
$\tilde{s}_n^i$	is the geodesic with end points $e^{2\pi i\frac{a+b}{2}}$ and $e^{2\pi ib}$.
$g_{n,i}$	is the unique parabolic transformation with fixed point $e^{2\pi i\frac{a+b}{2}}$, sending s_n^i to $\tilde{s}_n^i$ (the unicity is a consequence of the property, that s_n^i is the isometric circle of $g_{n,i}$).

In contrast to the last example, there are no infinite vertex cycles. Any cycle is of type $\{e^{2\pi i\frac{a+b}{2}}\}$ where $n\in\mathbb{N},0<i\leq 2^n$. Again by Poincaré, this gives a discrete group $G_P:=<g_{n,i}\mid n\in\mathbb{N},0<i\leq 2^n>$ where P is given by the half - planes with sides s_n^i resp.

$\tilde{s}_n^i$. Then:

A_P is of zero Liouiville - measure, V_P is uncountable and G_P is of first kind.

Proof: As P has no vertices in $\mathbb{B}$,

$$\begin{aligned}(V_P)^c &= \bigcup_{n\in\mathbb{N},0<i\le 2^n,(a,b)=B_n^i} \Big((e^{2\pi i a}, e^{2\pi i \frac{a+b}{2}}) \cup (e^{2\pi i \frac{a+b}{2}}, e^{2\pi i b})\Big) \\ &= \big(e^{2\pi i I_\infty} \cup \{e^{2\pi i x} \mid x \text{ is a midpoint of some } B_n^i\}\big)^c\end{aligned}$$

As the first set is uncountable, the first statement is proved. As the second set is countable, it suffices to show for the second statement that the set

$$\{(e^{2\pi i x}, e^{2\pi i y}, t) \mid x, y \in I_\infty, x \neq y, t \in \mathbb{R}\}$$

is of zero Liouiville measure. By the product structure of this measure, this can be reduced to show that $V^* := \{(e^{2\pi i x}, e^{2\pi i y}) \mid x, y \in I_\infty, x \neq y\}$ is of zero measure with respect to $2|d\xi||d\eta|/|\xi-\eta|^2$. Define for $\delta > 0$:

$$\begin{aligned}V^*(\delta) &:= \{(e^{2\pi i x}, e^{2\pi i y}) \mid x, y \in M(\delta)\} \text{ with} \\ M(\delta) &:= \{(x,y) \mid x, y \in I_\infty, |x-y| > \delta \text{ and } 1-|x-y| > \delta\}\end{aligned}$$

Now the transformation rule gives that

$$2\int_{V^*(\delta)} \frac{|d\xi||d\eta|}{|\xi-\eta|^2} = 4\pi^2 \int_{M(\delta)} \frac{dxdy}{1-\cos(2\pi|x-y|)}$$

As $\frac{1}{2} \le (1-\cos(2\pi|x-y|))^{-1} < (1-\cos(2\pi\delta))^{-1}$ on $M(\delta)$, the measure of V^* is zero for all $\delta > 0$. Now by choosing a sequence $\delta_n \downarrow 0$, the second statement follows. By proposition (3.10) the last assertion that G is of first kind follows. $\square$

Remark 3.17 The $V^*(\delta)$ - construction can be generalized to any subset of ∂B. As $A_P \cap \mathbb{B}$ is countable, the corresponding subset of V_P is of zero Liouiville - measure. Hence the following are equivalent:

- $A_P \cap \partial\mathbb{B}$ is of zero Lebesgue - measure (w.r.t. to $\partial\mathbb{B}$).
- V_P is of zero Liouiville - measure.

Example 3.18 Fix $0 < \alpha < \frac{1}{3}$ and define P resp. G_P analogous as in the last example. Then:

A_P is of positive Liouiville - measure and G_P is of first kind. As A_P is by the remark to definition (3.5) of **(CA)** flow - invariant, the flow on $\mathbb{B}/_{G_P}$ can not be ergodic.

Proof: By the same arguments and in the same notation as in the last example, it suffices to show that there is a measurable subset M' of $I_\infty \times I_\infty$ with $\int_{M'} \frac{dxdy}{1-\cos(2\pi|x-y|)} > 0$. But as $(1-\cos(2\pi|x-y|))^{-1} \geq \frac{1}{2}\ \forall x, y \in \mathbb{R}$ and $M' := ([0, \frac{1-\alpha}{2}] \times [\frac{1+\alpha}{2}, 1]) \cap (I_\infty \times I_\infty)$ is a set of positive Lebesgue measure, the statement follows. $\square$

Summarizing the results one gets:

Proposition 3.19 Assume G is not geometrically finite, then the following statements

(1) The flow on $\mathbb{H}/_G$ is ergodic.

(2) G satisfies condition **(CA)**.

(3) G is of first kind.

(4) There exists a fundamental polygon P for G with V_P countable.

have the following relations:

$$\begin{array}{ccccc} (1) & \overset{(i)}{\Longrightarrow} & (2) & \overset{(ii)}{\Longrightarrow} & (3) \\ & & \Uparrow (iii) & & \\ & & (4) & & \end{array}$$

$$\begin{array}{ccccc} (1) & \overset{(j)}{\not\Longleftarrow} & (2) & \overset{(jj)}{\not\Longleftarrow} & (3) \\ & & \not\Downarrow (jjj) & & \\ (1) & \overset{(j)}{\not\Longleftarrow} & (4) & & \end{array}$$

Proof: (i) is the remark to definition (3.5), (ii) is corollary (3.11) and (iii) is analogous to the remark to proposition (3.12) as GV_P is countable. (jj) is the last example (3.18) and (jjj) is example (3.16). Hence, what is left to show is (j): it is shown in [AD] that the flow on a $\mathbb{Z}^2$ - cover of the cofinite group $\Gamma(2)$ is not ergodic. But as for this cover (4) holds, the proposition is proved. $\square$

As there are geometrically infinite examples which are ergodic (cf. [AD], [Re1], [Re2]), it should be possible to find some new additional criteria implying (2) $\Rightarrow$ (1) resp. (4) $\Rightarrow$ (1).

4 The Bowen - Series Map

The topic of this section is to find one - dimensional Markov maps, which are associated to the section map of some special flow representation. This goes back to Bowen & Series (cf. [BS]). They constructed a Markov map $T : \partial\mathbb{B} \to \partial\mathbb{B}$ associated to a cofinite Fuchsian group G, which is orbit equivalent to the action of G on $\partial\mathbb{B}$ (i.e. $gx = y \iff \exists\, n, m > 0 : T^n x = T^m y$) and is expansive (i.e. the modulus of the derivative is bigger or equal to 1). This approach is highly dependent on the shape of some fundamental polygon P of G. Therefore, as any two Riemannian surfaces of genus g with k cusps (cf. [Ber] p. 275) are quasiconformal equivalent, they fix a model for such a surface with some extra properties. A polygon P is constructed which satisfies the assumptions of Poincaré's theorem with respect to some side - pairing in a way that the corresponding surface is of genus g with k cusps. The construction gives the following properties:

(1) Each side s of P is contained in the isometric circle of the side - pairing g_s.

(2) $G(\partial P)$ consists of complete geodesics, where G is the group defined by Poincaré's theorem

Now with respect to this model, Bowen & Series defined a transformation $T : \partial\mathbb{B} \to \partial\mathbb{B}$, which is piecewise Moebius. To avoid confusing notation T will be defined only for an example. In the situation of figure 2, $T|_{a_i} = g_i$, where g_i is the side pairing which maps s_i to some $s_{i'}$; the generalization for arbitrary cofinite G is obvious:

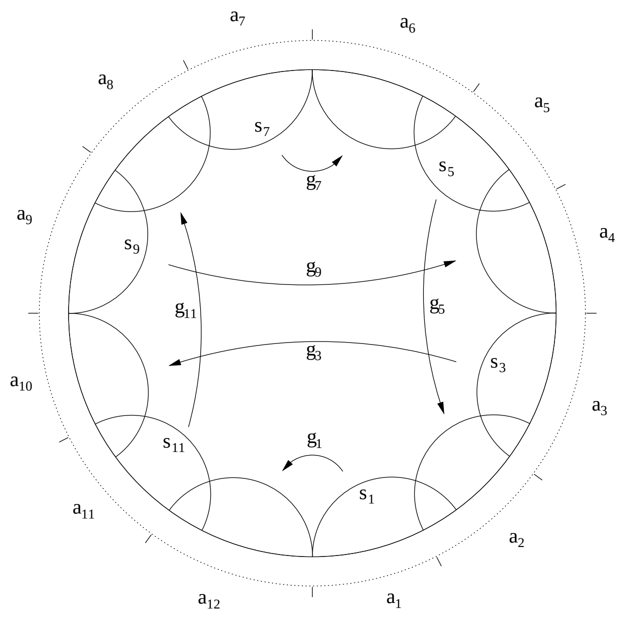

Figure 2: The Bowen - Series construction

By condition (2), it is shown that T is Markov with respect to a countable partition α. In addition, α is finite if and only if G is cocompact (resp. P has no vertices on $\partial\mathbb{B}$). The main result of Bowen & Series is as follows:

> If G is cocompact, then T^2 is expanding, i.e. there is a $\delta > 1$ with $|DT^2(x)| > \delta$ for all x in the interior of some element of α. In addition, $\sup_{x\in\partial\mathbb{B}} |D^2T(x)|/|DT(x)|^2 < \infty$. If G is not cocompact, then there is a set K which is a finite union of elements of α such that the induced transformation T_K has the same properties as in the compact case. Now by a Renyi - type result, Bowen and Series deduced that T resp. T_K admits a unique finite invariant measure, which is equivalent to Lebesgue - measure. By a mixing property of T, the ergodicity of T follows.

The property, that T and G are only linked via orbit equivalence, was improved by Adler & Flatto (cf. [AF]): they showed for G cocompact that T is a factor of an invertible map, which is conjugated to the section map defined in the last section. As in [BS], quasiconformal deformation was implicitly used to obtain a suitable fundamental polygon.

In the following, it will be shown that this factor property of T can be attained directly if G is **(CA)** and admits an exact fundamental polygon P such that the sides of P consists of complete geodesics:

Definition 4.1 G has the *complete geodesic property*, abbreviated by **(GC)**, if G admits an exact fundamental polygon P such that the sides of P consists of complete geodesics.

If G has property **(GC)**, the corresponding P is exact. Hence, P fulfills the assumptions of Poincaré's theorem. Now as P has no vertices in $\mathbb{H}$, G has to be a free group. Now with regard to the set V_P defined in (3.5) this gives:

- G is cofinite if and only if V_P is finite (cf. proposition (3.12)).
- G is **(CA)** if and only if the Lebesgue measure of V_P with respect to the disc model is zero (cf. remark (3.17)).
- G is of first kind if and only if V_P contains no interval (cf. proposition (3.10)).

For example, any subgroup of the modular group $\mathrm{PSL}_2\mathbb{Z}$ is a group with property **(GC)** (cf. [Ku],[St]).

4.1 The Bowen - Series Map

Assume for the rest of the section that G has **(GC)** and **(CA)**. Then the definition of such a group gives immediately a more explicit version of (3.8):

Proposition 4.2 The section map $(Y, \mathcal{B}, m, S)$ defined via an exact fundamental polygon P given by definition (4.1) with sides $\mathcal{S}$ is given by (cf. figure (3):

(1) $Y = \bigcup_{s\in\mathcal{S}} (a_s)^c \times a_s$

(2) $dm = 2|d\xi||d\eta|/|\xi - \eta|^2$ w. r. t. $\mathbb{B}$ and $dm = 2d\xi d\eta/(\xi - \eta)^2$ w. r. t. $\mathbb{U}$

(3) $S|_{(a_s)^c\times a_s}(\xi, \eta) = (g_s(\xi), g_s(\eta))$ m a. e..

(4) $S((a_s)^c \times a_s) = a_{s'} \times (a_{s'})^c \bmod m$.

where $a_s := \mathrm{Int}_{\partial\mathbb{H}} (\mathrm{Clos}_{\overline{\mathbb{H}}}(H_s) \cap \partial\mathbb{H})$ and H_s is the open hyperbolic half - space with $H_s \cap P = \emptyset$.

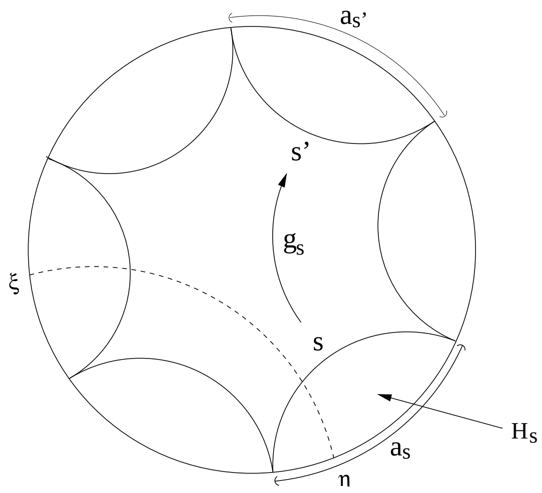

Figure 3: The section map S for a group with the factor assumption

Proof: By proposition (2.5), $P = (\bigcup_{s\in\mathcal{S}} H_s)^c$. By the definition of G, $\mathcal{S}$ consists of complete geodesics implying that $H_t \cap H_s = \emptyset$ for $t \neq s$. Hence, the geodesic $\gamma_{\xi,\eta}$ with $\xi \in a_t$ and $\eta \in a_s$ has to meet P, in particular $\gamma_{\xi,\eta}(t^+_{\xi,\eta}) \in t$ and $\gamma_{\xi,\eta}(t^-_{\xi,\eta}) \in s$. So $S(\xi, \eta) = g_s(\xi, \eta)$ which is statement (3). In addition, "$\supset$" for (1) is shown. As by convexity of H_s, $a_s \times a_s$ contains no admissible elements, $Y^c \supset \bigcup_{s\in\mathcal{S}} a_s \times a_s$. So, (1) follows by **(CA)**.
To prove (4), as g_s for $s \in \mathcal{S}$ is a homeomorphism $\partial\mathbb{H} \to \partial\mathbb{H}$ it is sufficient to show that $\mathrm{Clos}_{\mathbb{H}} g_s(H_s) = (H_s)^c$. As this is an immediate consequence of the side - pairing property of g_s, the proposition is shown. $\square$

As a consequence of this proposition, the following diagram commutes where $X = \bigcup_{s\in\mathcal{S}} a_s$, $T|_{a_s} = g_s$. In addition, for $\mu = m \circ \mathrm{pr}_2^{-1}$, T is a factor of S:

$$\begin{tikzcd} Y \arrow[r, "S"] \arrow[d, "\mathrm{pr}_2"] & Y \arrow[d, "\mathrm{pr}_2"] \\ X \arrow[r, "T"] & X \end{tikzcd}$$

Remark 4.3
(1) $X = \partial\mathbb{H}$ mod μ: as G is free, $V_P = (\bigcup_{s\in\mathcal{S}} a_s)^c$ (c.f. definition (3.5) and definition (4.1)). Now, by **(CA)**, $(V_P \times V_P) \subset Y$ is a set of zero m - measure which gives the statement. In particular, $\{a_s \mid s \in \mathcal{S}\}$ is a partition.
(2) $T(a_s) = \bigcup_{t\neq s'} a_t = a_{s'}^c$ mod μ by the last proposition.
(3) By integrating over the pr_2 - preimages, the density of μ with respect to Lebesgue measure can be calculated. Fix the upper half space model and a_s. Assume $A \subset a_s$ is measurable. Then:

$$\mu(A \times (a_s)^c) = \int_{A\times(a_s)^c} \frac{2d\xi d\eta}{(\xi-\eta)^2} = \int_A \left(\int_{(a_s)^c} \frac{2d\xi}{(\xi-\eta)^2} \right) d\eta$$

Whether ∞ is contained in a_s or ∂a_s (assume for notation, that $-\infty < a < b < \infty$), this gives:

$$\begin{array}{lcll} \text{if } a_s = (a,b) & : & d\mu = 2(\frac{1}{\eta-a} + \frac{1}{b-\eta})d\eta & \text{for } \eta \in a_s \\ \text{if } a_s = (a,\infty) & : & d\mu = 2\frac{1}{\eta-a}d\eta & \text{for } \eta \in a_s \\ \text{if } a_s = (-\infty,b) & : & d\mu = 2\frac{1}{b-\eta}d\eta & \text{for } \eta \in a_s \\ \text{if } a_s = [a,b]^c & : & d\mu = 2(\frac{1}{\eta-b} + \frac{1}{a-\eta})d\eta & \text{for } \eta \in a_s \end{array}$$

In addition, as $\mu(a_s) = \int_{a_s} d\mu = \infty$, it follows that μ is infinite and by definition T - invariant. Now fix the disc model and assume that $a_s = e^{2\pi i I}$ where I is an open interval in $\mathbb{R}$ so that $a_s \neq \partial\mathbb{B} = S^1$. Then for $A = e^{2\pi i J} \subset a_s$ with $J \subset I$, there is a closed interval I' such that $(a_s)^c = e^{2\pi i I'}$. Then the transformation rule gives:

$$\begin{aligned} \mu(A\times(a_s)^c) &= \int_{A\times(a_s)^c} \frac{2|d\xi||d\eta|}{|\xi-\eta|^2} \\ &= \int_J \int_{I'} \frac{8\pi^2 dsdt}{|e^{2\pi is} - e^{2\pi it}|^2} \\ &= \int_J \int_{I'} \frac{2\pi^2 ds}{\sin^2(\pi(s-t))} dt \end{aligned}$$

As $\frac{2\pi^2}{\sin^2(\pi(s-t))} > 2\pi^2$, it follows that the invariant density $\frac{d\mu}{d\eta}$ is bounded away from zero on I. To find an explicit representation it is sufficient to handle the case where $I = (0,a)$ and $I' = [a,1]$ where $0 < a < 1$. Then, as $\int \sin^{-2} x = -\cot x$:

$$d\mu = 2\pi(\cot(\pi(a-t)) - \cot(\pi(1-t)))dt \text{ for } t \in I$$

(4) Let g be the densitiy function given by $d\mu$. As $g(\eta) > 0\ \forall \eta \in \bigcup a_s$, μ and the Lebesgue measure λ are equivalent on $\bigcup a_s$. But as the set V_P has by **(CA)** Lebesgue measure zero (cf. remark (3.17)), the equivalence of μ and λ follows. Hence, the following is well defined:

Definition 4.4 The *Bowen - Series map*[5] associated to the group G via the exact fundamental polygon P is the transformation $(\partial\mathbb{H}, \mathcal{B}, \nu, T)$ where ν is either λ or μ.

In the following, it will be shown that the Bowen - Series map has the Markov property with respect to the partition $\alpha = \{a_s \mid s \in \mathcal{S}\}$, i.e.

(M1) $T|_{a_s}$ is one to one.

(M2) $T(a_s)$ is the union of elements of α (mod μ resp. λ.

(M3) $\sigma(\{T^{-i}\alpha \mid i \in \mathbb{N}\}) = \mathcal{B}$ mod μ resp. λ.

Therefore, the structure of the preimages of α has to be checked: assume $\{\beta \mid i \in I\}$ is a countable collection of partitions of the same space. Then $\bigvee_{i\in I}\beta_i$ is defined as the coarsest partition, which is finer than each of the β_i. Then:

Lemma 4.5

$$\begin{aligned}\alpha^{n+1} &:= \bigvee_{i=0}^{n} T^{-i}\alpha \\ &= \left\{g_{s_n}^{-1}\cdots g_{s_1}^{-1}a_{s_0} \mid s_i \in \mathcal{S} \text{ for } i = 0, \ldots n, s_i \neq s'_{i-1} \text{ for } i = 1, \ldots n\right\}\end{aligned}$$

Proof: Let $b \subset a_{s_0}$ for $a_{s_0} \in \mathcal{S}$. From the definition of T, it follows that

$$T^{-1}(b) = \bigcup_{s\in\mathcal{S}\,:\, s\neq s'_0} g_s^{-1}(b).$$

But as $T(a_s) = g_s(a_s) = \partial\mathbb{H}\backslash a_{s'} \supset a_{s_0}$ mod μ for $s \neq s'_0$, $g_s^{-1}(b)$ is a subset of a_s. Besides, as α is a partition, the sets $g_s^{-1}(b)$ with $s \neq s'_0$ are p.w. disjoint. Now induction gives the statement of the lemma. Assume that the following is already proved:

- $\beta := \left\{ g_{s_{n-1}}^{-1}\cdots g_{s_1}^{-1}a_{s_0} \mid \begin{array}{l} s_i \in \mathcal{S} \text{ for } i = 0, \ldots n-1, \\ s_i \neq s'_{i-1} \text{ for } i = 1, \ldots n-1 \end{array}\right\}$
  is a partition, which is finer than α.

- $g_{s_{n-1}}^{-1}\cdots g_{s_1}^{-1}a_{s_0} \subset a_{s_{n-1}} \;\; \forall\, s_i \in \mathcal{S}$ for $i = 0, \ldots n-1$ and $s_i \neq s'_{i-1}$

But then, this implies:

- $T^{-1}(\beta)$ is a partition

- Let $b := g_{s_{n-1}}^{-1}\cdots g_{s_1}^{-1}a_{s_0}$. As $b \subset a_{s_{n-1}}$, it follows that $T^{-1}(b)$ is the disjoint union of the sets $g_s^{-1}(b)$ with $s \neq s'_{n-1}$.

[5]This is a generalization: in [BS] and [AF], the Bowen - Series map is defined with respect to Lebesgue - measure and the disc model.

The next equality gives the result:

$$T^{-n}(a_{s_0}) = \bigcup_{\substack{s_1, \ldots s_n \in \mathcal{S} \\ s_i \neq s'_{i-1}}} g_{s_n}^{-1} \cdots g_{s_1}^{-1}(a_{s_0})$$

□

Corollary 4.6 $T^{n+1}|_{g_{s_n}^{-1} \cdots g_{s_1}^{-1}(a_{s_0})} = g_{s_0} g_{s_1} \cdots g_{s_n}$

Proof: This is an immediate consequence of the last proof and the fact, that $g_{s_1} \cdots g_{s_n} \circ g_{s_n}^{-1} \cdots g_{s_1}^{-1}(a_{s_0}) = a_{s_0}$. □

Recall the notion of a cylinder set $[a_{s_n} \ldots a_{s_0}] := g_{s_n}^{-1} \cdots g_{s_1}^{-1}(a_{s_0})$ for $s_i \in \mathcal{S}$. Now the last two results can be rewritten as follows:

$$\begin{aligned} \alpha &= \{[a_s] \mid s \in \mathcal{S}\} \\ \alpha^n &= \{[a_{s_n} \ldots a_{s_1}] \mid s_1 \ldots s_n \in \mathcal{S}, s_i \neq s'_{i+1} \text{ for } i = 1, \ldots n-1\} \end{aligned}$$

Define a word $(s_1 \ldots s_n)$ in $\mathcal{S}$ to be *admissible* if $s'_i \neq s_{i+1}$ for $i = 1, \ldots n-1$. Then the following holds for all admissible words $(s_1 \ldots s_n)$:

$$T^{n+1}|_{[a_{s_1} \ldots a_{s_n}]} = g_{s_n} g_{s_{n-1}} \cdots g_{s_1}$$

Theorem 4.7 Assume G has conditions **(CA)** and **(GC)**. Then:

- The Bowen - Series Map $(\partial\mathbb{H}, \mathcal{B}, T, \mu)$ resp. $(\partial\mathbb{H}, \mathcal{B}, T, \lambda)$ is a Markov map with respect to the partition $\alpha = \{a_s \mid s \in \mathcal{S}\}$.
- T preserves μ (and hence is nonsingular with respect to μ and λ).
- T is topologically mixing, i.e. $\forall\, U, V \subset \partial\mathbb{H}$ open, $U, V \neq \emptyset$, there exists n_0, such that $U \cap T^{-n}V \neq \emptyset\ \forall\, n > n_0$.

Proof: Without loss of generality, assume that the Bowen - Series map is $(\partial\mathbb{B}, T, \mathcal{B}, \mu)$. For the Markov property it remains to show that $\sigma(\bigvee_{i=0}^{\infty} T^{-i}\alpha) = \mathcal{B} \bmod \mu$. Hence, it suffices to show that the euclidean diameters of the partition α^n tend to zero as $n \to \infty$. By lemma (4.5), a side of $g_{s_1} \circ \cdots g_{s_n} P$ corresponds to some element of α^n. So this boils down to show that the euclidean distance of the endpoints of the sides of $g_n P$ tend to zero as $n \to \infty$ where $\{g_n\}_{n \in \mathbb{N}}$ is a sequence of elements in G with the property that the unique representation of g_n as word in the side - pairings has length n.
Assume this is not true. Then there is a $\delta > 0$ such that infinitely many $g_n P$ have a side where the distance between the endpoints is bigger than δ. Hence, the midpoints of these sides have an euclidean distance to the origin smaller than some $0 < \rho < 1$. This contradicts the local finiteness of the tesselation given by GP. This now finishes the proof

of the Markov property.
To show the mixing property, it is standard to show the aperiodicity of the so called incidence graph (cf. [Aa], section 4.2): The set of vertices are the elements in α and the set of (directed) edges are the pairs (a,b) with the property that $T(a) \supset b \bmod \mu$. By **(2)** in remark (4.3) this is equivalent to $b \neq a'$. Now it is easy to see that there are cycles $((a_0,a_1),(a_1,a_2),\ldots(a_{k-1},a_k),(a_k,a_0))$ of each length for all $a = a_0$. As this gives the aperiodicity, T is topologically mixing. $\square$

4.2 The Relation to the Flow

As the Bowen - Series Map is a factor of the section map S, ergodicity of S implies the ergodicity of T: Assume $A \subset \partial\mathbb{H}$ is T - invariant, i.e. $T^{-1}A \stackrel{\mu}{=} A$. Then by **(GC)**, $\mathrm{pr}_2^{-1}A$ is S - invariant. So if S is ergodic, $\mathrm{pr}_2^{-1}A$ has to be trivial mod m which forces A to be trivial. A transformation T is called *conservative* if there is no measurable set A of positive measure such that $\{T^{-n}A\}_{n\in\mathbb{N}}$ is a collection of pairwise disjoint sets. Such sets are called *wandering sets*. Now by the same arguments as before, if A is wandering with respect to T, $\mathrm{pr}_2^{-1}A$ has to be wandering for S. Hence T is conservative if S is conservative. To prove the other direction, one has to show some minimality conditions of $(Y,\mathcal{B}_Y,m,S)$ with respect to $(X,\mathcal{B}_X,\mu,T)$:

Definition 4.8 Let $(X,\mathcal{B}_X,\mu,T)$ be a measure preserving dynamical system of the σ - finite standard measure space X. A natural extension of T is a system $(Y,\mathcal{B}_Y,m_S,S)$ with S invertible and a measurable map $\pi : Y \to X$ such that:

- $\pi \circ S = T \circ \pi$
- $m \circ \pi^{-1} = \mu$
- $\bigvee_{n=1}^{\infty} S^n \pi^{-1}\mathcal{B}_X \stackrel{m}{=} \mathcal{B}_Y$

Now, it has to be shown that the section is the natural extension of the Bowen - Series map. The main argument here is the symmetry of the geodesic flow with respect of going backwards and forward: Analogous to the definition of T as a factor of S one can define $\tilde{T}$ as a factor of S^{-1}. As by proposition (4.2), (3) and (4) are implying that $S^{-1}|_{a_s\times(a_s)^c}(\xi,\eta) = (g_s\xi, g_s\eta)$, the following diagram commutes:

$$\begin{tikzcd} Y \arrow[r, "S^{-1}"] \arrow[d, "\mathrm{pr}_1"] & Y \arrow[d, "\mathrm{pr}_1"] \\ X \arrow[r, "\tilde{T}"] & X \end{tikzcd}$$

Besides, T and $\tilde{T}$ coincide. As in [AF] for the compact case, one can deduce:

Proposition 4.9 $(Y, \mathcal{B}_Y, m, S)$ is the natural extension of $(X, \mathcal{B}_X, \mu, T)$ with $\pi = \mathrm{pr}_2$.

Proof: It remains to validate the last condition of (4.8). As T is Markov, $\bigvee_{i\in\mathbb{N}} T^{-i} \overset{\mu}{=} \mathcal{B}_X$. Hence, $\bigvee_{n\in\mathbb{N}} S^n \mathrm{pr}_2^{-1}\mathcal{B}_X \overset{m}{=} \bigvee_{n,m\in\mathbb{N}} S^n \mathrm{pr}_2^{-1}\alpha^m$. This reduces the problem to the investigation of the S- images of the sets $\mathrm{pr}_2^{-1}a$ where $a \in \alpha^m$ for some m. By using the factor properties of T and $\tilde{T}$ one gets for $0 < m < n$:

$$\begin{array}{rl} & S^m(\mathrm{pr}_1^{-1}a_s) = \mathrm{pr}_1^{-1}(\tilde{T}^{-m}a_s) \\ \text{and} & S^m(\mathrm{pr}_2^{-1}[a_{s_1}\dots a_{s_n}]) \subset \mathrm{pr}_2^{-1}(T^m[a_{s_1}\dots a_{s_n}]) = \mathrm{pr}_2^{-1}[a_{s_{n-m}}\dots a_{s_n}] \\ \Rightarrow & S^m(a_s \times [a_{s_1}\dots a_{s_n}]) \subset \tilde{T}^{-m}a_s \times [a_{s_1}\dots a_{s_n}] \end{array}$$

By proposition (4.2), $S(\mathrm{pr}_2^{-1}[a_{s_1}\dots a_{s_n}]) = a_{s'_1} \times [a_{s_2}\dots a_{s_n}]$. Now the Markov property of T and $\tilde{T}$ with respect to α finishes the proof. $\square$

As S is the natural extension of T, a theorem of Parry (cf. [Aa] theorem 3.1.7) gives that S is conservative if T is conservative and that S is conservative and ergodic if T is conservative and ergodic. So by summarizing the results, one gets:

Proposition 4.10 Assume G is a Fuchsian group with condition **(CA)** and admits an exact fundamental polygon P which sides consist of complete geodesics. Then the following relations between the geodesic flow φ on $\mathbb{H}/_G$, the section map Y defined via P and the Bowen - Series map T hold:

- φ is ergodic $\iff$ S is ergodic $\Rightarrow$ T is ergodic
- T is ergodic and conservative $\iff$ S is ergodic and conservative $\Rightarrow$ φ is ergodic

5 Ergodic Properties of the Bowen – Series map

The aim of this section is to determine ergodic properties of the Bowen - Series map like conservativity, ergodicity or rational ergodicity. In contrast to [BS] and [AF], an invariant measure for T is explicitly given by remark (4.3) but is σ - finite. The first application of this property is to determine the wandering rate of T for some set A which gives the conservativity of T as a corollary. In addition, this rate is used later to determine the return sequence of T.

5.1 The Wandering Rate

Assume in the sequel, that G is a cofinite Fuchsian group with the property **(GC)**. If P is the corresponding fundamental polygon P, proposition (3.12) implies that the set V_P is finite and hence G has property **(CA)**. In addition, let $(\partial\mathbb{H}, \mathcal{B}, \mu, T)$ be the Bowen – Series map defined in (4.4) with Markov partition $\alpha := \{a_s \mid s \in \mathcal{S}\}$ where $\mathcal{S}$ is the set of sides of P. The turning point here is the close relation between the so called parabolic cycle condition and the preimages of the elements of α:

Assume $v = v_1$ is an ideal vertex of P in $\partial\mathbb{H}$ (i.e. an element of V_P). Then by Poincaré's theorem and as V_P is finite, this gives a finite cycle of vertices $v_1, \dots v_n$, of sides $s_1, \dots s_n$ and of side - pairings $g_1, \dots g_n$ such that:

- $g_i(v_i) = v_{i+1}$ for $0 < i < n$ and $g_n(v_n) = v_1$.
- s_i is adjacent to v_i and $g_i(s_i)$ and s_{i+1} have v_{i+1} in common for $0 < i < n$ resp. $g_n(s_n)$ and s_1 have v_1 in common.
- $g_n \circ g_{n-1} \cdots g_1$ fixes v_1 and is parabolic by Poincaré.

Now recall that $a_s \equiv [a_s]$ for $s \in \mathcal{S}$ is an open interval and define $\overline{T}|_{[a_s]}$ as the continuous extension of T to the closure of $[a_s]$, i.e $\overline{T}|_{[a_s]} = g_s$ on $\mathrm{Clos}_{\partial\mathbb{H}}[a_s]$. With regard to T, the cycle statement is hence the same as:

- $\overline{T}|_{[a_{s_i}]}(v_i) = v_{i+1}$ for $0 < i < n$ and $\overline{T}|_{[a_{s_n}]}(v_n) = v_1$.
- a_{s_i} is adjacent to v_i and the closures $a_{g_i(s_i)} = a_{s_i'}$ and $a_{s_{i+1}}$ have v_{i+1} in common for $0 < i < n$ resp. the boundaries of a_{s_n} and a_{s_1} have v_1 in common.

By (2) of remark (4.3), $\overline{T}|_{[a_{s_i}]}(a_{s_i}) = (a_{s_i'})^c$. As $s_i' \neq s_{i+1}$ resp. $s_n' \neq s_1$, $\overline{T}|_{[a_{s_i}]}(a_{s_i}) \subset a_{s_{i+1}}$. Hence, $T^n([a_{s_1} \dots a_{s_n}])$ is well defined and maps $[a_{s_1} \dots a_{s_n}]$ to $T(a_{s_n}) \overset{\mu}{=} (a_{s_n'})^c$. With respect to the continuation of $\overline{T}^n$ on $\mathrm{Clos}_{\partial\mathbb{H}}([a_{s_1} \dots a_{s_n}])$ the last property gives:

- v_1 is a parabolic fixed point $\overline{T}^n|_{[a_{s_1} \dots a_{s_n}]} = \overline{T}|_{[a_{s_n}]} \circ \overline{T}|_{[a_{s_{n-1}}]} \cdots \overline{T}|_{[a_{s_1}]}$, i.e. v_1 is a fixed point and the modulus of the derivative at v_1 equals one.

If N is the greatest common divisor of the lengths of all vertex cycles, it follows that any element[6] of V_P is a parabolic fixed point of $\overline{T}^N$ with respect to the corresponding element of α^N (for notation, cf. (4.6)). Now, for any $v \in V_P$, define

$$\begin{aligned} U(v) \quad := \quad & \underbrace{[a_{s_1} \dots a_{s_n} a_{s_1} \dots a_{s_n} \quad \dots \quad a_{s_1} \dots a_{s_n}]}_{N/n \text{ times}} \cup \\ & \underbrace{[a_{t_1} \dots a_{t_{n'}} a_{t_1} \dots a_{t_{n'}} \quad \dots \quad a_{t_1} \dots a_{t_{n'}}]}_{N/n' \text{ times}}, \end{aligned}$$

where $s_1, \dots s_n$ and $t_1 \dots t_{n'}$ are the edge cycles such that a_{s_1} and a_{t_1} are adjacent to v. Besides, by the cycle property, it follows that $n = n'$ and $t_i = g_{s_{i-1}}(s_{i-1})$ for $i = 2, \dots n$ resp. $t_1 = g_{s_n}(s_n)$. Now define the words $w(v)$ and $w'(v)$ of length N by

$$\begin{aligned} w(v) \quad &:= \quad [a_{s_1} \dots a_{s_n} a_{s_1} \dots a_{s_n} \quad \dots \quad a_{s_1} \dots a_{s_n}] \\ w'(v) \quad &:= \quad [a_{t_1} \dots a_{t_{n'}} a_{t_1} \dots a_{t_{n'}} \quad \dots \quad a_{t_1} \dots a_{t_{n'}}]. \end{aligned}$$

There is the following general fact about Markov maps: assume $a \in \alpha$ and that the word aa is admissible. Then:

$$\begin{aligned} \big(T^{-1}([\underbrace{a \dots a}_{k \text{ times}}])\big)^c \cup ([\underbrace{a \dots a}_{k \text{ times}}])^c \quad &= \quad T^{-1}\big(\bigcup_{b_1 \dots b_k \neq a \dots a} [b_1 \dots b_k]\big) \cup \bigcup_{b_1 \dots b_k \neq a \dots a} [b_1 \dots b_k] \\ &= \quad \bigcup_{b_1 \dots b_k \neq a \dots a,\ cb_1 \text{ adm.}} [cb] \quad \cup \quad \bigcup_{b \neq a} [b] \\ &= \quad [\underbrace{a \dots a}_{k+1 \text{ times}}]^c \end{aligned}$$

As T^N is Markov with respect to α^N and $w(v)w(v)$ is admissible for $v \in V_P$:

$$\begin{aligned} T^{-N}([\underbrace{w(v) \dots w(v)}_{k \text{ times}}])\big)^c \cup [\underbrace{w(v) \dots w(v)}_{k \text{ times}}]^c \quad &= \quad [\underbrace{w(v) \dots w(v)}_{k+1 \text{ times}}]^c \text{ resp.} \\ T^{-N}[w(v)] \cup [w(v)] \quad &= \quad [w(v)w(v)] \end{aligned}$$

and as k is arbitrary, by induction for $n > 0$:

$$\bigcup_{i=0}^{n} T^{-iN}[w(v)]^c = [\underbrace{w(v) \dots w(v)}_{n+1 \text{ times}}]^c$$

[6]This works only if V_P is finite. Hence there is no generalization to the geometrical infinite case.

By defining $A := \left(\bigcup_{v\in V_P}([w(v)]\cup[w'(v)])\right)^c = \left(\bigcup_{v\in V_P} U(v)\right)^c$, this gives:

$$\begin{aligned}
\bigcup_{i=0}^{n} T^{-iN}A &= \bigcup_{i=0}^{n} T^{-iN}\Big(\bigcup_{v\in V_P}([w(v)]\cup[w'(v)])\Big)^c \\
&= \bigcup_{i=0}^{n}\bigcap_{v\in V_P}\Big(\big(T^{-iN}[w(v)]^c\big)\cap\big(T^{-iN}[w'(v)]^c\big)\Big) \\
&= \bigcap_{v\in V_P}\Big(\big(\bigcup_{i=0}^{n} T^{-iN}[w(v)]^c\big)\cap\big(\bigcup_{i=0}^{n} T^{-iN}[w'(v)]^c\big)\Big) \\
&= \Big(\bigcup_{v\in V_P}[\underbrace{w(v)\ldots w(v)}_{n+1 \text{ times}}]\cup[\underbrace{w'(v)\ldots w'(v)}_{n+1\text{ times}}]\Big)^c
\end{aligned}$$

This result now makes it possible to calculate the wandering rate of the set A. By the relation to the parabolic cycle condition, $T^N|_{w(v)}$ has to be parabolic with fixed point v. Assume by conjugation without loss of generality that $v=\infty$ and $T^N|_{w(v)}(z) = z-1$. Then there is $a\in\mathbb{R}$ such that[7] $[w(v)] = (a,\infty)$. By (3) of remark (4.3) there is $b<a$ such that $d\mu = 2/(x-b)$. Now calculus gives:

$$\mu((a,a+n]) = \int_a^{a+n}\frac{2}{x-b}dx = 2\big(\log(a+n-b)-\log(a-b)\big),\ \text{whence:}$$

$$\frac{\mu((a,a+n]}{\log n} = 2\left(\frac{\log(a+n-b)}{\log n}-\frac{\log(a-b)}{\log n}\right)\overset{n\to\infty}{\longrightarrow} 2$$

Again by $T^N|_{w(v)}z = z-1$, this gives in terms of the partition α^N that :

$$(a,a+n] = [w(v)]\backslash[\underbrace{w(v)\ldots w(v)}_{n\text{ times}}] \implies \frac{\mu([w(v)]\backslash[\overbrace{w(v)\ldots w(v)}^{n\text{ times}}])}{\log n}\overset{n\to\infty}{\longrightarrow} 2$$

For each $v\in V_P$, $U(v)$ is a neighborhood of v. Hence, for all $a\in\alpha$, $A\cap a$ is bounded away from ∂a. By the structure of μ, $\mu(A\cap a)$ is finite. As α is finite (as the corresponding group is cofinite), $\mu(A)$ is finite. As the rate of convergence for $[w'(v)]$ is the same as for $[w(v)]$, the equality

$$\bigcup_{i=0}^{n}T^{-iN}A = A\cup\bigcup_{v\in V_P}([w(v)]\backslash[\underbrace{w(v)\ldots w(v)}_{n+1\text{ times}}]\cup[w'(v)]\backslash[\underbrace{w'(v)\ldots w'(v)}_{n+1\text{ times}}])$$

implies by the finiteness of $\mu(A)$, that

$$\frac{\mu\big(\bigcup_{i=0}^{n}T^{-iN}A\big)}{\log n}\overset{n\to\infty}{\longrightarrow} 4\quad \#V_P,$$

[7]The case, that $[w(v)] = (\infty,a)$ has to be excluded, as $T^{-N}|_{w(v)}[w(v)] = [w(v)w(v)]$ has to be a subset of $[w(v)]$ (cf. (7.7) in the appendix).

where $\#V_P$ is the cardinality of V_P. As the sequence $\{\bigcup_{i=0}^{n} T^{-i}A\}_{i\in\mathbb{N}}$ is monotonically increasing, the following proposition is proved:

Proposition 5.1
$$\frac{\mu\big(\bigcup_{i=0}^{n} T^{-i}A\big)}{\log n} \quad \stackrel{n\to\infty}{\longrightarrow} \quad 4 \quad \#V_P$$

In addition, as $(a, a+n] \stackrel{n\to\infty}{\longrightarrow} (a,\infty)$, $\bigcup_{i=0}^{n} T^{-iN}A \stackrel{n\to\infty}{\longrightarrow} \partial\mathbb{H}$. As in contrast to [BS] and [AF], the invariant measure is explicitly given, this gives for the induced transformation on A:

Proposition 5.2 $T_A : A \to A$ is well defined and preserves μ. As $\mu(A)$ is finite, T_A is conservative. In addition, T is conservative.

Proof: As $\mu(A) < \infty$, there is no wandering set of positive measure. Assume W is wandering for T. As $\bigcup_{n=0}^{\infty} T^{-n} = \partial\mathbb{H}$, there is $W^* \subset A$ of positive measure and n_W such that $T^{-n_W}W \cap W^*$ is of positive measure. Hence, W^* would be wandering for T_A which finishes the proof. □

5.2 Distortion Properties

Let $(X, \mathcal{B}, \mu, T, \alpha)$ be a nonsingular Markov map. As T is locally invertible, there are inverse branches of T^n for each $n \geq 1$: for $a = [a_1 \ldots a_n] \in \alpha^n$, $T^n|_a$ is one to one and $T^n a = T a_n$. Now define (cf. [Aa]):

$$\begin{aligned} \mathcal{D}(v_a) &:= T^n a \\ v_a &: \mathcal{D}(v_a) \to a \text{ by } T^n \circ v_a(x) = x \text{ for } x \in T^n a \\ v_a' &:= \frac{d\mu \circ v_a}{d\mu}, \end{aligned}$$

where v_a' is the Radon - Nikodym derivative. A distortion property is a feature of the multiplicative variation of v_a' on $T^n a$. Let

$$\tilde{\alpha}_+ := \{a \in \tilde{\alpha} \mid \mu(a) > 0\} \text{ where } \tilde{\alpha} := \bigcup_{n=1}^{\infty} \alpha^n$$

Definition 5.3 $(X, \mathcal{B}, \mu, T, \alpha)$ has the *strong distortion property* if there is $C > 1$ such that $\mathfrak{g}(C, T) = \tilde{\alpha}_+$ where

$$\mathfrak{g}(C,T) = \{a \in \tilde{\alpha}_+ \mid \frac{v_a'(x)}{v_a'(y)} \leq C \text{ for } \mu \times \mu \text{ - a.e. } (x,y) \in (\mathcal{D}(\nu_a))^2\}$$

The stronger Gibbs property is connected to estimates of $\mu(a)$ for $a \in \alpha^n$ and n large:

Definition 5.4 $(X, \mathcal{B}, \mu, T, \alpha)$ has the *Gibbs property* if there is $C > 1$ and $0 < r < 1$ such that $\mathfrak{g}_r(C,T) = \tilde{\alpha}_+$ where

$$\mathfrak{g}_r(C,T) := \{a \in \tilde{\alpha}_+ \mid \left|\log \frac{v_a'(x)}{v_a'(y)}\right| \leq C r^{t(x,y)} \text{ for } \mu \times \mu \text{ - a.e. } (x,y) \in (\mathcal{D}(v_a))^2\}$$
$$\text{with } t(x,y) := \min\{n \geq 1 \mid T^n x \in a \in \alpha, T^n y \in b \in \alpha \ : \ a \neq b\}$$

In many cases (e.g. if there are some parabolic fixed points) it is not possible to achieve the strong distortion property. But sometimes there is a weaker property:

Definition 5.5 Let $(X, \mathcal{B}, \mu, T, \alpha)$ a Markov map. A collection $\mathfrak{r} \subset \tilde{\alpha}_+$ is called a *Schweiger collection* if there is $C > 1$, such that

- $\mathfrak{r} \subset \mathfrak{g}(C,T)$
- $[b] \in \mathfrak{r}, [a] \in \tilde{\alpha}_+, [a,b] \in \tilde{\alpha}_+$ implies that $[a,b] \in \mathfrak{r}$
- $\bigcup_{b \in \mathfrak{r}} b = X \mod \mu$

$(X, \mathcal{B}, \mu, T, \alpha)$ has the *weak distortion property* if there exists a Schweiger collection for $(X, \mathcal{B}, \mu, T, \alpha)$.

In the sequel, these distortion properties will be discussed with respect to Lebesgue measure and local diffeomorphisms. Recall that in general, if J and J' are bounded intervals in $\mathbb{R}$, λ is Lebesgue measure and $f : J \to J'$ is a diffeomorphism:

$$\frac{d\lambda \circ f}{\lambda} = |Df|,$$

where Df is the usual derivative of f. Hence, if T is in addition a C^1 - endomorphism of a bounded interval:

$$v_a' = \frac{d\lambda \circ v_a}{\lambda} = |Dv_a|.$$

Definition 5.6 Assume that T is a C^2 - endomorphism of a bounded interval I. Then T has the *Renyi property* if there is $0 < C < \infty$ with:

$$\left|\frac{D^2T^n(z)}{(DT^n(z))^2}\right| < C \text{ for Lebesgue a.e. } z \in I$$

Remark 5.7 Distortion properties of T with respect to λ are related to the Renyi property (this is taken from [Aa], s.145). Let $a \in \alpha^n$. Then $T^n \circ v_a = id$ on $\mathcal{D}(v_a)$. Hence, $DT^n \circ v_a \cdot Dv_a = 1$. As $DT^n > 0$ a.e. by the nonsingularity of T:

$$\frac{D^2T^n \circ v_a}{(DT^n \circ v_a)^2} = -\frac{D^2 v_a}{Dv_a}$$

Let $x, y \in \mathcal{D}(v_a)$ and $x < y$. Then by $(\log f)' = f'/f$ for $f = Dv_a(x)/Dv_a(y)$:

$$\frac{d}{dx} \log \frac{Dv_a(x)}{Dv_a(y)} = \frac{D^2 v_a(x)}{Dv_a(x)} \Rightarrow \log \frac{Dv_a(x)}{Dv_a(y)} = \int_x^y \frac{D^2 v_a(t)}{Dv_a(t)} dt$$

Now by the Renyi property:

$$\left| \log \frac{Dv_a(x)}{Dv_a(y)} \right| = \left| \int_x^y \frac{D^2 v_a(t)}{Dv_a(t)} dt \right| \leq \int_x^y \left| \frac{D^2 v_a(t)}{Dv_a(t)} \right| dt \leq C|x - y|$$

As I is bounded, any $a \in \tilde{\alpha}_+$ is a bounded interval, say $\mathrm{diam}(a) < C_{\tilde{\alpha}_+}$ for all $a \in \tilde{\alpha}_+$. Hence, $\tilde{\alpha}_+ = \mathfrak{g}(CC_{\tilde{\alpha}_+}, T)$ resp. T has the strong distortion property. In addition, if $x = e^{\pm M}$ denotes that $x \in [e^{-M}, e^M]$ for $M > 0$, the last inequality gives for $M := CC_{\tilde{\alpha}_+}$:

$$\begin{aligned}
& v_a' = e^{\pm M} v_a'(y) \quad \forall x, y \in \mathcal{D}(v_a) \\
\Rightarrow \quad & \int_{\mathcal{D}(v_a)} v_a'(x) dy = e^{\pm M} \int_{\mathcal{D}(v_a)} v_a'(y) dy \\
\Rightarrow \quad & v_a'(x) \lambda(\mathcal{D}(v_a)) = e^{\pm M} \lambda(a) \\
\Rightarrow \quad & v_a'(x) = e^{\pm M} \frac{\lambda(a)}{\lambda(\mathcal{D}(v_a))} \quad \forall x \in \mathcal{D}(v_a)
\end{aligned}$$

These calculations now lead to the following proposition:

Proposition 5.8 Assume $(I, \mathcal{B}, \lambda, T, \alpha)$ is a nonsingular Markov map where I is a bounded intervall and T is a C^2 - endomorphism of I having the Renyi property. Then $(I, \mathcal{B}, \lambda, T, \alpha)$ has the strong distortion property. If there is in addition a constant C_λ with $\lambda(Ta) > C_\lambda$ for all $a \in \alpha$, then there is $N \in \mathbb{N}$ and $\rho > 1$ with $|DT^n(x)| > \rho$ for a.e. $x \in I$ and $n > N$.

Proof: As it was already shown that the Renyi property implies the strong distortion property, it is left to show the second assertion. For $a = [a_1 \ldots a_n] \in \alpha^n$, $\mathcal{D}(v_a) = T^n a = T(a_n)$. Hence, $\lambda(\mathcal{D}(v_a)) > C_\lambda$.

$$v_a'(x) = e^{\pm M} \frac{\lambda(a)}{\lambda(\mathcal{D}(v_a))} \quad \forall x \in \mathcal{D}(v_a)$$

implies now that

$$v_a'(x) < e^M \frac{\lambda(a)}{C_\lambda} \quad \forall x \in \mathcal{D}(v_a).$$

As T is Markov and λ is Lebesgue measure, $\sup_{a \in \alpha^n} \lambda(a) \overset{n \to \infty}{\to} 0$. Hence, there is $N \in \mathbb{N}$ and $\rho > 1$ with $v_a'(x) < \rho^{-1}$ for all $x \in \mathcal{D}(v_a)$ and $a \in \alpha^n$ for $n > N$. $\square$

Remark 5.9 Assume T is a transformation of the unit circle S^1. If T is piecewise conformal, it is well known, that the same assertions about distortion properties hold.

5.3 Distortion Properties for the Bowen - Series Map

The key estimate for the distortion of T relies on the so called cross ratio:

Definition 5.10 Assume u, v, x, y are four different points in $\mathbb{C}$. Then the *cross ratio* $[u, v, x, y]$ is given by

$$[u, v, x, y] := \frac{|u-x||u-v|}{|u-v||x-y|}$$

By setting $\infty/\infty = 1$, this definition extends to arbitrary u, v, x, y in the Riemann sphere.

As it is well known, cross ratios are preserved by Moebius transformations (cf. [Ra], theorem 4.3.1). This property now allows to prove:

Lemma 5.11 Fix the disc model and let T be the Bowen - Series map given by a polygon P with finite set of ideal vertices V_P. Then for B measurable with the property that there is $\epsilon > 0$ with $d(B, V_P) > \epsilon$ (e.g. $B = A$ as in (5.2)):
There is $0 < C < \infty$ such that for all n:

$$\left| \frac{D^2 T^n(z)}{(DT^n(z))^2} \right| < C \text{ for Lebesgue a.e. } z \text{ with } T^n(z) \in B$$

Proof: Fix $a = [a_{s_1} \dots a_{s_n}] \in \alpha^n$. Then $T^n|_a = g_{s_n} \dots g_{s_1}$. Define $g_a = g = g_{s_n} \dots g_{s_1}$. As it was shown in the appendix for $g \in G$:

$$\left| \frac{D^2 g(z)}{(Dg(z))^2} \right| = 2 \frac{|z - m_g|}{|m_g|^2 - 1}$$

where m_g is the center of the isometric circle $I(g)$ of g. Assume now that η_g is an element of $I(g)$. Then

$$[m_g, \eta_g, z, \infty] = [g(m_g), g(\eta_g), g(z), g(\infty)]$$

As $g(I(g)) = I(g^{-1})$, $g(\eta_g) \in I(g^{-1})$. Let r_g be the radius of $I(g)$. Then

$$r_g = |m_g - \eta_g| = |m_{g^{-1}} - g(\eta_g)| = r_{g^{-1}}.$$

As $g(m_g) = \infty$, $g(\infty) = m_{g^{-1}}$:

$$\begin{aligned} [m_g, \eta_g, z, \infty] &= [\infty, g(\eta_g), g(z), m_{g^{-1}}] \\ \Rightarrow \frac{|m_g - z|}{|m_g - \eta_g|} &= \frac{|m_{g^{-1}} - g(\eta_g)|}{|m_{g^{-1}} - g(z)|} \\ \Rightarrow \frac{|m_g - z|}{r_g} &= \frac{r_g}{|m_{g^{-1}} - g(z)|} \end{aligned}$$

Hence:

$$\left| \frac{D^2 g(z)}{(Dg(z))^2} \right| = \frac{1}{|m_{g^{-1}} - g(z)|}$$

To apply this equality it has to be distinguished whether a and $g(a) = T^n(a)$ are disjoint or not: Let $H(a)$ be the half - space given by

$$\mathrm{Clos}_{\bar{\mathbb{B}}}(H(a)) \cap \partial\mathbb{B} = \mathrm{Clos}_{\partial\mathbb{B}}(a).$$

As $T^n(a) = (a_{s'_n})^c$, $H(a) \overset{g}{\mapsto} H((a_{s'_n})^c)$. Hence $\partial H(a) \overset{g}{\mapsto} s'_n$. As the tesselation GP is locally finite, the collection $\{g^k s'_n\}_{k\in\mathbb{Z}}$ is locally finite. Hence, if $H(a) \subset g(H(a))$, $(g, H(a))$ has the side - pairing property (cf. definition (7.7). But this condition is equivalent to $s_1 \neq s'_n$.

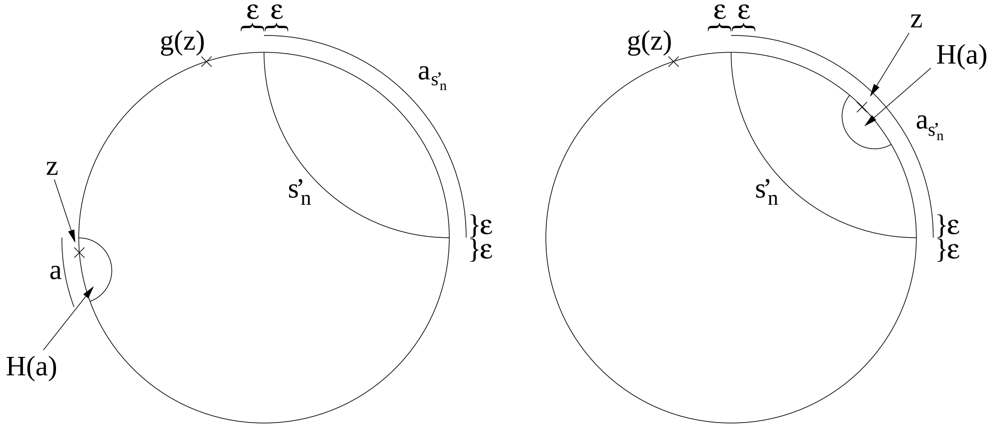

Figure 4: the case $s_1 \neq s'_n$ resp. $s_1 = s'_n$

(1) Assume $s_1 \neq s'_n$ and fix $z \in a \cap B$ with $g(z) \in B$. As $(g, H(a))$ has the side - pairing property, $(g^{-1}, (gH(a))^c)$ has the side - pairing property where $(gH(a))^c = (H(a_{s'_n})^c)^c = H(a_{s'_n})$. Now by propositions (7.8) and (7.9) the parabolic resp. the repelling hyperbolic fixed point $\psi_{g^{-1}}$ of g^{-1} has to be contained in $a_{s'_n}$ and by proposition (7.6), $|\psi_{g^{-1}} - m_{g^{-1}}| < r_{g^{-1}}$. As $g(z) \in (a_{s'_n})^c \cap B$ and $d(B, V_P) > \epsilon$, the triangle inequality gives: $|m_{g^{-1}} - g(z)| \geq \epsilon - r_{g^{-1}}$. Now by theorem 3.3.7 in [Ka]: assume $(g_1, g_2 \ldots)$ is a sequence of distinct elements of G, then $r_{g_k} \to 0$ as $k \to \infty$. Hence $r_g > \epsilon/2$ only for finitely many $g \in G$. Hence,

$$\left|\frac{D^2 g(z)}{(Dg(z))^2}\right| = \frac{1}{|m_{g^{-1}} - g(z)|} \leq \frac{2}{\epsilon}$$

for only finitely many $g \in G$.

(2) Assume now that $s_1 = s'_n$. Hence a and $g(a)$ are disjoint. Hence neither a nor $g(a)$ contain any fixed point of g. So the situation is similar as before: $g(z) \in g(a) = (a_{s'_n})^c$ and both fixed points of g are elements of $a_{s'_n}$. Hence for $z \in B$ with $g(z) \in B$, by the same arguments:

$$\left|\frac{D^2 g(z)}{(Dg(z))^2}\right| = \frac{1}{|m_{g^{-1}} - g(z)|} \leq \frac{2}{\epsilon}$$

for only finitely many $g \in G$. This finishes the proof. $\square$

Recall that the set from proposition (5.2) is given by:

$$A = \partial\mathbb{B} \backslash \big(\bigcup_{v \in V_P} U(v) \big) = \partial\mathbb{B} \backslash \big(\bigcup_{v \in V_P} ([w(v)] \cup [w'(v)]) \big)$$

where N is the smallest common multiple of the lengths of the edge cycles. Hence,

$$\beta := \{a \in \alpha^N \mid a \neq [w(v)], a \neq [w'(v)] \ \forall\ v \in V_P\}$$

is a partition for A. Now define

$$\begin{aligned} \tilde{\beta} \ &:= \ \{b = [b_1 \dots b_n] \in \tilde{\alpha} \mid b_1, \dots b_n \in \alpha, n \geq N, \\ &\quad\ \ [b_1 \dots b_N] \in \beta, [b_{n-N+1} \dots b_n] \in \beta, [b_i \cdots b_{i+N-1}] \notin \beta \ \forall i = 2, \dots n-N\} \\ &= \ \{b \in \tilde{\alpha} \mid \exists\ a_1, a_2 \in \beta \ :\ b \subset a_1, T_A : b \to a_2 \text{ is one to one}\} \end{aligned}$$

Now T_A can be shown to be Markov with the so called *big image property*:

Lemma 5.12 $\tilde{\beta}$ is a Markov partition for T_A and the first return time ϕ_A is measurable with respect to $\tilde{\beta}$. In addition T_A has the big image property, i.e. there is $\delta > 0$ with $\lambda(T_A(b)) > \delta$ for all $b \in \tilde{\beta}$.

Proof: As T_A is defined a.e. on A, $\tilde{\beta}$ is a partition of A mod λ and is by definition finer than β. Hence for all $b \in \tilde{\beta}$, $T_A(b) \in \beta$ is the disjoint union of elements of $\tilde{\beta}$. Now assume that $[a_1 \dots a_n] \in \tilde{\alpha}$ with $[a_1 \dots a_n] \subset A$ and $n > N$. Hence $[a_1 \dots a_N] \in \beta$. In addition, there is $[b_1 \dots b_N] \in \beta$ such that $[a_1 \dots a_N b_1 \dots b_N]$ is admissible (with respect to T). As α is generating $\mathcal{B}$ and $[a_1 \dots a_N b_1 \dots b_N] \subset [b_1 \dots b_N]$, $\tilde{\beta}$ is generating $A \cap \mathcal{B}$. So $\tilde{\beta}$ is a Markov partition for T_A. As the measurability of ϕ_A follows immediately from the definition of $\tilde{\beta}$, only the last assertion has to be shown. But this is a consequence of the facts, that $T_A(b) \in \beta$ for all $b \in \tilde{\beta}$ and that β is a finite partition consisting of elements of positive measure. $\square$

As $\tilde{\beta}$ is a Markov partition for T_A, the following distortion properties can be derived:

Proposition 5.13 Let G be cofinite with property (GC). Then the following holds for T resp. T_A:

(1) T_A has the Renyi property, where A is defined as above (resp. as in (5.2)).

(2) T_A has the strong distortion property with respect to the partition $\tilde{\beta}$ and Lebesgue measure.

(3) There is $N' \in \mathbb{N}$ and $\rho > 1$ such that for all $n \geq N'$ and a.e. $z \in \partial\mathbb{B}$:

$$|DT_A^n(z)| > \rho > 1$$

(4) T has the weak distortion property with respect to Lebesgue measure.

Proof: The first assertion is an immediate consequence of lemma (5.11). As $\tilde{\beta}$ is a Markov partition for T_A and T_A has the big image property with respect to $\tilde{\beta}$, (2) and (3) follow by proposition (5.8). Define

$$\mathfrak{r} := \{[ab] \in \tilde{\alpha} \mid a \in \tilde{\alpha} \text{ and } b \in \beta\}.$$

It is left to show, that $\mathfrak{r}$ is a Schweiger collection (cf. definition (5.5)). By conservativity of T, $\bigcup_{b\in\mathfrak{r}} = \partial\mathbb{B} \bmod \lambda$. The second condition follows directly from the definition of $\mathfrak{r}$. To show that there is $0 < C < \infty$ such that $\mathfrak{r} \subset \mathfrak{g}(C, T)$ it suffices by remark (5.7) to find an upper bound for

$$\left|\frac{D^2T^n(x)}{(DT^n(x))^2}\right| \text{ for Lebesgue a.e. } x \text{ and } n \text{ with } x \in [a_1 \ldots a_n] \in \mathfrak{r}$$

Recall that β is a partition of A consisting of words in α of length N. As $T^{n-N}x \in A$ for all $x \in [a_1 \ldots a_n] \in \mathfrak{r}$, by lemma (5.11) there is a $C' > 0$ such that:

$$\begin{aligned}
\left|\frac{D^2T^n(x)}{(DT^n(x))^2}\right| &= \left|\frac{D^2(T^N \circ T^{n-N})(x)}{(D(T^N \circ T^{n-N})(x))^2}\right| \\
&= \left|\frac{D^2T^N(T^{n-N}(x))}{(DT^N(T^{n-N}(x)))^2} + \frac{D^2T^{n-N}(x)}{DT^N(T^{n-N}(x))(DT^{n-N}(x))^2}\right| \\
&\le \left|\frac{D^2T^N(T^{n-N}(x))}{(DT^N(T^{n-N}(x)))^2}\right| + \frac{C'}{|DT^N(T^{n-N}(x))|} \\
&\quad \text{for all } x \in [a_1 \ldots a_n] \in \mathfrak{r}
\end{aligned}$$

It is shown in the appendix (corollary (7.5)), that for $g \in \mathrm{Iso}^+\mathbb{B}$ there are $0 < m_1, m_2 < \infty$ with $1/m_1 < |Dg| < m_1$ and $|D^2g/(Dg)^2| < m_2$. Hence for any finite collection $\mathcal{H}$ of elements of $\mathrm{Iso}^+\mathbb{B}$, there are $0 < m_1, m_2 < \infty$ with $1/m_1 < |Dg| < m_1$ and $|D^2g/(Dg)^2| < m_2 \ \forall\, g \in \mathcal{H}$. As β is finite,

$$\mathcal{H} := \{g \in \mathrm{Iso}^+\mathbb{B} \mid \exists\, b \in \beta \text{ with } T^N|_b = g|_b\}$$

is also a finite collection. Hence there is $0 < m_1, m_2 < \infty$ such that

$$\left|\frac{D^2T^n(x)}{(DT^n(x))^2}\right| \le m_2 + m_1C' \quad \forall\, x \in [a_1 \ldots a_n] \in \mathfrak{r}$$

This finishes the proof. $\square$

By property (3) of the last proposition it is now possible to give an estimate for the Lebesgue measure of an element of $a \in \tilde{\beta}^n$. Assume $n \ge kN'$. Then:

$$\lambda(a) < \rho^{-k} \max_{b\in\beta}(\{\lambda(b)\}) < 2\pi\rho^k$$

Hence for $a \in \tilde{\beta}^{kN'+l}$ with $l = \{1, \ldots N'-1\}$ and $\tilde{\rho} =: \rho^{1/N'}$:

$$\begin{aligned} \lambda(a) &< 2\pi\tilde{\rho}^{-kN'} = 2\pi\tilde{\rho}^{l}\tilde{\rho}^{-(kN'+l)} \\ &< 2\pi\tilde{\rho}^{N'}\tilde{\rho}^{-(kN'+l)} = \text{const } \tilde{\rho}^{-(kN'+l)} \end{aligned}$$

Now the exponential decay of $\lambda(a)$ for $a \in \tilde{\beta}^n, n \to \infty$ leads to

Theorem 5.14 Assume G is cofinite and of type **(GC)**. Let A be defined as in proposition (5.1). Then T_A has the Gibbs property with respect to Lebesgue measure and with respect to μ.

Proof: To show the Gibbs property with respect to Lebesgue measure λ, $C > 0$ and $0 < r < 1$ have to be found such that

$$\left|\log \frac{v_a'(x)}{v_a'(y)}\right| \leq C r^{t(x,y)} \text{ for } \lambda \times \lambda \text{ - a.e. } (x,y) \in (\mathcal{D}(v_a))^2.$$

By remark (5.7) and the Renyi property of T_A, there is C_1 with:

$$\left|\log \frac{v_a'(x)}{v_a'(y)}\right| = \left|\log \frac{Dv_a(x)}{Dv_a(y)}\right| \leq C_1 d_{S^1}(x,y) \text{ for a.e. } x,y \in \mathcal{D}(v_a)$$

So assume that $t(x,y) = n$. Hence there is an element $b \in \tilde{\beta}^n$ with $x, y \in b$. By the last calculation, there is $C_2 > 0$ and $\rho > 1$ with:

$$\begin{aligned} & d_{S^1}(x,y) \leq \lambda(b) \leq C_2\rho^{-n} \\ \Rightarrow \quad & \left|\log \frac{v_a'(x)}{v_a'(y)}\right| \leq C_1 C_2 \rho^{-t(x,y)} \end{aligned}$$

It is left to show that T_A has the the Gibbs property with respect to μ. By proposition 4.7.1 in [Aa], it is sufficient to show that $\log(\frac{d\mu}{d\lambda})$ is Lipschitz continuous on A. But in remark (4.3), $\frac{d\mu}{d\lambda} =: g$ was already explicitly given. As α is a finite partition, it is sufficient to show the Lipschitz continuity for $g|_{a\cap A}$ for arbitrary $a \in \alpha$. Assume w.l.o.g. that $a = e^{2\pi i I}$ where $I = (0, x)$ for $0 < x < 1$. Then

$$\begin{aligned} g(t) &= 2\pi(\cot(\pi(x-t)) - \cot(\pi(1-t))) \text{ for } t \in I \\ \Rightarrow \frac{d}{dt}\log(g(t)) &= \frac{g'(t)}{g(t)} = \pi\frac{(\sin(\pi(x-t)))^{-2} - (\sin(\pi(1-t)))^{-2}}{\cot(\pi(x-t)) - \cot(\pi(1-t))} \end{aligned}$$

As it was mentioned before, $g|_{a\cap A}$ is bounded away from zero. As A is bounded away from V_P, $A \cap a$ can be written as $A \cap a = e^{2\pi i J}$ with $\mathrm{Clos}(J) \subset I$. Hence $\frac{d}{dt}\log(g(t))$ is a bounded continuous function on J and hence $\log\frac{d\mu}{d\lambda}$ is a Lipschitz continuous function on A. $\square$

5.4 Ergodic Properties

In the sequel, the Gibbs property of T_A will be used to derive further results with respect to T. The first direct consequence is:

Proposition 5.15 T and T_A are exact (and hence ergodic) with respect to μ and Lebesgue measure.

Proof: By a result of Aaronson, Denker and Urbanski (cf. [Aa], theorem 4.4.7), a topologically mixing Markov map having the weak distortion property is exact if this map is conservative. As T and T_A are conservative by proposition (5.2) and as exactness is implying ergodicity, it has to be shown that both maps are topologically mixing and have the weak distortion property. But as T is topologically mixing by theorem (4.7), T_A is also topologically mixing. In addition, the weak distortion was already shown in proposition (5.13). $\square$

For conservative, ergodic, infinite measure preserving transformations like T with respect to μ, there is a further classification (for reference see [Aa]):

Definition 5.16 A conservative, ergodic, measure preserving transformation T of $(X, \mathcal{B}, \mu)$ is called *rationally ergodic* if there is a set $A \in \mathcal{B}$ with $0 < \mu(A) < \infty$ and a constant $M > 0$ with

$$\int_A (\sum_{i=0}^{n-1} 1_A \circ T^i)^2 d\mu \leq M \left(\int_A \sum_{i=0}^{n-1} 1_A \circ T^i d\mu \right)^2 \quad \forall n \geq 1$$

If T is rationally ergodic, there is a sequence $a_n(T) \equiv a_n \nearrow \infty$ unique up to asymptotic equality (cf. theorem 3.3.1 in [Aa]) such that: assume $A \in \mathcal{B}$ is a set corresponding to the last definition, then:

$$\frac{1}{a_n} \sum_{i=0}^{n-1} \mu(B \cap T^{-i} C) \overset{n \to \infty}{\to} m(B) m(C) \; \forall B, C \in \mathcal{B} \cap A$$

This sequence is called the *return sequence* of T. The next definition is based on the *transfer operator* $\widehat{T} : L^1(\mu) \to L^1(\mu)$, defined by:

$$\int_X \widehat{T} f \cdot g d\mu = \int_X f \cdot g \circ T d\mu \quad \forall f \in L^1(\mu), g \in L^\infty(\mu)$$

Definition 5.17 A conservative, ergodic, measure preserving transformation T of $(X, \mathcal{B}, \mu)$ is called *pointwise dual ergodic* if there is a sequence of constants a_n such that

$$\frac{1}{a_n} \sum_{i=0}^{n-1} \widehat{T}^i f \to \int_X f d\mu \text{ a.e. as } n \to \infty \; \forall f \in L^1(X).$$

By proposition (3.7.5) in [Aa], a pointwise dual ergodic transformation T is rationally ergodic. In addition, the sequence a_n from the definition of pointwise dual ergodicity is a return sequence for T. Applying standard results to the Bowen - Series map T for G cofinite, torsionfree and with property **(GC)** gives:

Proposition 5.18 The Bowen - Series Map T is pointwise dual ergodic and rationally ergodic with respect to μ.

Proof: Recall the properties of T_A: T_A is topologically mixing and has the Gibbs property by (5.14). In addition, as $\mu(A)$ is finite, T_A is finite measure preserving. By lemma (5.12), $\mu(T_A(b)) > 0$ for all $b \in \tilde{\beta}$ and ϕ_A is measurable with respect to $\tilde{\beta}$. Hence, corollary (4.7.8) in [Aa] gives that T_A is continued fraction mixing (cf. definition 3.7.4 in [Aa]). This now implies via lemma (3.7.4) in [Aa], that A is a so called Darling - Kac set, i.e. there are constants $b_n > 0$ such that

$$\frac{1}{b_n}\sum_{i=0}^{n-1}\widehat{T}^i 1_A \to \mu(A) \text{ almost uniformly on } A.$$

Now by proposition (3.7.5) in [Aa], T is pointwise dual ergodic and hence by proposition (3.7.1) in [Aa] rationally ergodic. $\square$

As it was shown in the proof, A is a Darling - Kac set. Now by the Chacon - Ornstein theorem:

$$\frac{\sum_{i=0}^{n-1}\widehat{T}^i(1_A)}{\sum_{i=0}^{n-1}\widehat{T}^i(f)} \overset{n\to\infty}{\to} \frac{\int 1_A d\mu}{\int f d\mu} \quad \text{a.e.} \quad \forall f \in L^1(\mu), f > 0$$

So the sequence b_n from the last proof is a return sequence for T. Assume without loss of generality that $a_n = b_n$. As A a is Darling - Kac set, A is *uniform* for the indicator 1_A:

$$\frac{1}{a_n}\sum_{i=0}^{n-1}\widehat{T}^i 1_A \to \int_{\mathbb{B}} 1_A d\mu \text{ almost uniformly on } A.$$

Now the return sequence of T can be determined via the wandering rate $L_A(\mu) = \mu(\bigcup_{i=0}^{n-1} T^{-i}A)$. Recall, that a measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is called *regularly varying at* ∞ if for all $y > 0$, the limit $\lim_{x\to\infty}\frac{f(xy)}{f(x)}$ exists and is bigger than 0. In this case, by the functional equation given by this limit, there is $\alpha \in \mathbb{R}$, called the *index of regularly variation* with

$$\lim_{x\to\infty}\frac{f(xy)}{f(x)} = y^{\alpha} \quad \forall y > 0.$$

By proposition (5.1), the wandering rate of A is proportional to $\log(n)$ and hence $L_A(n)$ is regularly varying at ∞ with index 0. Now 3.8.7 in [Aa] states: Assume T is pointwise dual ergodic, A is uniform for some $f \in L^1(\mu), f > 0$ and $L_A(n)$ is regularly varying with index $\alpha \in [0,1]$. Then

$$a_n \sim \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)} \frac{n}{L_A(n)}$$

This gives:

Theorem 5.19 The Bowen - Series map T for G cofinite and of type **(GC)** is pointwise dual ergodic and the return sequence a_n is given by

$$a_n \sim \frac{n}{\log(n)} \cdot \frac{1}{4\#V_P}$$

resp. as $\text{Area}(P) = (\#V_P - 2)\pi$:

$$a_n \sim \frac{\pi n}{4\log(n)(\text{Area}(\mathbb{H}/_G) + 2\pi)}$$

6 Summary

If G is torsionfree and not necessarily finitely generated, the conditions **(CA)** and **(GC)** are implying that the Bowen - Series map T is a factor of a section $S : Y \to Y$. By theorem (4.7), T is a topologically mixing, infinite measure preserving Markov map. In proposition (4.9) S is shown to be the natural extension of T. Hence S is conservative and ergodic if and only if T is conservative and ergodic.

If G is cofinite, torsionfree and **(GC)**, it was mentioned before that G is not cocompact and that G is a free group with property **(CA)**. In this case it is shown that there is a set $A \subset \partial\mathbb{B}$ with $0 < \mu(A) < \infty$ (cf. proposition 5.1)) and:

- The wandering rate of A is $4\#V_P \log(n)$.
- T_A is a finite measure preserving Markov map and has the Gibbs property.

A first application of these two results is that the Bowen - Series map T is ergodic and conservative (and hence the geodesic flow is ergodic by (4.9) and (3.2)). But in addition more sophisticated results for the infinite measure preserving map T can be deduced:

- T is rationally ergodic and pointwise dual ergodic.
- The return sequence of T is $\frac{n}{\log(n)} \cdot \frac{1}{4\#V_P}$.

As S is the natural extension of T, S is also rationally ergodic with the same return sequence. Hence:

Proposition 6.1 Assume G is torsionfree and cofinite with **(GC)**. Then the geodesic flow on $\mathbb{H}/_G$ admits a section, which is rationally ergodic with return sequence $\frac{n}{\log(n)} \cdot \frac{1}{4\#V_P}$.

It has to be pointed out that this section is infinite measure preserving as T has this property. The reason for that is the existence of parabolic periodic points. Inducing on a set not containing any of those points now leads to a finite measure preserving section for the flow: By (4.2), the following diagram commutes:

$$\begin{tikzcd} Y \arrow[r, "S"] \arrow[d, "\mathrm{pr}_2"] & Y \arrow[d, "\mathrm{pr}_2"] \\ \partial\mathbb{H} \arrow[r, "T"] & \partial\mathbb{H} \end{tikzcd}$$

Recall, that $Y = \bigcup_{s\in\mathcal{S}} (a_s)^c \times a_s$ and $S|_{(a_s)^c \times a_s}(\cdot,\cdot) = (g_s(\cdot), g_s(\cdot))$ where $\mathcal{S}$ is the set of sides of a polygon P with property **(GC)**. Then with A defined as in (5.1):

$$B := \mathrm{pr}_2^{-1}(A) = \bigcup_{s\in\mathcal{S}} (a_s)^c \times (a_s \cap A)$$

As $S^n(\eta, \xi) \in B \iff T^n(\xi) \in A$, the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, "S_B"] \arrow[d, "\mathrm{pr}_2"] & B \arrow[d, "\mathrm{pr}_2"] \\ A \arrow[r, "T_A"] & A \end{tikzcd}$$

By proposition (3.4), S_B is a candidate for another section for the flow:

Proposition 6.2 S_B is a section for the flow on $\mathbb{H}/_G$ where G is torsionfree, cofinite but not cocompact with **(GC)**. S_B itself is a conservative, ergodic and finite measure preserving transformation of $(B, \mathcal{B}_Y \cap B, m|_B)$.

Proof: As S is conservative, S_B is conservative. Hence, the first return map ϕ_B is finite a.e. To apply proposition (3.4), it remains to show that $\bigcup_{n\in\mathbb{Z}} S^n(B) = Y \bmod m$. But as S is ergodic, S_B is ergodic. As $\bigcup_{n\in\mathbb{Z}} S^n(B)$ is S - invariant, this set has to be equal to Y mod m. To finish the proof, the finiteness of $m(B)$ has to be shown. As $\mu = m \circ \mathrm{pr}_2^{-1}$ and $\mu(A) < \infty$, the assertion follows. $\square$

This is in some sense an analog to a result of [AF] for the cocompact case: they constructed for some Fuchsian model $\mathbb{B}/_G$ of a compact surface of genus $g \geq 2$ such a section. Besides, they found a conjugate map having a factor coincident to the Bowen - Series map defined in [BS]. By [BS], this factor is Markov and has the Gibbs property[8].
For the noncompact case treated here, this factor property follows without further conjugation. This is a consequence of the existence of a fundamental polygon P which has no vertices in $\mathbb{H}$. This is connected to the correspondence of cutting sequences and boundary expansions as follows (cf. [Se]): the bijection given by Series is in this case the identity (mod Liouiville measure). In addition, the Gibbs - Markov property for the factor T_A follows without choosing some special model for a given surface. This is of some importance as a quasiconformal deformation is in common nonsingular with respect to the Liouiville measure.

[8]Bowen & Series in fact proved, that this factor f is a C^2 - Markov map with $|D(f^2)| > \delta > 1$. From these properties, it is standard to derive the Renyi and the Gibbs property (cf. [Th]).

7 Appendix: Isometric Circles & Side - Pairings

The aim of this last section is to describe relations between the side - pairings of some polygon P with respect to the locus of its isometric circles. This will lead to estimates of $|D^2T|/|DT|^2$, where T is the Bowen - Series map. Therefore, the notion of an inversion has to be introduced (cf. [Kat],[Ra]):

Definition 7.1 Let $S(a,r)$ be the euclidean circle in $\mathbb{C}$ around $a \in \mathbb{C}$ with radius $r > 0$. Then the *inversion* $\sigma_{S(a,r)}$ in $S(a,r)$ is the self - mapping of the Riemann sphere, given by:

$$\sigma_{S(a,r)}(z) = \frac{a\bar{z} - |a|^2 + r^2}{\bar{z} - \bar{a}}$$

As it is well known, $\sigma_{S(a,r)}$ is an antiholomorphic diffeomorphism, fixing $S(a,r)$ pointwise and mapping a to ∞. In addition, these elements together with the usual reflections in lines generate the Moebius group $\mathrm{Moeb}(\hat{\mathbb{C}})$. Besides, $\mathrm{Moeb}^+(\hat{\mathbb{C}}) = \mathrm{PSL}_2(\mathbb{C})$ is the subgroup of orientation preserving transformations. The next two results are standard (cf. [Ka], §3.3 and [Ra], §4.3):

Proposition 7.2 Assume $g \in \mathrm{PSL}_2(\mathbb{C}), g(\infty) \neq \infty$. Then there is an euclidean circle $I(g)$ such that g acts as an euclidean isometry on $I(g)$. This circle is called the *isometric circle* of g.

Proposition 7.3 Assume $g \in \mathrm{Iso}^+(\mathbb{B})$ and g is not an euclidean isometry (i.e. $g(0) \neq 0$). Then $I(g)$ is unique g and has a representation $g = \tau\sigma$, where τ is a reflection at a straight line through the origin and σ is the inversion in $I(g)$. In addition, $I(g)$ is perpendicular to S^1 and hence corresponds to a geodesic.

As an immediate consequence of this statement, the modulus of the derivative (cf. the next remark) can be calculated with respect to $I(g)$. Assume that $I(g) = S(m_g, r)$. Now the property, that $I(g)$ is perpendicular to S^1, is equivalent to $|m_g|^2 = r^2 + 1$. Hence for fixed $m_g \in \mathbb{C}$:

$$\sigma_g(z) := \sigma_{I(g)}(z) = \frac{m_g\bar{z} - 1}{\bar{z} - \bar{m}_g}$$

As the modulus of the derivative is invariant under multiplication from the left with elements of O(2), i.e. with euclidean isometries fixing the origin, it follows:

$$\begin{aligned} |Dg(z)| &= |D\sigma_g(z)| = |D\overline{\sigma_g(z)}| = \left|\frac{|m_g|^2 - 1}{|z - m_g|^2}\right| \\ \Longrightarrow &\quad \begin{cases} |Dg(z)| = 1 & \Longleftrightarrow \quad |z - m_g|^2 = 1 \\ |Dg(z)| > 1 & \Longleftrightarrow \quad |z - m_g|^2 < 1 \\ |Dg(z)| < 1 & \Longleftrightarrow \quad |z - m_g|^2 > 1 \end{cases} \end{aligned}$$

Remark 7.4 Here, the derivative Dg of g is defined as the usual derivative of a holomorphic resp. antiholomorphic self - mapping of the Riemann sphere (e.g. if $g \in \text{Moeb}(\mathbb{C})$). Then Dg itself is holomorphic resp. antiholomorphic. Hence, the second derivative is well defined. Here $|Dg(z)|$ resp. $|D^2g(z)|$ denotes the 2 - norm of $Dg(z)$ resp. $D^2g(z)$.

Furthermore, to get estimates of $|D^2g|/|Dg|^2$:

$$\begin{aligned} |D^2g(z)| &= \left| D\left(\frac{|m_g|^2-1}{(z-m_g)^2}\right)\right| = \left|\frac{2(|m_g|^2-1)}{(z-m_g)^3}\right| \\ \Rightarrow \left|\frac{D^2g(z)}{(Dg(z))^2}\right| &= 2\frac{|z-m_g|}{|m_g|^2-1} \end{aligned}$$

As $m_g \notin \mathbb{B} \cup \partial\mathbb{B}$, it follows that:

Corollary 7.5 For g there is $0 < m_1, m_2 < \infty$ such that

$$\frac{1}{m_1} < |Dg(z)| < m_1 \text{ and } \frac{1}{m_2} < \left|\frac{D^2g(z)}{(Dg(z))^2}\right| < m_2 \quad \forall\, z \in \mathbb{B} \cup \partial\mathbb{B}$$

Now by the structure of $|Dg|$, the following can be shown:

Proposition 7.6 If g is a parabolic element of $\text{Iso}^+\mathbb{B}$, the unique fixed point z_g of g is contained in $I(g) \cap S^1$. In addition, if $g = \tau_g\sigma_g$, τ_g is the reflection on the line joining 0 and z_g. In addition, $I(g)$ and $I(g^{-1})$ intersect in z_g. If g is hyperbolic with fixed points z_g and z_g', then the geodesic joining the two fixed points intersects $I(g)$. In addition, $I(g)$, $I(g^{-1})$ and the reflection axis of τ_g do not intersect. Each of them is intersecting the geodesic joining z_g and z_g' perpendicular. See figure (5) for illustration.

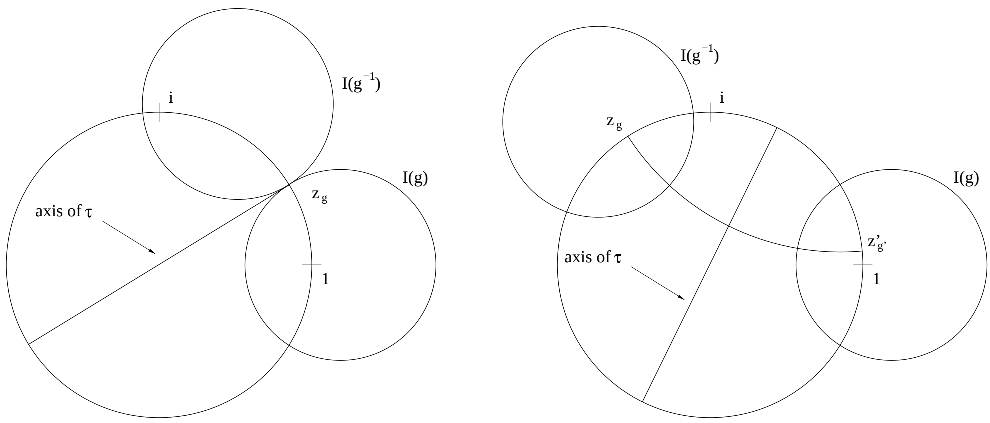

Figure 5: $I(g)$ for g parabolic resp. hyperbolic

Proof: It is well known that $|Dg(z_g)| = 1$, if g is parabolic. Hence, $z_g \in I(g) \cap S^1$, implying that $\sigma_g(z_g) = z_g$. Therefore, τ_g has to fix z_g which gives the required property of τ_g. The

statement about $I(g^{-1})$ now follows from the fact that $\sigma_g(I(g)) = I(g^{-1})$.
In the hyperbolic case, it is well known that $|Dg(z_g)| < 1 < |Dg(z_g')|$ (w.l.o.g. $|Dg(z_g)| < |Dg(z_g')|$). Hence, the geodesic joining z_g and z_g' intersects $I(g)$. If the reflection axis of τ_g would intersect $I(g)$, then g would have either a fixed point in $\mathbb{B}$ or a parabolic fixed point in $\partial\mathbb{B}$. Hence, this axis and $I(g)$ has to be disjoint and τ_g maps $I(g)$ to the connected component of $\mathbb{B}$ without the reflection axis of τ_g. To prove the perpendicularity, switch to the upper half plane and let $\tilde{g}, \tilde{\tau}_g$ and $\tilde{\sigma_g}$ be the elements corresponding to g, τ_g and σ_g. Then $\tilde{\sigma_g}$ and $\tilde{\tau}_g$ are fixing the corresponding geodesics. Assume w.l.o.g. that $\tilde{g}(z) = \lambda z$ with $\lambda > 0$. Then neither the reflection axis of τ_g nor of σ_g contains a fixed point of g. But this has to be also true for $\tilde{\sigma_g}$ and $\tilde{\tau}_g$. Hence, $\tilde{\sigma_g}$ and $\tilde{\tau}_g$ are inversions at geodesics not containing ∞. Let m_{τ_g} resp. m_{σ_g} those points in $\mathbb{R}$ such that $\tilde{\tau}_g$ resp. $\tilde{\sigma_g}$ are the inversions in a circle with center m_{τ_g} resp. m_{σ_g} . As $\tilde{g}$ fixes 0 and ∞, $\tilde{\tau}_g \circ \tilde{\sigma_g}$ has to fix these points. As $\tilde{\sigma_g}(\infty) = m_{\sigma_g}$ and , $\tilde{\tau}_g(m_{\tau_g}) = \infty$, $m_{\tau_g} = m_{\sigma_g}$. In addition, $\tilde{\tau}_g\tilde{\sigma_g}(m_{\sigma_g}) = \tilde{\tau}_g(\infty) = m_{\tau_g} = m_{\sigma_g}$. Hence, $m_{\tau_g} = m_{\sigma_g} = 0$. Now the conformal equivalence of $\mathbb{U}$ and $\mathbb{B}$ finishes the proof. □

In the following relations between a side s, its side -pairing g_s and its isometric circle I_{g_s} will be described: assume P is a polygon with set of sides $\mathcal{S}$ such that each side is a complete geodesic and P satisfies the conditions of Poincaré's theorem with respect to some side - pairings $\{g_s\}_{s\in\mathcal{S}}$. Hence, $G :=< g_s \mid s \in \mathcal{S} >$ has to be free, implying that G has no torsion and therefore only contains parabolic and hyperbolic elements. In addition, as P is an exact fundamental polygon for G:

(1) $H_s \subset g_s(H_s)$, where H_s is a half space with $\partial H_s = s$ (c.f. (4.2)).

(2) The collection $\{g_s^n(s)\}_{n\in\mathbb{Z}}$ is locally finite.

The aim is now not only to show properties of side - pairings, but also for transformations in some sense similar to side - pairings. So define the following:

Definition 7.7 (g_s, H_s) has the *side - pairing property* if:

(0) g_s is a hyperbolic or parabolic transformation and and s is a geodesic.

(1) There is a halfspace H_s with $H_s \subset g_s(H_s)$ and $\partial H_s = s$.

(2) The collection $\{g_s^n(s)\}_{n\in\mathbb{Z}}$ is locally finite.

The parabolic case: Assume g_s is parabolic and has the side - pairing property. Without loss of generality, $g_s \in \mathrm{Iso}^+\mathbb{U}$ and $g_s(z) = z + 1$. Then $H_s \subset g_s(H_s)$. But as this fails if both endpoints of s are in $\mathbb{R}$, it follows that one of them has to be ∞. Hence, s is a line perpendicular to $\mathbb{R}$, say $s = \gamma_{x,\infty}$. In addition, again by $H_s \subset g_s(H_s)$, H_s has to be the half - space to the left of s (see figure (6) for illustration). Hence, after switching back to the disc model, s and s' have the fixed point z_{g_s} of g_s in common. By proposition (7.6), z_g is also contained in the isometric circle $I(g_s)$ and in the reflection axis of τ_{g_s}. Now again

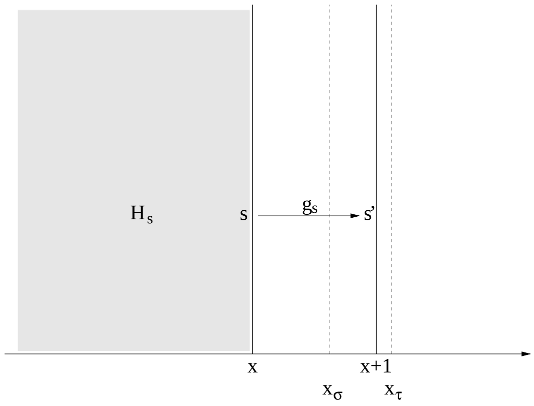

Figure 6: H_s, s and s' for g_s

in the upper half space model, this gives that $(z \mapsto z+1) = \tilde{\tau}\tilde{\sigma}$, where $\tilde{\tau}$ resp. $\tilde{\sigma}$ are reflections in the geodesics corresponding to the reflection axis of τ_{g_s} resp. $I(g_s)$ in $\mathbb{B}$. As ∞ has to be an endpoint of those geodesics, they have to be lines perpendicular to $\mathbb{R}$. Assume that $\tilde{\tau}$ resp. $\tilde{\sigma}$ is the reflection in $\{z \mid \mathrm{Im} z = x_\tau\}$ resp. in $\{z \mid \mathrm{Im} z = x_\sigma\}$, where $x_\tau, x_\sigma \in \mathbb{R}$. Hence:

$$\begin{aligned} & \tilde{\tau}(z) = -\bar{z} + 2x_\tau \text{ and } \tilde{\sigma}(z) = -\bar{z} + 2x_\sigma \\ \Rightarrow \quad & g_s(z) = \tilde{\tau}\tilde{\sigma}(z) = z + 2(x_\tau - x_\sigma) \\ \Rightarrow \quad & z + 1 = z + 2(x_\tau - x_\sigma) \\ \Rightarrow \quad & x_\tau - x_\sigma = \tfrac{1}{2} \end{aligned}$$

Hence, $x_\tau > x_\sigma$. With respect to the disc model, the following can be concluded: Define the interval a_s^I similar to the definition of a_s as follows: let $U_{I(g_s)}$ be the bounded connected component of $\mathbb{C} \backslash I(g_s)$. Then

$$a_s^I := U_{I(g_s)} \cap \partial\mathbb{B}.$$

As switching between the two models $\mathbb{U}$ and $\mathbb{B}$ preserves orientation, the property that $x_\tau > x_\sigma$ gives:

- If $x \geq x_\sigma$, then $a_s^I \subset a_s$
- If $x < x_\sigma$, then $a_s^I \supset a_s$

In addition, the fixed point of g_s is a common endpoint of a_s and a_s^I. This gives as $U_{I(g_s)} = \{z \mid |Dg(z)| > 1\}$:

Proposition 7.8 Assume g_s is parabolic and has the side - pairing property. Then the unique fixed point of g_s is a boundary point of a_s as well as of a_s^I. In addition, $a_s \cap a_s^I$ is a nonempty interval which has the property that $|Dg_s(z)| \geq 1$ for $z \in a_s \cap a_s^I$.

The hyperbolic case: If g_s is hyperbolic, an analog of the last proposition can be shown. Analogously to the parabolic case, it is assumed without loss of generality, that $g_s(z) = \lambda z$ for $\lambda > 0$. As $I(g^{-1}) = gI(g)$, $s' = g_s(s)$ and $g_{s'} = g_s^{-1}$, assume without loss of generality that $\lambda > 1$ (otherwise changing to the inverse of g_s gives the wanted property). So assume that $s = \gamma_{x,\infty}$. Then

$$\{g_s^n(s) \mid n \in \mathbb{Z}\} = \{\gamma_{\lambda^n,\infty} \mid n \in \mathbb{Z}\}$$

meets any neighbourhood of i infinetly often. This is a contradiction to (2) in the last definition. Hence, $s = \gamma_{x,y}$ with $x, y \neq \infty$. By the same argument, $x, y \neq 0$. Now as $H_s \subset g_s(H_s)$, it follows that 0 has to be contained in the open interval (x, y) (w.l.o.g. $x < y$) and that H_s is the bounded component of $\mathbb{U}\backslash\gamma_{x,y}$ (see figure (7) for illustration). By proposition (7.6) and with the same notation as above, the geodesics corresponding

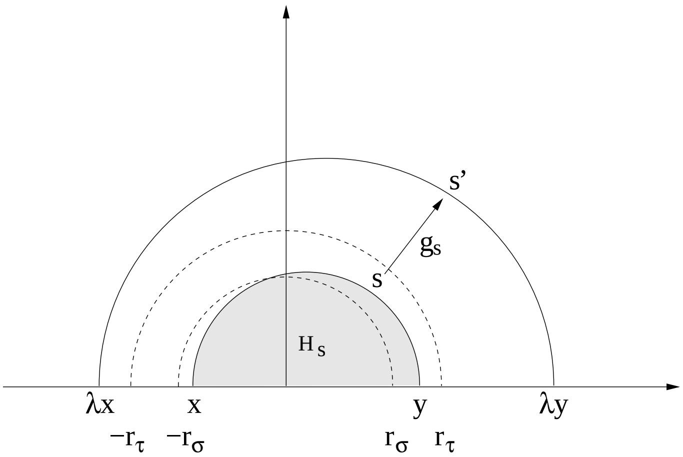

Figure 7: H_s, s and s' for g_s

to the reflection axis of τ_{g_s} resp. to the isometric circle $I(g_s)$ have to be perpendicular to $\gamma_{0,\infty}$. Hence, $\tilde{\tau}$ and $\tilde{\sigma}$ are inversions at circles around the origin. Now by definition (7.1) for appropriate $r_\tau > 0$ and $r_\sigma > 0$,

$$\begin{aligned} & \tilde{\tau}(z) = \frac{r_\tau^2}{\bar{z}} \text{ and } \tilde{\sigma}(z) = \frac{r_\sigma^2}{\bar{z}} \\ \Rightarrow \quad & g_s(z) = \tilde{\tau}\tilde{\sigma}(z) = r_\tau^2 r_\sigma^{-2} z \\ \Rightarrow \quad & r_\tau^2 r_\sigma^{-2} = \lambda > 1 \end{aligned}$$

By the same reasons as above, it was proved:

Proposition 7.9 Assume g_s is hyperbolic and has the side - pairing property. Then the unique fixed point z'_{g_s} of g_s with $|Dg(z_{g_s}| \geq 1$ is a point in the interior of a_s as well as of a_s^I. In addition, $a_s \cap a_s^I$ is a nonempty interval, which has the property that $|Dg_s(z)| \geq 1$ for $z \in a_s \cap a_s^I$.

8 Notational Conventions

$\mathbb{N}$	the natural numbers $\{1, 2 \ldots\}$
$\mathbb{Z}$	the integers
$\mathbb{R}$	the real numbers
$\mathbb{C}$	the complex numbers
$\mathbb{B}$	the disk model of the hyperbolic plane
$\mathbb{U}$	the upper half space model of the hyperbolic plane
$\mathbb{H}$	the hyperbolic plane (with no model spec.)
S^1	the unit circle
$\mathrm{Iso}(\mathbb{H})$	the group of isometries of $\mathbb{H}$
$\mathrm{Iso}^+(\mathbb{H})$	the group of orientating preserving isometries of $\mathbb{H}$
$\mathrm{Moeb}(D)$	the group of Moebius transformations of the domain D
$\mathrm{Moeb}^+(D)$	the group of orientating preserving Moebius transformations of the domain D
$\mathrm{O}(2)$	the orthogonal group of $\mathbb{R}^2$
$a_n \sim b_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$
$\mathrm{diam}(\cdot)$	the euclidean diameter of a set
$d_{S^1}(\cdot, \cdot)$	the metric given by the arc length of S^1
$\lvert \cdot \rvert$	the 2 - norm on $\mathbb{C}$
$A_{\mathbb{H}}$	the hyperbolic area.
λ	Lebesgue measure
$\overset{m}{=}$	equality modulo the measure m
$T\vert_A$	the restriction of T on A
T_A	the induced transformation on A
ϕ_A	the return time to A
$\mathcal{S}$	the set of sides of a polygon
$[a_1 \ldots a_n]$	the cylinder set given by $a_1, \ldots a_n$
a_s	the interval on $\partial\mathbb{H}$ corresponding to the side $s \in \mathcal{S}$
g_s	the sidepairing which maps $s \mapsto s'$

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