Bump detection in heterogeneous Gaussian regression

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We analyze the bump detection problem for a signal in a heterogeneous Gaussian regression model. To this end we allow for a simultaneous bump in the variance and specify its impact on the difficulty to detect the null signal against a single bump. This is done by calculating lower and upper bounds, both based on the likelihood ratio. The analysis of the corresponding likelihood ratio test relies on a generalized deviation inequality for weighted sums of non-central chi-squared random variables.

Lower and upper bounds together lead to explicit characterizations of the detection boundary in several subregimes depending on the asymptotic behavior of the bump heights in mean and variance. In particular, we explicitly identify those regimes, where the additional information about a simultaneous bump in variance eases the detection problem for the signal. This effect is made explicit in the constant and / or the rate, appearing in the detection boundary.

We also discuss the case of an unknown bump height and provide an adaptive test and some upper bounds in that case.

Keywords: minimax testing theory, heterogeneous Gaussian regression, change point detection, chi-squared deviation bound.

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1. Introduction

Assume that we observe random variables \( Y = (Y_1, ..., Y_n) \) through the regression model

\[
Y_i = \mu_n \left( \frac{i}{n} \right) + \lambda_n \left( \frac{i}{n} \right) Z_i, \quad i = 1, ..., n, \tag{1}
\]

where \( Z_i \overset{i.i.d.}{\sim} \mathcal{N}(0, 1) \) are observational errors, \( \mu_n \) denotes the (unknown) signal and \( \lambda_n^2 \) is the (unknown) variance function sampled at \( n \) equidistant points \( i/n \), say. The aim of this paper is to analyze the effect of a simultaneous change in the variance on the difficulty to detect \( \mu_n \equiv 0 \) against a signal of the form

\[
\mu_n (x) = \Delta_n I_n (x) = \begin{cases} 
\Delta_n & \text{if } x \in I_n, \\
0 & \text{otherwise}
\end{cases}, \tag{2}
\]

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i.e. a bump with height $\Delta_n > 0$ on an interval $I_n \subset [0, 1]$ as location. We will not assume that the location $I_n$ of the bump is known, but we assume that its width $|I_n|$ is known. For the moment, let us also assume that $\Delta_n$ is known, this will be relaxed later on. The assumption that $\Delta_n > 0$ is posed only for simplicity here, if $\Delta_n$ is arbitrary or $|\mu_n|$ is considered, similar results can be obtained analogously. Further, we will model a simultaneous change of the variance by

$$
\lambda^2_n(x) = \sigma_0^2 + \sigma_n^2 1_{I_n}(x), \quad x \in [0, 1]
$$

(3)

with a known “baseline” variance $\sigma_0^2 > 0$. The additional change in the variance $\sigma_n^2 \geq 0$ is also known and may only occur when the signal has a bump of size $\Delta_n$ on $I_n$. The model arising from (1)–(3) will be called the heterogeneous bump regression (HBR) model, which is illustrated in Figure 1.

![Figure 1: The HBR model: Data together with parameters $I_n, \Delta_n, \sigma_0$ and $\sigma_n$ in (1)–(3) from the HBR. Here $\Delta_n = 4, \sigma_0^2 = 1, \sigma_n^2 = 4$ and $n = 512$.](image)

The HBR model may serve as a prototypical simplification of more complex situations, where many bumps occur and have to be tested or estimated. In fact, in many practical situations it is to be expected that the variance changes when a change of the signal is present; see e.g. [34] for a discussion of this in the context of CGH array analysis. Another example is the detection of opening and closing states in ion channel recordings (see [35] and the references therein), when the so-called open channel noise is present which arises for large channels, e.g. porins [39, 44].

This model can also be viewed as a heterogeneous extension of the “needle in a haystack” problem as $|I_n| \downarrow 0$ (see [2, 12, 13]), i.e. to identify a small cluster of nonzero components in a high dimensional multivariate normal vector [1, 5, 15, 18, 28]. For a more thorough discussion in the light of our subsequent results, see Section 5.

In this paper, the analysis of the HBR model will be addressed in the context of minimax testing theory, and we refer e.g. to a series of papers by Ingster [27] and to Tsybakov [46] for the general methodology. Following this paradigm we aim for determining the detection boundary which marks the borderline between asymptotically detectable and undetectable signals of type (2). Obviously, a fixed signal can be detected always with the sum of first and second kind error
tending to 0 as \( n \to \infty \). Thus we are interested in asymptotically vanishing signals, hence we want to investigate which asymptotic behavior of \( \Delta_n \), \( \sigma_n \) and \( |I_n| \) \( \gamma \to 0 \) implies the following twofold condition:

**lower detection bound:** no test at level \( \alpha \) under the null-hypothesis \( \mu \equiv 0, \lambda_n \equiv \sigma_0 \) can differentiate between faster vanishing signals with power strictly greater than \( \alpha \).

**upper detection bound:** there exists a test which can differentiate between the null hypothesis \( \mu \equiv 0, \lambda_n \equiv \sigma_0 \) and slower vanishing signals at level \( \alpha \) with a power greater or equal \( 1 - \alpha \).

In fact it turns out that both conditions do not depend asymptotically on the specific choice of \( \alpha \in (0,1) \) as we will explain later. Typically the terms faster and slower are measured by changing the constant appearing in the detection boundary by an \( \pm \varepsilon_n \) term. If upper and lower bounds coincide (asymptotically), we speak of the (asymptotic) detection boundary.

Methods for detecting changes or bumps in a sequence of observations have been studied extensively in statistics and related fields [4, 8, 11, 19, 23, 24, 30, 36, 37, 42, 43, 47]. We refer to Csörgö & Horváth [17], Siegmund [41], Carlstein et al. [14] or Frick et al. [22] for a survey. The special question of minimax testing in such a setting has been addressed e.g. in [16, 20, 21, 22, 29, 31]. Nevertheless, all these papers only address the case of a homogeneous variance, i.e. \( \lambda_n \equiv \sigma_0 \).

There, it is well-known that the detection boundary is determined by the equations

\[
\sqrt{n} |I_n| \Delta_n = \sqrt{2} \sigma_0 \sqrt{-\log (|I_n|)},
\]

and the terms “faster” and “slower” in the above definition are expressed by replacing \( \sqrt{2} \sigma_0 \) by \( \sqrt{2} \sigma_0 \pm \varepsilon_n \) with a sequence \( (\varepsilon_n)_{n \in \mathbb{N}} \) such that \( \varepsilon_n \to 0, \varepsilon_n \sqrt{-\log (|I_n|)} \to \infty \) as \( n \to \infty \) (for details we again refer to [16, 20, 21, 22, 29]).

In the model (1)–(3) mainly two situations have to be distinguished, see also [32].

**HBR** The first situation emerges when it is not known whether the variance changes (but it might). Then as in (3) we explicitly admit \( \sigma_n^2 = 0 \) and the variance is a nuisance parameter rather than an informative part of the model for identification of the bump in the signal. Hence, no improvement of the bound (4) for the case of homogeneous variance is to be expected. This model has been considered for example in [3, 10, 26], although a rigorous minimax testing theory remains elusive and will be postponed to further investigations.

**SHBR** The second situation, which we treat in this paper, is different and will be denoted as the problem of **strongly heterogeneous bump regression (SHBR):** We will always assume \( \sigma_n^2 > 0 \), which potentially provides extra information on the location of the bump and detection of a change might be improved. This is not the case in the HBR model, as the possibility of a change in variance can only complicate the testing procedure, whereas \( \sigma_n^2 > 0 \) gives rise to a second source of information (the variance). The central question is: Does the detection boundary improve due to the additional information and if so, how much?

In fact we will make the improvement precise and determine the detection boundary in the SHBR model as \( \sigma_n \to 0 \). Thereby we find that this is characterized by three different regimes, essentially depending on the asymptotic behavior of the coefficient of variation \( \sigma_n^2 / \Delta_n \). These three regimes are separated by two distinctive phase transitions. Our results are achieved on the one hand by modifying a technique from Dümbgen & Spokoiny [20] providing lower bounds in the homogeneous model. We will generalize this to the case of a non-central chi-squared likelihood ratio which appears due to the change in variance. On the other hand we will analyze the likelihood ratio test which then provides upper bounds. Doing so we derive a generalized deviation inequality (Lemma 2.3) for the weighted sum of non-central chi-squared random variables, which might be of interest by its own as it generalizes and unifies various results from the literature [6, 7, 9, 25, 38, 45].

Note, that if the location \( I_n \) of the bump was known (and not only its width \( |I_n| \)), then the whole problem reduces to a hypothesis test with simple alternative, i.e. the Neyman-Pearson test will be optimal. In fact, as a byproduct of our analysis we obtain (cf. Remark 2.5) that the rates change in this setting (the \( \sqrt{-\log (|I_n|)} \)-terms in (4) and its analogs vanish), but the improvement
in the constant over the homogeneous model stays exactly the same. This situation may be viewed as a variation of the celebrated Behrens-Fisher problem (see e.g. [33]) to the situation when the variance changes with a mean change.

To state the results more precisely, let us now formalize the considered testing problem. W.l.o.g. assume in the following that \( l_n := 1/|I_n| \in \mathbb{N} \) for all \( n \in \mathbb{N} \) is a given sequence and define

\[
\mathcal{A}_n := \{(j-1)|I_n|, j|I_n| \mid 1 \leq j \leq l_n\}.
\]

(5)

Then for fixed \( n \in \mathbb{N} \) we want to test

\[
H_0 : \quad \mu_n \equiv 0, \lambda_n \equiv \sigma_0, \quad \sigma_0 > 0 \text{ fixed}
\]

against

\[
H^n_1 : \exists \ I_n \in \mathcal{A}_n \text{ s.t. } \mu_n = \Delta_n 1_{I_n}, \quad \lambda_n^2 = \sigma^2_0 + \sigma^2_0 1_{I_n}, \quad \sigma_n > 0.
\]

(6)

In the following we consider tests \( \Phi_n : \mathbb{R}^n \rightarrow \{0,1\} \), where \( \Phi_n(Y) = 0 \) means that we accept the hypothesis \( H_0 \) and \( \Phi_n(Y) = 1 \) means that we reject the hypothesis. The sequence of tests \( \Phi_n \) is said to have asymptotic level \( \alpha \in (0,1) \) under \( H_0 \), if \( \limsup_{n \rightarrow \infty} \mathbb{E}_{H_0}[\Phi_n(Y)] = \limsup_{n \rightarrow \infty} \mathbb{P}_{H_0}(\Phi_n(Y) = 1) \leq \alpha \), i.e. if the probability of a first kind error is less or equal \( \alpha \) in the limit \( n \rightarrow \infty \). The second kind error is given by

\[
\mathbb{P}_{H^n_1}(\Phi_n(Y) = 0) := \sup_{I_n \in \mathcal{A}_n} \mathbb{P}_{\mu_n = \Delta_n 1_{I_n}, \lambda_n^2 = \sigma^2_0 + \sigma^2_0 1_{I_n}}(\Phi_n(Y) = 0)
\]

and the quantity

\[
\liminf_{n \rightarrow \infty} \mathbb{E}_{H^n_1}[\Phi_n(Y)] = 1 - \limsup_{n \rightarrow \infty} \mathbb{P}_{H^n_1}(\Phi_n(Y) = 0)
\]

will be called the asymptotic power of the sequence of tests \( \Phi_n \). As \( n \rightarrow \infty \), it turns out that the sum of first and second kind error either tend to 0 or 1, which implies that the definition of lower detection bounds, upper detection bounds and the detection boundary itself do not depend on the choice of the test level \( \alpha \in (0,1) \) (cf. [27]).

In the SHBR model, our signal has a change in mean determined by \( \Delta_n \) and also a change in its variance, which will be described by the parameter

\[
\kappa_n := \frac{\sigma_n}{\sigma_0} > 0.
\]

In fact, we will see that the detection boundary is effectively determined by the ratio of \( \kappa_n^2 \) and \( \Delta_n \), which leads to three different regimes:

**DSR Dominant signal regime.** If \( \kappa_n^2 \) vanishes faster than \( \Delta_n \), the signal will asymptotically dominate the testing problem. In this case, the additional information will asymptotically vanish too fast, so that we expect the same undetectable signals as for the homogeneous case. The dominant signal regime consists of all cases in which \( \kappa_n^2/\Delta_n \rightarrow 0, \ n \rightarrow \infty \).

**ER Equilibrium regime.** If \( \kappa_n^2 \) has a similar asymptotic behavior as \( \Delta_n \), we expect a gain from the additional information. The equilibrium signal regime consists of all cases in which \( \kappa_n \rightarrow 0, \Delta_n \rightarrow 0 \) and \( c := \lim_{n \rightarrow \infty} \kappa_n^2/(\Delta_n/\sigma_0) = \lim_{n \rightarrow \infty} \sigma^2_n/(\Delta_n \sigma_0) \) satisfies \( 0 < c < \infty \).

**DVR Dominant variance regime.** If \( \Delta_n \) vanishes faster than \( \kappa_n^2 \), the variance will asymptotically dominate the testing problem. In this case, we expect the same detection boundary as for the case of testing for a jump in variance only. The dominant variance regime consists of all cases in which \( \kappa_n \rightarrow 0 \) and \( \kappa_n^2/\Delta_n \rightarrow \infty, \ n \rightarrow \infty \).

Throughout the following we need notation for asymptotic inequalities. For two sequences \( (a_n)_{n \in \mathbb{N}} \) and \( (b_n)_{n \in \mathbb{N}} \), we say that \( a_n \) is asymptotically less or equal to \( b_n \) and write \( a_n \lesssim b_n \) if there exists \( N \in \mathbb{N} \) such that \( a_n \leq b_n \) for all \( n \geq N \). Similarly we define \( a_n \gtrsim b_n \). If \( a_n/b_n \rightarrow c \) as \( n \rightarrow \infty \) for some \( c \in \mathbb{R} \setminus \{0\} \) we write \( a_n \sim b_n \). If \( c = 1 \) we write \( a_n \asymp b_n \). We use the terminology
to say that $a_n$ and $b_n$ have the same asymptotic rate (but probably different constant) if $a_n \sim b_n$, and that they have the same rate and constant if $a_n \asymp b_n$. Consequently, the relation in (4) determining the detection boundary in the homogeneous case becomes

$$\sqrt{n}|T_n| \Delta_n \asymp \sqrt{2}\sigma_0 \sqrt{- \log (|T_n|)}.$$  

The notations $\sim$ and $\asymp$ coincide with standard notations. The definitions of $\gtrsim$ and $\lesssim$ can be seen as extensions of the classical minimax notations by Ingster [27] or Dümbgen & Spokoiny [20], where the null hypothesis is tested against the complement of a ball around 0 with varying radius $\rho = \rho(n)$. The results are typically presented by stating that if $\rho(n)$ tends to 0 faster than the boundary function $\rho^*(n)$, or if their ratio converges to some constant $< 1$, then the null and the alternative hypotheses are indistinguishable.

In our setting, the asymptotic conditions are defined in terms of sums rather than ratios. Thus we introduced $\lesssim$ and $\gtrsim$ to have similar notations.

With this notation, we can collect the most important results of this paper in the following Table 1. The corresponding detection boundaries in different regimes are also illustrated in Figure 2.

|                | rate $\sqrt{n}|T_n| | \Delta_n \sim \sqrt{- \log (|T_n|)}$ | constant $\Delta_n$ known | $\Delta_n$ unknown |
|----------------|--------------------------|-------------------------------|--------------------------|-------------------|
| DSR            | $\sqrt{2}\sigma_0 - \varepsilon_n$ | $\sqrt{2}\sigma_0 + \varepsilon_n$ | $\sqrt{2}\sigma_0 + \varepsilon_n$ | $\sqrt{2}\sigma_0 + \varepsilon_n$ |
|                | Thm. 3.1                  | Thm. 4.1                      | Thm. 4.4                  | Thm. 4.4          |
| ER             | $2\sqrt{2}\sigma_0 \sqrt{\frac{1}{2+\varepsilon_n}} - \varepsilon_n$ | $2\sqrt{2}\sigma_0 \sqrt{\frac{1}{2+\varepsilon_n}} + \varepsilon_n$ | $\sigma_0 \sqrt{\frac{1}{1+\varepsilon_n}} + \varepsilon_n$ | $\sigma_0 \sqrt{\frac{1}{1+\varepsilon_n}} + \varepsilon_n$ |
|                | Thm. 3.2                  | Thm. 4.2                      | Thm. 4.4                  | Thm. 4.4          |
| DVR            | $2\sqrt{\frac{c^2}{2+\varepsilon_n}} - \varepsilon_n$ | $2\sqrt{\frac{c^2}{2+\varepsilon_n}} + \varepsilon_n$ | $\varepsilon_n$ | $\varepsilon_n$ |
|                | cf. (29)                  | analog to (29)                | Thm. 3.3                  | Thm. 4.3          |

Here $(\varepsilon_n)$ is any sequence such that $\varepsilon_n \to 0$, $\varepsilon_n \sqrt{- \log (|T_n|)} \to \infty$.

The second column depicts the rates obtained in the different regimes, the columns three to five give the constants in different situations. ER is stated twice, because the results are shown w.r.t. the different rates from DSR and DVR respectively.

Exemplary, the lower bound entry in the DSR denotes that signals are no longer detectable if $\sqrt{n}|T_n| \Delta_n \gtrsim (\sqrt{2}\sigma_0 - \varepsilon_n) \sqrt{- \log (|T_n|)}$. Vice versa, the upper bound entry in the DSR means that signals are detectable as soon as $\sqrt{n}|T_n| \Delta_n \lesssim (\sqrt{2}\sigma_0 + \varepsilon_n) \sqrt{- \log (|T_n|)}$, no matter if $\Delta_n$ is known or needs to be estimated from the data.

$\Delta_n$ known. It can readily be seen from Table 1 that the lower and the upper bounds with known $\Delta_n$ coincide up to the $\pm \varepsilon_n$ term in all regimes. This directly implies that the detection boundaries are determined by these constants, which we will describe in more detail now.
**DSR:** A comparison of the lower and upper bounds in Table 1 yields that the detection boundary is given by

$$\sqrt{n |I_n| \Delta_n} \approx \sqrt{2\sigma_0} \sqrt{-\log(|I_n|)}.$$  \hfill (7)

We point out that the detection boundary in the dominant signal regime hence coincides with the detection boundary in the homogeneous model (cf. (4)). More precisely, if the additional information $\kappa_n^2$ about a jump in the variance vanishes faster than the jump $\Delta_n$ in mean, we will not profit from this information.

**ER:** It follows similarly that the detection boundary is given by

$$\sqrt{n |I_n| \Delta_n} \approx \sqrt{2\sigma_0} \sqrt{\frac{2}{2 + c^2}} \sqrt{-\log(|I_n|)},$$

where $c = \sigma_0^{-1} \lim_{n \to \infty} \sigma_n^2 / \Delta_n$. The characterization (8) shows that we do always profit from the additional information in the ER case as $\sqrt{\frac{2}{2 + c^2}} < 1$ whenever $c > 0$. Compared to the homogeneous model or the dominant signal regime the constant improves but the rate stays the same. Note, that the improvement can be quite substantial, e.g. if $c = 1$, which amounts to the same magnitude of $\sigma_n$ and $\Delta_n$, the additional bump in the variance leads to a reduction of 33% sample size to achieve the same power compared to homogeneous bump detection. If $c = 2$, we obtain a reduction of 66%.

As $\Delta_n$ and $\kappa_n^2$ are of the same order in the ER, we may replace $\Delta_n$ by $\kappa_n^2$ in (8) and obtain the equivalent characterization

$$\sqrt{n |I_n| \kappa_n^2} \approx 2 \sqrt{\frac{c^2}{2 + c^2}} \sqrt{-\log(|I_n|)}.$$ \hfill (9)

This formulation allows for a comparison with the DVR below and is also depicted in Table 1, second ER row.

**DVR:** We find from entries three and four in the DVR row of Table 1 that the detection boundary is given by

$$\sqrt{n |I_n| \kappa_n^2} \approx 2 \sqrt{-\log(|I_n|)}.$$ \hfill (10)

If the jump in variance asymptotically dominates the jump in mean, we exactly obtain the detection boundary for a jump in variance only. This is a natural analog to (7) and coincides with the findings in the literature, cf. [17, Sect. 2.8.7.].

Finally note that if $c$ in (9) tends to $\infty$, we end up with (10). In this spirit a constant $c < \infty$ can be also seen as an improvement over the pure DVR where the rate stays the same but the constant decreases, which is an analog to the comparison of the DSR and the ER, see above.

A summary of the obtained detection boundary statements is illustrated in Figure 2.

**$\Delta_n$ unknown.** In case that $\Delta_n$ is unknown, it has to be estimated from the data $Y$ and the corresponding likelihood ratio test will be introduced in Section 4.4. The change in the considered statistics leads to different upper bounds. It is readily seen from Table 1 that the obtained upper bounds do not coincide with the lower bounds in all regimes.

**DSR:** It follows from the last entry in the DSR row that (7) also characterizes the adaptive detection boundary for unknown $\Delta_n$.

**ER:** If the parameter $\Delta_n$ is unknown, it can be seen from the last entry in the ER row that it is unclear if the lower bound remains sharp. We emphasize that we do not have an adaptive lower bound which would allow for an explicit statement here.

**DVR:** In this regime it is again unclear if the lower bound is also sharp in case that $\Delta_n$ is unknown (again we do not have an adaptive one), but we consider it likely that the obtained upper bound $1 + 3$ (cf. the last entry in the DVR row) is sharp.
This loss of a constant can be interpreted as the price for adaptation and is quantified by the ratio \( r(c) \) of (37) and (30), (38) and (31) and (39) and (33) respectively. The ratios are given by

\[
 r(c) = \begin{cases} 
 1 & \text{DSR, } c = 0, \\
 \sqrt{2\tau\sigma^2/c} & \text{ER, } 0 < c < \infty, \\
 \frac{1+\sqrt{3}}{2} & \text{DVR, } c = \infty, 
\end{cases}
\]

which are displayed in Figure 3. Note that \( r \) is a continuous function. Remarkably, the price for adaptation is never larger than \( \sqrt{2} \), as \( r \) attains its maximum at \( c = \sqrt{2} \), such that \( r(\sqrt{2}) = \sqrt{2} \). As \( c \to \infty \), \( r \) tends to \((1 + \sqrt{3})/2\), as \( c \to 0 \), it tends to 1, meaning that adaptation has no cost at all in this situation. Note, that this is in line with findings for the homogeneous Gaussian model [22].

If we relax the regimes ER and DVR by allowing for \( \sigma_n \to \sigma > 0 \), things become more complicated. We will prove lower bounds for those cases as well, which include logarithmic terms in
the likelihood ratio function is then given by \( \sup \)

\[ I \]

\( \Phi \)

\( S \)

We will now determine classes \( \kappa \)

\( \Delta \)

the underlying deviation inequality (cf. Lemma 2.3) does not include logarithmic terms in \( \kappa \), neither do any from the literature. We will discuss this issue in detail in Subsection 2.2.

The paper is organized as follows: The methodology and the corresponding general results are stated in Section 2. Afterwards we derive lower bounds in Section 3 and upper bounds in Section 4. There we also discuss upper bounds for the case that \( \Delta \) is unknown and provide a likelihood ratio test which adapts to this situation. Finally we will discuss some open questions in Section 5. To ease the presentation all proofs will be postponed to the Appendix.

2. General methodological results

Fixing \( I_n \) in (6), we obtain the likelihood ratio by straightforward calculations as

\[
L_n (\Delta_n, I_n, \kappa_n; Y) = (\kappa_n^2 + 1)\exp \left( \sum_{i \in I_n} \frac{\kappa_n^2 I_n^2 + 2 Y_i \Delta_n - \Delta_n^2}{2 (1 + \kappa_n^2) \sigma_0^2} \right) \]

\[
= (\kappa_n^2 + 1)\exp \left( \frac{\kappa_n^2}{2 (1 + \kappa_n^2) \sigma_0^2} \sum_{i \in I_n} Y_i^2 - \frac{\Delta_n^2}{\sigma_0^2} \right). \quad (11)
\]

For the general testing problem (6) where only the size \(|I_n|\) is known but the location \( I_n \) not, the likelihood ratio function is then given by \( \sup_{I_n \in A_n} L_n (\Delta_n, I_n, \kappa_n; Y) \).

2.1. Lower bounds

Let \( \Phi \) be any test with asymptotic significance level \( \alpha \) under the null-hypothesis \( \mu \equiv 0, \lambda_n \equiv \sigma_0 \).

We will now determine classes \( S := (S_n)_{n \in \mathbb{N}} \) of bump functions, such that \( \Phi \) is not able to differentiate between the null hypothesis \( \mu \equiv 0, \lambda_n \equiv \sigma_0 \) and functions \((\mu_n, \lambda_n) \in S_n \) with power \( \geq \alpha \). To this end, we construct a sequence \( S \) such that

\[
\mathbb{E}_{\mu, \lambda} [\Phi (Y)] \leq \alpha + o(1) \quad \Rightarrow \quad \inf_{(\mu, \lambda) \in S} \mathbb{E}_{\mu, \lambda} [\Phi (Y)] \leq \alpha + o(1). \quad (12)
\]

For such classes \( S \) we say that \( S \) is undetectable.

In the following, the sequence \( S \) will always be characterized by asymptotic requirements on \( \Delta_n, \kappa_n \) and \(|I_n|\). To keep the notation as simple as possible, only the asymptotic requirements will be stated below, meaning that the sequence \( S \)

\[
S_n = \left\{ \mu_n = \Delta_n 1_{I_n}, \lambda_n = \sigma_0^2 + \sigma_n^2 1_{I_n} \mid \Delta_n, |I_n|, \kappa_n \text{ satisfy the specified requirements} \right\}
\]

is undetectable.

To prove lower bounds we use the following estimate, which has been employed in [20, 22] as well:

**Lemma 2.1.** Assume that (1)–(3) hold true and \( A_n \) is given by (5). If \( \Phi \) is a test with asymptotic level \( \alpha \in (0, 1) \) under the null-hypothesis \( \mu \equiv 0 \), i.e. \( \mathbb{E}_{\mu, \lambda} [\Phi (Y)] \leq \alpha + o(1) \), then

\[
\inf_{(\mu_n, \lambda_n) \in S_n} \mathbb{E}_{\mu, \lambda} [\Phi (Y)] - \alpha \leq \mathbb{E}_{\mu, \lambda} \left[ \frac{1}{|I_n|} \sum_{I_n \in A_n} L_n (\Delta_n, I_n, \kappa_n, Y) - 1 \right] + o(1), \quad (13)
\]

as \( n \to \infty \).
Recall that we want to construct $S$ such that the right-hand side of (13) tends to 0. This can be achieved (as in [20, Lemma 6.2]) by the weak law of large numbers. By controlling the moments of $L_n (Δ_n, t_n, κ_n, Y)$, this leads to the following theorem characterizing undetectable sets $S$:

**Theorem 2.2.** Assume the SHBR model is valid with known parameters $Δ_n, κ_n$ and $I_n$. The sequence $S$ determining the asymptotic behavior of $Δ_n, κ_n$ and $I_n$ is undetectable, if $|I_n| ↘ 0$ and there exists a sequence $δ_n > 0$, satisfying $δ_n < 1/κ_n^2$ such that for $n → ∞$,

$$n |I_n| Δ_n^2 (1 + δ_n) δ_n - δ_n n |I_n| \log (1 + κ_n^2) - n |I_n| \log (1 + δ_n κ_n^2) + δ_n \log (|I_n|) → -∞. \quad (14)$$

**2.2. A chi-squared deviation inequality**

It turns out that the analysis of the likelihood ratio test and the adaptive version for unknown $Δ_n$ in Subsection 4.4 requires a specific deviation inequality for the weighted sum of two non-central $χ^2$ distributed random variables. Recall that $X ∼ χ^2_d (a^2)$ with non-centrality parameter $a^2$ and $d$ degrees of freedom, if

$$X = \sum_{j=1}^{d} (ξ_j + a_j)^2 \quad \text{where} \quad ξ_j \overset{\text{i.i.d.}}{\sim} N(0, 1) \quad \text{and} \quad a^2 = \sum_{j=1}^{d} a_j^2.$$  

In this case $E[X] = d + a^2$ and $V[X] = 2(d + 2a^2)$.

In the following we consider the weighted sum of $k$ non-central chi-squared variables $Z = \sum_{i=1}^{k} b_i X_i$, where $b_i ≥ 0$ and $X_i ∼ χ^2_{d_i}(a_i^2)$ are independent with $d_i ∈ N, a_i^2 ≥ 0$, $i = 1, …, k$. Note that

$$E[Z] = \sum_{i=1}^{k} b_i (d_i + a_i^2) \quad \text{and} \quad V[Z] = 2 \sum_{i=1}^{k} b_i^2 (d_i + 2a_i^2).$$

**Lemma 2.3.** Let $Z = \sum_{i=1}^{k} b_i X_i$, where $b_i ≥ 0$ and $X_i ∼ χ^2_{d_i}(a_i^2)$ are independent with $d_i ∈ N, a_i^2 ≥ 0$, $i = 1, …, k$. Let $∥b∥_∞ = \max_{1 ≤ i ≤ k} |b_i|$. Then the following deviation bounds hold true for all $x > 0$:

$$P \left( Z ≤ E[Z] − \sqrt{2V[Z]}x \right) ≤ \exp(-x), \quad (15a)$$

$$P \left( Z > E[Z] + \sqrt{2V[Z]}x + 2∥b∥_∞x \right) ≤ \exp(-x). \quad (15b)$$

The proof of Lemma 2.3 can be found in Appendix A and is based on the fact that $Z − E[Z]$ is a sub-Gamma random variable, see [9, Sec. 2.4].

In the following we will briefly discuss other deviation inequalities for $χ^2$ distributed random variables and their connections to (15). At first we mention an inequality by Birgé [7, Lemma 8.1], which deals with $k = 1$ and coincides with our result in that case. Also related to our result is Prop. 6 in Rohde & Dumbgen [38],

$$P \left( Z > E[Z] + \sqrt{2V[Z]}x + 4∥b∥_∞^2x^2 + 2∥b∥_∞x \right) ≤ \exp(-x), \quad x > 0. \quad (16)$$

This differs from our bound (15b) by the additional $4∥b∥_∞^2x^2$ term in the square root, revealing (15b) as slightly sharper.

Other deviation results were proven for second order forms $f_w(s) = s^T B^T B s + w^T s, s ∈ \mathbb{R}^d$ of Gaussian vectors with $B ∈ \mathbb{R}^{d×d}, w ∈ \mathbb{R}^d$. To relate this with our situation, let $d = \sum_{i=1}^{k} d_i$, choose any $a_{j,i} ∈ \mathbb{R}$ such that $\sum_{j=1}^{d_i} a_{j,i}^2 = a_i^2$ for all $1 ≤ i ≤ k$ and set

$$B := \text{diag} \left( \sqrt{b_1} \cdot 1_{d_1}, \sqrt{b_2} \cdot 1_{d_2}, ..., \sqrt{b_k} \cdot 1_{d_k} \right),$$

$$w := (b_1 a_{1,1}, b_1 a_{2,1}, ..., b_{1, a_{1,1}}, b_2 a_{1,2}, ..., b_{2, a_{2,2}}, b_3 a_{1,3}, ...)^T.$$
Here \( \mathbf{1}_m = (1, \ldots, 1) \in \mathbb{R}^m \). Let \( \text{Tr}(A) \) be the trace of a matrix \( A \). Then \( \text{Tr}(B^TB) = \text{Tr}(B^2) = \sum_{i=1}^{k} b_i d_i \), \( \text{Tr}(B^4) = \sum_{i=1}^{k} b_i^4 d_i \), \( \|B^2\|_\infty = \|b\|_\infty \) and \( w^T w = \sum_{i=1}^{k} b_i^2 a_i^2 \). This yields

\[
 f_w (\zeta) = Z - \sum_{i=1}^{k} b_i a_i^2 = Z - \text{E}[Z] + \text{Tr}(B^2), \quad \zeta \sim \mathcal{N}(0, I_d)
\]

(17)

with \( Z \) as in Lemma 2.3.

A deviation inequality for the case \( w \neq 0 \) can be found in the monograph by Ben-Tal et al. [6, Prop. 4.5.10], which provides in our setting the bound

\[
 P \left( f_w (\zeta) > \text{Tr}(B^2) + 2 \sqrt{\|B^2\|_\infty^2 x^2 + 2(\text{Tr}(B^4) + w^T w)x + 2\|B^2\|_\infty^2} \right) \leq \exp(-x)
\]

(18)

for all \( x > 0 \), where \( \|B^2\| \) denotes the spectral norm of \( B^2 \) given by the square root of the largest eigenvalue. Taking (17) into account, this yields

\[
P \left( Z > \text{E}[Z] + 2 \sqrt{\sum_{i=1}^{k} b_i^2 (d_i + a_i^2) x + \|b\|_\infty^2 x^2 + 2\|b\|_\infty^2 x} \right) \leq \exp(-t), \quad t > 0
\]

for \( Z \) as in Lemma 2.3. As \( 8 \sum_{i=1}^{k} b_i^2 (d_i + a_i^2) > 2\text{V}[Z] \), this inequality is strictly weaker than the Dümbgen-Rohde estimate (16) in the present setting.

For the case \( w = 0 \) (corresponding to centered \( \chi^2 \) random variables or \( a_i = 0 \) for \( 1 \leq i \leq k \) in our notation), Lemma 2.3 reduces to the deviation inequality in Hsu et al. [25]:

\[
P \left( f_0 (\zeta) > \text{Tr}(B^2) + 2 \sqrt{\text{Tr}(B^4)} x + 2\|B^2\|_\infty^2 x \right) \leq \exp(-x), \quad x > 0.
\]

(19)

The situation \( w = 0 \) has also been considered by Spokoiny & Zhilova [45], where the following inequality is proven:

\[
P \left( f_0 (\zeta) > \text{Tr}(B^2) + \max \left\{ 2 \sqrt{2 \text{Tr}(B^4)} x, 6\|B^2\|_\infty^2 x \right\} \right) \leq \exp(-x), \quad x > 0
\]

(20)

Comparing (19) with (20) gives that (20) is sharper for moderate deviations, i.e. if

\[
(3 - 2\sqrt{2}) \frac{\text{Tr}(B^4)}{\|B^2\|_\infty^2} \leq x \leq \frac{1}{4} \frac{\text{Tr}(B^4)}{\|B^2\|_\infty^2},
\]

for large and small deviations it is weaker. We are actually interested in large deviations, and furthermore the restriction \( a_i = 0 \) makes (20) not applicable for our purpose.

### 2.3. Upper bounds

To construct upper bounds we will now consider the likelihood ratio test. Recall that \( |I_n| \) is known, but the true location of the bump is unknown. Motivated by (11) it is determined by

\[
 T_{n,|I_n|}^* (Y) := \sup_{I_n \in A_n} \frac{1}{\sigma_0^2} \sum_{i \in I_n} \left( Y_i + \frac{\Delta_n}{\kappa_n^2} \right)^2
\]

(21)

as test statistic.

Furthermore let

\[
 \Phi_n (Y) := \begin{cases} 
 1 & \text{if } T_{n,|I_n|}^* (Y) > c_{n,n}^*, \\
 0 & \text{else}
 \end{cases}
\]

(22)

be the corresponding test where \( c_{n,n}^* \in \mathbb{R} \) is determined by the test level.
In the following we will be able to analyze the likelihood ratio test (22) in DSR, ER and DVR with the help of Lemma 2.3. For the relaxed situations where \( \kappa_n \neq 0 \) we will prove lower bounds including logarithmic terms in Theorems 3.2 and 3.3. Upper bounds including logarithmic terms cannot be obtained by the deviation inequalities in Lemma 2.3, and we furthermore found no deviation inequality including logarithmic terms in the literature. Even worse we are not in position to prove such an inequality here, and thus we will not be able to provide upper bounds which coincide with the lower ones in the relaxed regimes.

Now we are able to present the main theorem of this section:

**Theorem 2.4.** Assume the SHBR model with \( |I_n| \searrow 0 \), \( \sigma_n > 0 \) holds true and let \( H_0, H_1^n \) be as in (6). Let \( \alpha \in (0,1) \) be a given significance level and

\[
c_{\alpha,n} = n |I_n| + \frac{n |I_n| \Delta_n^2}{\sigma_0^2 \kappa_n} - 2 \log (\alpha |I_n|) + 2 \sqrt{n |I_n| \left( 1 + 2 \frac{\Delta_n^3}{\sigma_0^2 \kappa_n^2} \right) \log \left( \frac{1}{\alpha |I_n|} \right)}.
\]

Assume furthermore that \( \Delta_n, |I_n| \) and \( \kappa_n \) satisfy the following condition:

\[
n |I_n| \left( \frac{\kappa_n^4 + 2 \frac{\Delta_n^2}{\sigma_0^2} n |I_n|}{\sigma_0^2} \right) + \frac{\kappa_n^2 \Delta_n^2 n |I_n|}{\sigma_0^2} \geq 2 \kappa_n^2 \log \left( \frac{1}{|I_n|} \right) + 2 \kappa_n^2 \log \left( \frac{1}{\alpha} \right) + 2 \sqrt{n |I_n| \left( \kappa_n^4 + 2 \frac{\Delta_n^2}{\sigma_0^2} \right) \log \left( \frac{1}{\alpha |I_n|} \right)}
\]

\[
+ 2 \left( 1 + \kappa_n^2 \right) \sqrt{n |I_n| \left( \kappa_n^4 + 2 (1 + \kappa_n^2) \frac{\Delta_n^2}{\sigma_0^2} \right) \log \left( \frac{1}{\alpha} \right)}
\]

Then the test (22) with the statistic given \( T_n^\kappa_{\alpha,|I_n|} \) defined in (21) and the threshold (23) satisfies

\[
E_{H_0} [\Phi_n (Y)] \leq \alpha \quad \text{and} \quad E_{H_1^n} [\Phi_n (Y)] \geq 1 - \alpha.
\]

This theorem allows us to analyze the upper bounds obtained by the likelihood ratio test in the regimes DSR, ER and DVR.

**Remark 2.5.** Suppose for a second that not only the width \( |I_n| \), but also the location \( I_n \) of the bump is known. In this case, the alternative becomes simple, and the analyzed likelihood ratio test will be optimal as it is the Neyman-Pearson test. It can readily be seen from the proofs that all \( \log (|I_n|) \)-terms vanish in this situation, whereas the other expressions stay the same. Thus our analysis will also determine the detection boundary in this case.

### 3. Lower bounds for the detection boundary

We will now determine lower bounds in the three different regimes by analyzing (14).

#### 3.1. Dominant signal regime (DSR)

As mentioned before, we expect the same lower bounds as for the homogeneous situation here.

**Theorem 3.1.** Assume the SHBR model with \( |I_n| \searrow 0 \) and let \( (\varepsilon_n) \) be any sequence such that \( \varepsilon_n \to 0, \varepsilon_n \sqrt{-\log |I_n|} \to \infty \).

1. If \( \sigma_n^2 / \Delta_n \to 0 \) and \( \sigma_n^2 = o (\varepsilon_n) \) as \( n \to \infty \), then the sequence \( S \) with

\[
\sqrt{n |I_n| \Delta_n} \lesssim \left( \sqrt{2} \sigma_0 - \varepsilon_n \right) \sqrt{-\log |I_n|}
\]

is undetectable.

2. If \( \sigma_n^2 / \Delta_n \to 0 \) where \( \sigma_n^2 = \sigma^2 (1 + o(\varepsilon_n)) \) and \( 1 / \Delta_n^2 = o (\varepsilon_n) \) as \( n \to \infty \), then the sequence \( S \) with (25) is undetectable.
3.2. Equilibrium regime (ER)

Now let us consider the case that \( \sigma_n^2 \) and \( \Delta_n \) are asymptotically of the same order. Consequently \( \kappa_n^2 \) and \( \Delta_n \) will be of the same order. In this situation we expect a gain by the additional information coming from \( \sigma_n^2 > 0 \). In fact, the following Theorem states that the constant in the detection boundary changes, but the detection rate stays the same:

**Theorem 3.2.** Assume the SHBR model with \( |I_n| \searrow 0 \) and let \( (\varepsilon_n) \) be any sequence such that \( \varepsilon_n \to 0, \varepsilon_n \sqrt{-\log (|I_n|)} \to \infty \).

1. Let \( \sigma_n^2 = \sigma_0 \Delta_n (1 + o(\varepsilon_n)) \), \( c > 0 \) and \( \sigma_n^2 = o(\varepsilon_n) \) as \( n \to \infty \). Then the sequence \( S \) with
\[
\sqrt{n} |I_n| \Delta_n \lesssim (C - \varepsilon_n) \sqrt{-\log (|I_n|)}, \quad C := \sqrt{2} \sigma_0 \sqrt{\frac{2}{1 + c^2}} \tag{26a}
\]
as \( n \to \infty \) is undetectable.

2. If \( \sigma_n^2 = \sigma^2 (1 + o(\varepsilon_n)) \) and \( \Delta_n = \sigma^2 \sigma_0 (1 + o(\varepsilon_n)) \) as \( n \to \infty \), then with \( \kappa := \sigma/\sigma_0 \) the sequence \( S \) with
\[
\sqrt{n} |I_n| \Delta_n \lesssim (C - \varepsilon_n) \sqrt{-\log (|I_n|)}, \quad C := \frac{1}{\sqrt{\frac{2\kappa}{2\kappa^2 + c^2}}} \tag{26b}
\]
as \( n \to \infty \) is undetectable.

In the first case, for \( c = 0 \), we have \( C = \sqrt{2} \sigma_0 \), which corresponds to no change in the variance and hence reduces to the homogeneous model. But if \( c > 0 \), we always obtain \( C < \sqrt{2} \sigma_0 \), more precisely we improve by the multiplicative factor \( \sqrt{2/(2 + c^2)} < 1 \).

As \( C = (\frac{\kappa^2}{2\kappa^2} (\kappa^2 + c^2) - \frac{1}{2} \log (1 + \kappa^2))^{-1/2} \) is not a multiple of \( \sqrt{2} \sigma_0 \), the gain in the second case (26b) is not that obvious. But using \( \log (1 + \kappa^2) \leq \kappa^2 \) with equality if and only if \( \kappa = 0 \) implies that \( C \leq \sqrt{2} \sigma_0 \) with equality if and only if \( \kappa = 0 \).

Finally, if \( \sigma_n \to 0 \), then by a Taylor expansion it follows that
\[
\frac{\kappa^2}{2\kappa^2} (\kappa^2 + c^2) - \frac{1}{2} \log (1 + \kappa^2) = \Delta_n^2 \frac{2 + c^2}{4 \sigma_0} + O(\sigma_n^2), \quad \sigma_n \to 0,
\]
and hence (26b) will reduce to (26a) in this situation.

3.3. Dominant variance regime (DVR)

Now let us consider the case that the jump in the signal \( \Delta_n \) vanishes faster than the jump in the variance \( \sigma_n^2 \). In this situation we again expect a further gain by the additional information. Somewhat surprisingly, we will even obtain a gain in the detection rate compared to the ER in (26).

**Theorem 3.3.** Assume the SHBR model with \( |I_n| \searrow 0 \) and let \( (\varepsilon_n) \) be any sequence such that \( \varepsilon_n \to 0, \varepsilon_n \sqrt{-\log (|I_n|)} \to \infty \).

1. If \( \sigma_n \Delta_n = \sigma_n^2 \theta_n \) with sequences \( \Delta_n, \sigma_n, \theta_n \to 0 \) as \( n \to \infty \) where \( \sigma_n^2 = o(\varepsilon_n) \) and \( \theta_n^2 = o(\varepsilon_n) \), then the sequence \( S \) with
\[
\sqrt{n} |I_n| \Delta_n \lesssim (2\sigma_0 - \varepsilon_n) \sqrt{-\log (|I_n|)} \theta_n \tag{27a}
\]
is undetectable.
2. If $\sigma_n = \sigma (1 + o (\varepsilon_n))$ and $\Delta_n^2 = o (\varepsilon_n)$ as $n \to \infty$, then the sequence $S$ with

$$\sqrt{n |I_n|} \lesssim (C - \varepsilon_n) \sqrt{-\log (|I_n|)}, \quad C := \frac{1}{\sqrt{\frac{\kappa_n^2}{2} - \frac{1}{2} \log (1 + \kappa_n^2)}}$$

is undetectable.

First note that (27a) can be equivalently formulated as

$$\sqrt{n |I_n|} |\kappa_n^2| \lesssim (2 - \varepsilon_n) \sqrt{-\log (|I_n|)}$$

which is also meaningful if $\Delta_n \equiv 0$ and hence determines the detection boundary if only a jump in the variance with homogeneous signal $\mu \equiv 0$ occurs. We want to emphasize that the exponent 2 of $\kappa_n$ in (28) seems natural, as testing the variance for a change is equivalent to testing the mean of the transformed quantities $(Y_i - \Delta_n 1_{I_n} (i/n))^2$ (see [17, Sect. 2.8.7.]). Consequently, the detection rate improves compared to the DSR (25), as we assume $\kappa_n^2 / \Delta_n \to \infty$.

Note that we can also rewrite (26a) in terms of the $\kappa_n^2$-rate, which gives the equivalent expression

$$\sqrt{n |I_n|} |\kappa_n^2| \lesssim (C - \varepsilon_n) \sqrt{-\log (|I_n|)}$$

This makes the ER comparable with the DVR. As expected $C \leq 2$, and thus we see a clear improvement in the constant over (28). If $c \to \infty$, then $C$ tends to 2 which coincides with (28).

Again, if $\kappa \to 0$ it can be seen by a Taylor expansion that (27b) reduces to (27a). Furthermore, (27b) can be seen as the natural extension of (26b) as $c \to \infty$.

3.4. Discussion

The condition $\sigma_n^2 = o (\varepsilon_n)$ seems to be unavoidable as soon as a Taylor expansion (cf. (50c)) is about to be used.

In several assumptions we require a convergence behavior of the form $1 + o (\varepsilon_n)$. Note that this cannot readily be replaced by $1 + o (1)$, because the detection boundary is well-defined only up to a term of order $\varepsilon_n$. Thus if the convergence behavior is slower, at least the constants will need to change depending on the specific convergence behavior.

4. Upper bounds for the detection boundary

After determining lower bounds for the detection boundary, we now aim for proving that those are sharp in the following sense: There exists a level $\alpha$ test which detects with asymptotic power $\geq 1 - \alpha$ bump functions having asymptotic behavior of the form (25), (26) and (27) where $(C - \varepsilon_n)$ on the right-hand side is replaced by $(C + \varepsilon_n)$.

4.1. Dominant signal regime (DSR)

Let us first consider the case $\sigma_n^2 / \Delta_n \to 0$ and $\sigma_n^2 = o (\varepsilon_n)$ as $n \to \infty$. The following theorem states that our lower bounds are optimal by exploiting the likelihood ratio test:

**Theorem 4.1.** Assume the SHBR model with $|I_n| \to 0$ and let $H_0, H_n^\alpha$ be defined by (6). Suppose $\varepsilon_n \to 0$ is any sequence such that $\varepsilon_n \sqrt{-\log (|I_n|)} \to \infty$ as $n \to \infty$ and $\alpha \in [0, 1]$ is a given significance level.

Furthermore, let

1. either $\sigma_n^2 / \Delta_n \to 0$ and $\sigma_n^2 = o (\varepsilon_n)$
2. or $\sigma_n^2 / \Delta_n \to 0$ and $\sigma_n^2 = \sigma^2 (1 + o (\varepsilon_n))$
as \( n \to \infty \). If
\[
\sqrt{n|I_n|} \Delta_n \gtrsim \left( \sqrt{2\sigma_0 + \varepsilon_n} \right) \sqrt{- \log |I_n|}, \tag{30}
\]
then the test (22) with \( T_n^{\varepsilon_n,|I_n|} \) defined in (21) and the threshold (23) satisfies
\[
\lim_{n \to \infty} \mathbb{E}_{H_0} [\Phi_n (Y)] \leq \alpha \quad \text{and} \quad \lim_{n \to \infty} \mathbb{E}_{H_1} [\Phi_n (Y)] \geq 1 - \alpha.
\]

4.2. Equilibrium regime (ER)

**Theorem 4.2.** Assume the SHBR model with \( |I_n| \to 0 \) and let \( H_0, H_1^n \) be defined by (6). Suppose \( \varepsilon_n > 0 \) is any sequence such that \( \varepsilon_n \sqrt{- \log |I_n|} \to \infty \) as \( n \to \infty \) and \( \alpha \in [0, 1] \) is a given significance level.

Furthermore let \( \sigma_n^2 = c \sigma_0 \Delta_n (1 + o(\varepsilon_n)) \) with \( c > 0 \) and \( \sigma_n^2 = o(\varepsilon_n) \) as \( n \to \infty \) and
\[
\sqrt{n|I_n|} \Delta_n \gtrsim (C + \varepsilon_n) \sqrt{- \log |I_n|}, \quad C := \sqrt{2\sigma_0} \sqrt{\frac{2}{2 + c^2}}.
\]

Then the test (22) with \( T_n^{\varepsilon_n,|I_n|} \) defined in (21) and the threshold (23) satisfies
\[
\lim_{n \to \infty} \mathbb{E}_{H_0} [\Phi_n (Y)] \leq \alpha \quad \text{and} \quad \lim_{n \to \infty} \mathbb{E}_{H_1} [\Phi_n (Y)] \geq 1 - \alpha.
\]

Comparing Theorems 3.2 and 4.2 one may note that our upper bounds so far do not handle the case where \( \kappa_n^2 \) does not tend to 0. In fact we can also obtain an upper bound from (24) for
\[
\sigma_n^2 = \sigma^2 (1 + o(\varepsilon_n)) \quad \text{and} \quad \Delta_n = \frac{\sigma^2}{\sigma_0} (1 + o(\varepsilon_n)) \quad \text{as} \quad n \to \infty, \text{i.e.}
\]
\[
\sqrt{n|I_n|} \gtrsim (C + \varepsilon_n) \sqrt{- \log |I_n|}, \quad C := \sqrt{2\kappa^2 + \frac{4\kappa^2}{c^2} + \frac{2\kappa^2}{c^2} + 1 + \frac{2}{c^2} + \sqrt{1 + \frac{2}{c^2}}}.
\]

Obviously, this bound does not coincide with the lower bounds from Theorem 3.2. This is due to the fact that our tail estimates for the \( \chi^2 \) distribution does not include any logarithmic terms, but our lower bounds do. Thus it is impossible to obtain the same bounds using Lemma 2.3 with our technique.

4.3. Dominant variance regime (DVR)

**Theorem 4.3.** Assume the SHBR model with \( |I_n| \to 0 \) and let \( H_0, H_1^n \) be defined by (6). Suppose \( \varepsilon_n > 0 \) is any sequence such that \( \varepsilon_n \sqrt{- \log |I_n|} \to \infty \) as \( n \to \infty \) and \( \alpha \in [0, 1] \) is a given significance level.

Furthermore let \( \sigma_0 \Delta_n = \sigma_n^2 \theta_n \) with sequences \( \Delta_n, \sigma_n, \theta_n \to 0 \) as \( n \to \infty \) where \( \sigma_n^2 = o(\varepsilon_n) \), \( \theta_n = o(\varepsilon_n) \) and
\[
\sqrt{n|I_n|} \Delta_n \gtrsim (2\sigma_0 + \varepsilon_n) \sqrt{- \log |I_n|} \theta_n \tag{33}
\]

Then the test (22) with \( T_n^{\varepsilon_n,|I_n|} \) defined in (21) and the threshold (23) satisfies
\[
\lim_{n \to \infty} \mathbb{E}_{H_0} [\Phi_n (Y)] \leq \alpha \quad \text{and} \quad \lim_{n \to \infty} \mathbb{E}_{H_1} [\Phi_n (Y)] \geq 1 - \alpha.
\]

Note again that we can also obtain upper bounds in case that \( \sigma_n = \sigma (1 + o(\varepsilon_n)) \) and \( \Delta_n^2 = o(\varepsilon_n) \) as \( n \to \infty \), i.e.
\[
\sqrt{n|I_n|} \gtrsim (C + \varepsilon_n) \sqrt{- \log |I_n|}, \quad C := \frac{\sqrt{2\kappa^2 + 1} + 1}{\kappa^2},
\]

which again cannot coincide with the lower bounds from Theorem 3.3 for the same reason as in the equilibrium regime.
### 4.4. Adaptation

In this section we will discuss an adaptive version of the likelihood ratio test for which $\Delta_n$ does not need to be known. In this case it is natural to replace $\Delta_n$ by its empirical version $\hat{\Delta}_n := (n | I_n |)^{-1} \sum i : \varepsilon \in I_n Y_i$. This leads to the marginal likelihood ratio

$$L_n (I_n, \kappa_n; Y) = (\kappa_n^2 + 1)^{-n | I_n | / 2} \exp \left( \frac{\kappa_n^2}{2\sigma_0^2 (\kappa_n^2 + 1)} \sum_{i : \varepsilon \in I_n} Y_i^2 + \frac{n | I_n |}{2\sigma_0^2} \Delta_n^2 \right)$$

$$= (\kappa_n^2 + 1)^{-n | I_n | / 2} \exp \left( \frac{\kappa_n^2}{2\sigma_0^2 (\kappa_n^2 + 1)} \sum_{i : \varepsilon \in I_n} (Y_i - \hat{\Delta}_n)^2 + \frac{n | I_n |}{2\sigma_0^2} \hat{\Delta}_n^2 \right)$$

and to its corresponding test statistic

$$T_{n, | I_n |} (Y) := \sup_{A_n \in A_n} \left( \frac{\kappa_n^2}{\sigma_0^2 (\kappa_n^2 + 1)} \sum_{i : \varepsilon \in A_n} (Y_i - \hat{\Delta}_n)^2 + \frac{1}{\sigma_0^2 n | I_n |} \left( \sum_{i : \varepsilon \in A_n} Y_i \right)^2 \right).$$

(35)

An analysis of the likelihood ratio test (22) with $T_n$ as in (35) can be carried out similarly to the non-adaptive case. The main difference is that we now need Lemma 2.3 with $k = 2$. Thus, the following result can be proven:

**Theorem 4.4.** Assume the SHBR model with $| I_n | \rightarrow 0$ and let $H_0, H_1^n$ be defined by (6). Suppose $\varepsilon_n > 0$ is any sequence such that $\varepsilon_n \sqrt{-\log (| I_n |)} \rightarrow \infty$ as $n \rightarrow \infty$ and $\alpha \in [0, 1]$ is a given significance level. Define the corresponding threshold by

$$c_n^\alpha := \frac{\kappa_n^2}{\kappa_n^2 + 1} (n | I_n | - 1) + 1 + 2 \left[ \frac{\kappa_n^2}{\kappa_n^2 + 1} \log (\frac{1}{\alpha | I_n |}) + 2 \log (\frac{1}{\alpha | I_n |}) \right]$$

(36)

Now suppose that we are in one of the following three situations:

- **DSR:** $\kappa_n^2 / \Delta_n \rightarrow 0$ and $1 / \Delta_n = o (\varepsilon_n)$ as $n \rightarrow \infty$ and
  $$\sqrt{n | I_n |} \Delta_n \gtrsim \left( \sqrt{2\sigma_0} + \varepsilon_n \right) \sqrt{-\log (| I_n |)}. \quad (37)$$

- **ER:** $\sigma_n^2 = c_0 \Delta_n (1 + o (\varepsilon_n))$ with $c > 0$ and $\sigma_n^2 = o (\varepsilon_n)$ as $n \rightarrow \infty$ and
  $$\sqrt{n | I_n |} \Delta_n \gtrsim \left( C + \varepsilon_n \right) \sqrt{-\log (| I_n |)}, \quad C := \frac{c c_0 + \sqrt{2 + 3c^2}}{1 + c^2}. \quad (38)$$

- **DVR:** $\sigma_0 \Delta_n = \sigma_n^2 \theta_n$ with sequences $\Delta_n, \sigma_n, \theta_n \rightarrow 0$ as $n \rightarrow \infty$ where $\sigma_n^2 = o (\varepsilon_n)$, $\theta_n = o (\varepsilon_n)$ and furthermore
  $$\sqrt{n | I_n |} \Delta_n \gtrsim \left( \left( 1 + \sqrt{3} \right) \sigma_0 + \varepsilon_n \right) \sqrt{-\log (| I_n |)} \theta_n. \quad (39)$$

In any of these cases the test (22) with the statistic given $T_{n, | I_n |}^\alpha$ defined in (35) and the threshold (36) satisfies

$$\lim_{n \rightarrow \infty} \mathbb{E}_{H_0} [\Phi_n (Y)] \leq \alpha \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathbb{E}_{H_1^n} [\Phi_n (Y)] \geq 1 - \alpha.$$
5. Conclusions and future work

Summary. We obtain asymptotic minimax testing rates for the problem of detecting a bump with unknown location \( I_n \) in a heterogeneous Gaussian regression model, the SHBR model. The lower bound condition (14) guides us for a distinction into three possible regimes in the SHBR model. It allows to quantify the influence of the presence of an additional information about the change in the variance on the detection rate. Both cases of known and unknown bump height \( \Delta_n \) are considered. In three possible regimes of the SHBR model the exact constants of the asymptotic testing risk are obtained for the case of known \( \Delta_n \). In the case of adaptation to an unknown \( \Delta_n \) the obtained rates remain the same as for the case of known \( \Delta_n \). The optimal constant coincides in the DSR with the case for known \( \Delta_n \), else it is unknown (see Table 1 for details). The obtained results are based on non-asymptotic deviation inequalities for the type I and type II errors of testing. We would like to stress that the thresholds (23) and (36) of both tests are non-asymptotic results are based on non-asymptotic deviation inequalities for the type I and type II errors of the detection boundary. We were not able to prove coinciding upper bounds due to limitations in the variance heterogeneity in the SHBR model. It allows to quantify the influence of the presence of an additional information about the change in the variance on the detection rate. Both cases of known and unknown bump height \( \Delta_n \) are considered. In three possible regimes of the SHBR model the exact constants of the asymptotic testing risk are obtained for the case of known \( \Delta_n \). In the case of adaptation to an unknown \( \Delta_n \) the obtained rates remain the same as for the case of known \( \Delta_n \). The optimal constant coincides in the DSR with the case for known \( \Delta_n \), else it is unknown (see Table 1 for details). The obtained results are based on non-asymptotic deviation inequalities for the type I and type II errors of testing. We would like to stress that the thresholds (23) and (36) of both tests are non-asymptotic in the sense that they guarantee the non-asymptotic significance level \( \alpha \). This allows to apply the proposed tests even for finite samples. We also provide non-asymptotic upper detection bound conditions (24) and (54). Lemma 2.1 on the lower bound estimate can be easily proven for the case of a non-asymptotic type I error as well.

Relation to inhomogeneous mixture models. Obviously, the HBR model is related to a Gaussian mixture model

\[
Z_i \sim \left(1 - \varepsilon\right) \mathcal{N} \left(0, \sigma_0^2\right) + \varepsilon \mathcal{N} \left(\Delta, \sigma^2\right), \quad i = 1, \ldots, n,
\]

(40)

which has been introduced to analyze heterogeneity in a linear model with a different focus, see e.g. [2, 12, 13]. For \( \varepsilon \sim n^{-\beta} (0 < \beta < 1) \) different (asymptotic) regimes occur. If \( \beta > 1/2 \), the non-null effects in the model are sparse, which leads to a different behavior as if they are dense (\( \beta \leq 1/2 \)). Although it is not possible to relate (40) with the SHBR model in general, it is insightful to highlight some distinctions and commonalities in certain scenarios (for the homogeneous case see the discussion part of [22]). A main difference to our model is that the non-null effects do not have to be clustered on an interval \( I_n \). It is exactly this clustering, which provides the additional amount of information due to the variance heterogeneity in the SHBR model.

A further difference is that the definition of an i.i.d. mixture as in (40) intrinsically relates variance and expectation of the \( Z_i \). In fact

\[
\mathbb{V}[Z_i] = \left(1 - \varepsilon\right)^2 \sigma_0^2 + \varepsilon^2 \sigma^2 = \sigma_0^2 - 2\varepsilon \sigma_0^2 + \varepsilon \left(\sigma_0^2 + \sigma^2\right),
\]

i.e. the analog \( R := \left(\mathbb{V}[Z_i] - \sigma_0^2\right) / \left[\sigma_0^2 \mathbb{E}[Z_i]\right] \) to the ratio \( \kappa_n^2 / \Delta_n \) in the SHBR model behaves like \( R \sim -2/\Delta \) as \( \varepsilon \downarrow 0 \). This shows that for fixed \( \Delta > 0 \) (40) is related to a modification of the SHBR model which allows for negative changes in the variance, i.e. \( \lambda_n^2 = \sigma_0^2 - \sigma_n^2 I_n > 0 \). For such a situation our analysis will yield the same results as in Table 1, as the quantity \( c \) enters e.g. the detection boundary (8) quadratically. Nevertheless, the case that \( \Delta > 0 \) is fixed is usually not considered for mixture models as in (40) so far. In the sparse regime (\( \beta > 1/2 \)), the common calibration is \( \Delta \sim \sqrt{\log(n)} \), which corresponds to the DSR and asymptotically behaves as the homogeneous problem. Hence, in this case the additional gain in information due to the variance heterogeneity does not occur in the mixture model (40). In the dense regime (\( \beta \leq 1/2 \)), one considers \( \Delta \sim n^{-r}, 0 < r < 1/2 \). This appears similar to the DVR, but no explicit comparison is possible here, as \( R \to -\infty \).

Open issues and extensions. Our results are incomplete concerning the situation if \( \kappa_n \to \kappa > 0 \) in the equilibrium and the dominant variance regime. We applied the same techniques to prove lower bounds in this case, and we conjecture that these lower bounds are optimal, i.e. determine the detection boundary. We were not able to prove coinciding upper bounds due to limitations in the used deviation inequalities. To do so we would need deviation inequalities for \( \chi^2 \) random variables involving the logarithmic terms from the moments (cf. (48)) which we estimated by polynomials, but such a result is beyond the scope of this paper.
Furthermore, we only investigated adaptation to $\Delta_n$, but it would also be interesting to see what happens if the "baseline" variance $\sigma_0$ or $\sigma_n$ is unknown. By using the empirical variance on $I_n$ as an estimator, this would lead to a likelihood ratio test with a test statistic involving fourth powers of Gaussian random variables. For the analysis good tail inequalities for such random variables are on demand.

Additionally, one could also ask for adaptation w.r.t. $|I_n|$ and corresponding lower bounds, but this is far beyond the scope of this paper and requires further research. See, however, [22] for the homogeneous case.

Another important open question is concerned with the situation of multiple jumps. In the homogeneous case this has been addressed in [22], and upper bounds with constants 4 for a bounded and 12 for an unbounded number of change points have been proven. Even though it seems likely that these constants are optimal, no corresponding lower bounds have been proven so far. In the present HBR model this is an open question as well.

As stressed in the Introduction we currently only deal with the SHBR model, i.e. that a jump in mean also enforces a jump in variance ($\kappa_n > 0$). But often it is more reasonable to consider the situation that whenever a jump in mean happens, there can also be a jump in variance, but not necessarily. This would be modeled by letting $\kappa_n \in [0, \infty)$. Still it is unclear what the detection boundary should be in this situation. In fact it is not even clear if the additional uncertainty leads to a loss of information.

We believe that lower bounds can be constructed in the same way (and it is likely that they stay the same), but the calculation of upper bounds seems quite more involved.

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\appendix

\section*{A. Proofs of Section 2}

\textbf{Proof of Lemma 2.1.} We follow [20],

$$\inf_{(\mu_n, \lambda_n) \in S_n} E_{\mu_n, \lambda_n} [\Phi_n (Y)]$$

\begin{equation}
\leq \inf_{I_n \in A_n} E_{\mu_n=\Delta_n I_n, \lambda_n^2=\sigma_0^2+\sigma_n^2 I_n} [\Phi_n (Y)] - \alpha
\end{equation}

\begin{equation}
\leq \frac{1}{n} \sum_{I_n \in A_n} \left[ E_{\mu_n=\Delta_n I_n, \lambda_n^2=\sigma_0^2+\sigma_n^2 I_n} [\Phi_n (Y)] - \alpha \right]
\end{equation}

\begin{equation}
\leq \frac{1}{n} \sum_{I_n \in A_n} E_{\mu_n=\Delta_n I_n, \lambda_n^2=\sigma_0^2+\sigma_n^2 I_n} [\Phi_n (Y) - E_{\mu \equiv 0} [\Phi_n (Y)]] + o(1)
\end{equation}

$$=E_{\mu \equiv 0, \lambda \equiv \sigma_n} \left[ \left( \frac{1}{n} \sum_{I_n \in A_n} L_n (\Delta_n, I_n, \kappa_n, Y) - 1 \right) \Phi_n (Y) \right] + o(1)$$

$$\leq E_{\mu \equiv 0, \lambda \equiv \sigma_n} \left[ \left| \frac{1}{n} \sum_{I_n \in A_n} L_n (\Delta_n, I_n, \kappa_n, Y) - 1 \right| \right] + o(1),$$

as $n \to \infty$. \hfill \Box

\textbf{Proof of Theorem 2.2.} By Lemma 2.1 we have to prove that (13) tends to 0 under the given conditions.
Suppose that
\[
\lim_{n \to \infty} \mathbb{E}_{\mu=0, \lambda=\sigma_0} \left[ 1 \{ |L_n(\Delta_n, I_n, \kappa_n, Y) - 1| \geq \xi \} \right] = 0
\]  
(41)
for any $\xi > 0$. Then the weak law of large numbers for triangular arrays (the condition (41) implies the conditions imposed in [40, Ex. 2.2]) is applicable (as $|I_n| \not\sim$ 0 and hence $L_n \not\sim\infty$), i.e., $m^{-1} \sum_{i=1}^{m} Z_i \to 1$ in probability with $m = l_n$ and $Z_i = L_n(\Delta_n, [(i-1) |I_n|, i |I_n|], \kappa_n, Y)$. As $Z_i > 0$ we have
\[
\mathbb{E} \left[ \frac{1}{m} \sum_{i=1}^{m} Z_i \right] = \frac{1}{m} \sum_{i=1}^{m} \mathbb{E} |Z_i| = \frac{1}{m} \sum_{i=1}^{m} \mathbb{E} [Z_i] = 1 \quad \text{for all } m \in \mathbb{N},
\]
which then implies $m^{-1} \sum_{i=1}^{m} Z_i \to 1$ in expectation, i.e., (13) tends to 0.

The only thing left to prove is that (41) holds true under the imposed conditions. To show this we will use the moments of $L_n = L_n(\Delta_n, I_n, \kappa_n, Y)$ under the null hypothesis. Note that under the null hypothesis $\mu \equiv 0$ it holds $Y_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_0^2)$ and thus due to independence for $\eta > 0$
\[
\mathbb{E}_{\mu=0, \lambda=\sigma_0} [L_n^2] = (1 + \kappa_n^2)^{-\eta \frac{n |I_n|}{2}} \mathbb{E}_{\mu=0, \lambda=\sigma_0} \left[ \exp \left( \eta \sum_{i \in i \in A_{n,j}} \frac{\kappa_n^2 Y_i^2 + 2 \eta \Delta_n - \Delta_n^2}{2 (1 + \kappa_n^2) \sigma_0^2} \right) \right],
\]
\[
= (1 + \kappa_n^2)^{-\eta \frac{n |I_n|}{2}} \prod_{i \in i \in A_{n,j}} \mathbb{E} \left[ \exp \left( \frac{\kappa_n^2 \sigma_0^2 X_i^2 + 2 \eta X_i \Delta_n - \Delta_n^2}{2 (1 + \kappa_n^2) \sigma_0^2} \right) \right],
\]
\[
= (1 + \kappa_n^2)^{-\eta \frac{n |I_n|}{2}} \left( \mathbb{E} \left[ \exp \left( \frac{\kappa_n^2 \sigma_0^2 X^2 + 2 \eta X \Delta_n - \Delta_n^2}{2 (1 + \kappa_n^2) \sigma_0^2} \right) \right] \right)^{n |I_n|},
\]
(42)
where $X \sim \mathcal{N}(0, 1)$. Thus we need to calculate the expectation of
\[
\exp \left( \frac{\kappa_n^2 \sigma_0^2 X^2 + 2 \eta X \Delta_n - \Delta_n^2}{2 (1 + \kappa_n^2) \sigma_0^2} \right) = \exp \left( -\frac{\eta \Delta_n^2}{2 \sigma_0^2 \kappa_n^2} \right) \cdot \exp \left( \eta s (X + \lambda)^2 \right)
\]
(43)
where we abbreviated
\[
s = \frac{\kappa_n^2}{2 (1 + \kappa_n^2)}
\]
\[
\lambda = \frac{\Delta_n}{\sigma_0 \kappa_n^2}.
\]
(44)
(45)
The right-hand side in (43) corresponds to the Laplace transform of a non-central $\chi^2$ distributed random variable given by
\[
\mathbb{E} \left[ \exp \left( t (X + \lambda)^2 \right) \right] = \frac{\exp \left( \frac{\lambda^2}{1 - 2t} \right)}{\sqrt{1 - 2t}}
\]
(46)
if $\lambda \in \mathbb{R}, t < 1/2$, otherwise the Laplace transform does not exist. Note that $t = \eta s < 1/2$ if and only if $\eta < 1 + 1/\kappa_n^2$. In this case the expectation is given by
\[
\mathbb{E} \left[ \exp \left( \frac{\kappa_n^2 \sigma_0^2 X^2 + 2 \eta X \Delta_n - \Delta_n^2}{2 (1 + \kappa_n^2) \sigma_0^2} \right) \right] = \exp \left( -\frac{\eta \Delta_n^2}{2 \sigma_0^2 \kappa_n^2} \right) \mathbb{E} \left[ \exp \left( \eta s (X + \lambda)^2 \right) \right]
\]
\[
= \frac{\sqrt{1 + \kappa_n^2}}{\sqrt{1 + (1 - \eta) \kappa_n^2}} \exp \left( \frac{\eta (\eta - 1) \Delta_n^2}{2 \sigma_0^2 (1 + (1 - \eta) \kappa_n^2)} \right).
\]
By (42) this implies
\[ E_{\mu \equiv 0, \lambda \equiv \sigma_0} [L_n^\eta] \]
\[ = (1 + \kappa_n^2)^{-\eta \frac{|I_n|}{n}} \left\{ \frac{\sqrt{1 + \kappa_n^2}}{\sqrt{1 + (1 - \eta) \kappa_n^2}} \exp \left( \frac{\eta (\eta - 1) \Delta_n^2}{2 \sigma_0^2 (1 + (1 - \eta) \kappa_n^2)} \right) \right\}^{n |I_n|} \]
\[ = \left( \frac{1 + \kappa_n^2}{(1 + (1 - \eta) \kappa_n^2)^{\frac{\eta |I_n|}{2}}} \right) \exp \left( \frac{\eta (\eta - 1) n |I_n| \Delta_n^2}{2 \sigma_0^2 (1 + (1 - \eta) \kappa_n^2)} \right). \]

To prove (41) we now note for any \( n \) such that \( I_n \geq 1/\xi \) as \( L_n > 0 \) we have
\[ E_{\mu \equiv 0, \lambda \equiv \sigma_0} [1_{\{|L_n - 1| \geq \xi l_n\}} |L_n - 1|] \]
\[ = \int_{\xi l_n}^\infty E_{\mu \equiv 0, \lambda \equiv \sigma_0} (L_n \geq x + 1) \, dx + \xi l_n E_{\mu \equiv 0, \lambda \equiv \sigma_0} (L_n \geq \xi l_n + 1) \]
\[ \leq \int_{\xi l_n}^\infty E_{\mu \equiv 0, \lambda \equiv \sigma_0} (L_n \geq x) \, dx + \xi l_n E_{\mu \equiv 0, \lambda \equiv \sigma_0} (L_n \geq \xi l_n) \]
\[ = E_{\mu \equiv 0, \lambda \equiv \sigma_0} [1_{\{|L_n \geq \xi l_n\}} L_n] \]
\[ \leq E_{\mu \equiv 0, \lambda \equiv \sigma_0} [(|L_n|)^{1+\delta} (\xi l_n)^{-\delta}] \]
\[ = \exp \left( n \frac{|I_n| \Delta_n^2 (1 + \delta) \delta}{2 \sigma_0^2} - \delta \frac{n |I_n|}{2} \log (1 + \kappa_n^2) - \frac{n |I_n|}{2} \log (1 - \delta \kappa_n^2) + \delta \log (|I_n|) - \delta \log (\xi) \right). \]

(47)

Now, for (47) to tend to 0, the exponent must converge to \( -\infty \). Consequently, the weak law of large numbers is applicable and finally (13) tends to 0 if (14) holds true.

**Proof of Lemma 2.3.** The following proof exploits the findings in [9, Sec. 2.4] for (15b) and closely follows Birgé’s proof [7, Lemma 8.1] for (15a).

Let \( X = Z - \xi \mathbb{E} [Z] \). We will show first that \( X \) is a sub-Gamma random variable, i.e.
\[ \log (\mathbb{E} [\exp(\lambda X)]) \leq \frac{\lambda^2 \nu}{2(1 - c \lambda)} 0 < \lambda < \frac{1}{c}, \]
with the variance factor \( \nu = \mathbb{V} [Z] \) and the scale parameter \( c = 2 ||b||_\infty \). Indeed, if \( 0 < \lambda < 1/(2\mu_i) \) for any \( i = 1, \ldots, k \), we have
\[ \log (\mathbb{E} [\exp(\lambda X)]) = \sum_{i=1}^k \log \mathbb{E} [\exp(\lambda Z_i)] - \lambda \mathbb{E} [Z] \]
\[ = \sum_{i=1}^k \left( \frac{\lambda^2 b_i^2}{1 - 2 \lambda b_i} - \frac{d_i}{2} \log (1 - 2 \lambda b_i) \right) - \lambda \sum_{i=1}^k b_i (a_i^2 + d_i) \]
\[ = \sum_{i=1}^k \frac{2 \lambda^2 b_i^2 a_i^2}{1 - 2 \lambda b_i} - \sum_{i=1}^k d_i \left( \frac{1}{2} \log (1 - 2 \lambda b_i) + \lambda b_i \right) \]
\[ \leq \sum_{i=1}^k \frac{2 \lambda^2 b_i^2 a_i^2}{1 - 2 \lambda b_i} + \sum_{i=1}^k d_i \frac{\lambda^2 b_i^2}{1 - 2 \lambda b_i} \]
\[ = \lambda^2 \sum_{i=1}^k \frac{b_i^2 (2a_i^2 + d_i)}{1 - 2 \lambda b_i} \leq \frac{\lambda^2 \mathbb{V} [Z]}{2(1 - 2\lambda c)}, \]

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Proof of Theorem 2.4. Recall that the test statistic is given by

\[ z = \sqrt{2 \nu x + cx} \]

where we used \(-2^{-1} \log (1 - 2y) - y \leq y^2 (1 - 2y)\) which holds true for all \(0 < y < 2^{-1}\). Thus, the second inequality (15b) immediately follows from the deviation bound for sub-Gamma random variables (see [9, P. 29]):

\[ P \left( X > \sqrt{2 \nu x + cx} \right) \leq \exp(-x), \quad x > 0. \]

Let us pass to the first inequality (15a). We have for any \(z > 0\):

\[ P \left( Z \leq E[Z] - z \right) = P \left( -X \geq z \right) \leq \inf_{\lambda \geq 0} E \left[ \exp(\lambda (X + z)) \right]. \tag{49} \]

Whenever \(\lambda \leq 0\) we have

\[
\log E \left[ \exp(\lambda X) \right] = \sum_{i=1}^{k} \frac{2 \lambda^2 b_i^2 a_i^2}{1 - 2 \lambda b_i} - \sum_{i=1}^{k} d_i \left( \frac{1}{2} \log(1 - 2 \lambda b_i) + \lambda b_i \right)
\leq \sum_{i=1}^{k} \frac{2 \lambda^2 b_i^2 a_i^2}{1 - 2 \lambda b_i} + \sum_{i=1}^{k} \lambda^2 b_i^2 d_i = \lambda^2 \sum_{i=1}^{k} \left( \frac{2 a_i^2 b_i}{b_i - 2 \lambda} + d_i b_i^2 \right)
\leq \lambda^2 \sum_{i=1}^{k} b_i^2 (2 a_i^2 + d_i) = \frac{\lambda^2}{2} \nu \left[ Z \right].
\]

Here we used the fact that \(-2^{-1} \log (1 - 2y) - y \leq y^2\) and that \(1/(\rho - 2\lambda) \leq 1/\rho\) for all \(\lambda \leq 0\) and \(\rho > 0\). Combining this estimate with (49) we find

\[
\log (P \left( Z \leq E[Z] - z \right)) \leq \inf_{\lambda \leq 0} \left( \frac{\lambda^2}{2} \nu \left[ Z \right] + \lambda z \right) = -\frac{z^2}{2 \nu [Z]}. \]

Taking \(z = \sqrt{2 \nu [Z]} x\) yields the first inequality (15a). \(\square\)

Proof of Theorem 2.4. Recall that the test statistic is given by

\[ T_{\alpha, n, |I_n|} (Y) = \sup_{A_n \in A_n} S(A_n), \]

with the inner part

\[ S(A_n) := \frac{1}{\sigma_0^2} \sum_{i \in I_n} \left( Y_i + \frac{\Delta_n}{\kappa_n} \right)^2. \]

We readily see that \(S(A_n)\) obeys the following distributions

Under \(H_0\) : \(S(A_n) \sim \chi^2_{|I_n|} \left( \frac{n |I_n| \Delta^2_n}{\sigma_0^2 \kappa^4_n} \right),\)

Under \(H_1^o\) : \(S(I_n) \sim (1 + \kappa^2_n) \chi^2_{|I_n|} \left( 1 + \kappa^2_n \right)^2 \left( \frac{n |I_n| \Delta^2_n}{\sigma_0^2 \kappa^4_n} \right).\)

Here \(I_n\) in the alternative denotes the true position of the jump.

Using (15b) with \(k = 1, b_1 = 1, d_1 = n |I_n|\) and \(a^2_1 = \frac{n |I_n| \Delta^2_n}{\sigma_0^2 \kappa^4_n}\) we find

\[
P_{H_0} \left( \sup_{I_n \in A_n} S(I_n) > c^*_{\alpha, n} \right) \leq \frac{1}{|I_n|} P_{H_0} \left( S(I_n) > c^*_{\alpha, n} \right)
\leq \frac{1}{|I_n|} \sup_{I_n \in A_n} \left( \frac{n |I_n| \Delta^2_n}{\sigma_0^2 \kappa^4_n} \right) > c^*_{\alpha, n}
\leq \alpha.
\]
For the type II error denote

\[ y_{n,\alpha} = (1 + \kappa_n^2) n |I_n| + (1 + \kappa_n^2)^2 \frac{n |I_n| \Delta_n^2}{\sigma_0^2 \kappa_n} - 2(1 + \kappa_n^2) \sqrt{n |I_n| + 2 (1 + \kappa_n^2) \frac{n |I_n| \Delta_n^2}{\sigma_0^2 \kappa_n}} \log \frac{1}{\alpha}. \]

Then we calculate using (15a) with \( k = 1 \), \( b_1 = (1 + \kappa_n^2) \), \( d_1 = n |I_n| \) and \( a^2 = (1 + \kappa_n^2) \frac{n |I_n| \Delta_n^2}{\sigma_0^2 \kappa_n} \) that

\[ \mathbb{P}_{H^*} \left( \sup_{A_n \in A_n} S(A_n) < y_{n,\alpha} \right) = \sup_{A_n \in A_n} \mathbb{P}_{B_n} \left( \sup_{A_n \in A_n} S(A_n) < y_{n,\alpha} \right) \leq \sup_{B_n \in A_n} \inf_{A_n \in A_n} \mathbb{P}_{B_n} (S(A_n) < y_{n,\alpha}) \leq \sup_{B_n \in A_n} \mathbb{P}_{B_n} (S(B_n) < y_{n,\alpha}) = \mathbb{P} \left( (1 + \kappa_n^2) \chi^2_{n|I_n|} (1 + \kappa_n^2) \frac{n |I_n| \Delta_n^2}{\sigma_0^2 \kappa_n} < y_{n,\alpha} \right) \leq \alpha. \]

Thus the claim is proven if \( y_{n,\alpha} \geq c_{n,\alpha}^*, \) i.e.

\[ (1 + \kappa_n^2) n |I_n| + (1 + \kappa_n^2)^2 \frac{n |I_n| \Delta_n^2}{\sigma_0^2 \kappa_n} - 2(1 + \kappa_n^2) \sqrt{n |I_n| + 2 (1 + \kappa_n^2) \frac{n |I_n| \Delta_n^2}{\sigma_0^2 \kappa_n}} \log \left( \frac{1}{\alpha} \right) \geq n |I_n| + \frac{n |I_n| \Delta_n^2}{\sigma_0^2 \kappa_n} - 2 \log (|I_n|) + 2 \sqrt{n |I_n| \left( 1 + 2 \frac{\Delta_n^2}{\sigma_0^2 \kappa_n^2} \right) \log \left( \frac{1}{\alpha} \right)}. \]

But it can be easily seen by rearranging terms and multiplying by \( \kappa_n^2 \) that this is equivalent to (24).

**B. Proofs of Section 3**

If \( \delta_n \kappa_n^2 \to 0 \) and / or \( \kappa_n \to 0 \), Taylor’s formula can be used to simplify some of the terms in (14), which we will use in the following:

\[ \frac{1 + \delta_n}{1 - \delta_n} \kappa_n^2 = \delta_n + \frac{\delta_n^2}{2} \kappa_n^2 + \frac{\delta_n^3 \kappa_n^4}{6} \left( 1 + \kappa_n^2 \right) + \frac{\delta_n^4 \kappa_n^6}{24} + O \left( \frac{\delta_n^6 \kappa_n^8}{720} \right), \]

\[ - \log \left( 1 - \delta_n \kappa_n^2 \right) = \delta_n \kappa_n^2 + \frac{\delta_n^2 \kappa_n^4}{2} + O \left( \frac{\delta_n^3 \kappa_n^6}{6} \right), \]

\[ - \delta_n \log \left( 1 + \kappa_n^2 \right) = - \delta_n \kappa_n^2 + \frac{\delta_n \kappa_n^4}{2} + O \left( \delta_n \kappa_n^6 \right). \]

**Proof of Theorem 3.2.** Note that \( \delta_n < \frac{\lambda}{\kappa_n^2} \) is satisfied for free here as the right-hand side diverges or is constant and \( \delta_n \to 0 \). Furthermore in both cases the assumption \( n |I_n| \Delta_n = O (\log (|I_n|)) \) is satisfied as well. Thus we can apply Theorem 2.2.

1. Under the assumptions of this part of the theorem, we may use (50a)–(50c) and insert \( \kappa_n^2 = c \Delta_n / \sigma_0 \left( 1 + o (\varepsilon_n) \right) \) to find that (14) is satisfied if

\[ \delta_n \left[ \frac{n |I_n| \Delta_n^2}{2 \sigma_0^2} \left( 1 + \frac{c^2}{2} \right) \log (|I_n|) \right] + \delta_n^2 \frac{n |I_n| \Delta_n^2}{2 \sigma_0^2} \left( 1 + \frac{c^2}{2} \right) + o \left( \delta_n \varepsilon_n n |I_n| \Delta_n^2 \right) \to - \infty. \]
Here we used that $\Delta_n = o(\varepsilon_n)$. Then with $\delta_n := \sigma_0^{-1}\varepsilon_n$ and using (26a) we obtain

$$
\delta_n \left[ \frac{n |I_n| \Delta_n^2}{2\sigma_0^2} \left( 1 + \frac{\varepsilon_n^2}{2} \right) + \log |I_n| \right] = \sigma_0^{-1}\varepsilon_n \log |I_n| \left( 1 - (C - \varepsilon_n)^2 C^{-2} \right)
$$

$$
= \frac{2\sigma_0^{-1}\varepsilon_n^2}{2} \log |I_n| - \sigma_0^{-1}C^{-2}\varepsilon_n^2 \log |I_n|
$$

and furthermore

$$
\delta_n^2 \frac{n |I_n| \Delta_n^2}{2\sigma_0^2} \left( 1 + \frac{\varepsilon_n^2}{2} \right) + \sigma_0^{-1}\varepsilon_n^2 \log |I_n|
$$

$$
= \sigma_0^{-1}\varepsilon_n^2 \log |I_n| \left( -\sigma_0^{-1}(C - \varepsilon_n)^2 C^{-2} + 2C^{-1} \right)
$$

$$
= \left( \sigma_0^{-2} \left( \sqrt{2 + \varepsilon_n^2} - 1 \right) \varepsilon_n^2 + O(\varepsilon_n^3) \right) \log |I_n|.
$$

It can be readily seen that $C_1 := \sqrt{2 + \varepsilon_n^2} - 1 \geq \sqrt{2} - 1 > 0$ for all $c \geq 0$. Thus under (26a) we have

$$
\delta_n \left[ \frac{n |I_n| \Delta_n^2}{2\sigma_0^2} \left( 1 + \frac{\varepsilon_n^2}{2} \right) + \log |I_n| \right] + \delta_n^2 \frac{n |I_n| \Delta_n^2}{2\sigma_0^2} \left( 1 + \frac{\varepsilon_n^2}{2} \right)
$$

$$
= (C_1 \varepsilon_n^2 + o(\varepsilon_n^2)) \log |I_n| \to -\infty
$$

as $n \to \infty$. Thus (51) is satisfied.

2. Here only (50a) and (50b) can be applied. Thus (14) is satisfied if

$$
\delta_n \left[ \frac{n |I_n| \Delta_n^2}{2\sigma_0^2} \left( \frac{1 + \varepsilon_n^2}{2} \right) + \log |I_n| \right] + \sigma_0^{-1}\varepsilon_n^2 \log |I_n| \to \infty
$$

(52)

Inserting the assumption $\Delta_n = \frac{\delta_0^2}{\sigma_0^2} (1 + o(\varepsilon_n)) = \frac{\delta_0^2}{\sigma_0^2} (1 + o(\varepsilon_n))$, (26b) and $\delta_n := \eta\varepsilon_n$ with $\eta > 0$ to be chosen later we find

$$
\delta_n \left[ \frac{n |I_n| \Delta_n^2}{2\sigma_0^2} \left( \frac{1 + \varepsilon_n^2}{2} \right) + \log |I_n| \right] + \sigma_0^{-1}\varepsilon_n^2 \log |I_n| \to \infty
$$

$$
= \eta \varepsilon_n \log |I_n| \left( 1 - (C - \varepsilon_n)^2 \left( \frac{\kappa^2}{2c^2} (\kappa^2 + c^2) - \frac{1}{2} \log (1 + \kappa^2) \right) + o(\varepsilon_n) \right)
$$

$$
= 2\eta C^{-1}\varepsilon_n^2 \log |I_n| (1 + o(1))
$$

and

$$
\delta_n^2 \frac{n |I_n| \Delta_n^2}{2\sigma_0^2} \left( 1 + \kappa^2 \right) + \frac{k^4}{4} + o(\varepsilon_n)
$$

$$
+ 2\eta C^{-1}\varepsilon_n^2 \log |I_n| (1 + o(1)) + O(\delta_n^2 n |I_n|)
$$

$$
= \eta^2 \varepsilon_n^2 \log (|I_n|) \left( 2\eta C^{-1} - (C - \varepsilon_n)^2 \left( \frac{k^4}{2c^2} (1 + \kappa^2) + \frac{k^4}{4} \right) + o(1) \right)
$$

$$
= \eta^2 \varepsilon_n^2 \log (|I_n|) \left( 2\eta C^{-1} - C^2 \left( \frac{\kappa^4}{2c^2} (1 + \kappa^2) + \frac{\kappa^4}{4} \right) + o(1) \right) \to -\infty
$$

if we choose e.g. $\eta^{-1} = C^3 \left( \frac{\kappa^4}{2c^2} (1 + \kappa^2) + \frac{\kappa^4}{4} \right)$. Thus (14) is satisfied.

\[\square\]
Proof of Theorem 3.1. 1. This follows directly from Theorem 3.2 with \( c = c_n \to 0 \).

2. Similarly as in the proof of Theorem 3.2 we have to show that (52) is satisfied. Therefore note that with \( \delta_n := \eta \varepsilon_n \) with \( \eta > 0 \) to be chosen later we have
\[
\delta_n \left[ n |I_n| \left( \frac{\Delta_n}{2\sigma_0^2} - \log \left(1 + \kappa^2\right) + \frac{\kappa^2}{2} + o(\varepsilon_n) \right) + \log(|I_n|) \right]
= \eta \varepsilon_n \log(|I_n|) \left[ 1 - \left( \sqrt{2} \sigma_0 - \varepsilon_n \right)^2 \left( \frac{1}{2\sigma_0^2} + \frac{1}{\Delta_n^2} \right) \right]
= \sqrt{2} \eta \varepsilon_n^{-1} \varepsilon_n^2 \log(|I_n|) (1 + o(1))
\]
by the assumption that \( 1/\Delta_n^2 = o(\varepsilon_n) \). Furthermore
\[
\delta_n^2 n |I_n| \left[ \frac{\Delta_n^2}{2\sigma_0^2} (1 + \kappa^2) + \frac{\kappa^4}{4} \right] + \sqrt{2} \eta \varepsilon_n^{-1} \varepsilon_n^2 \log(|I_n|) (1 + o(1))
= \varepsilon_n^2 \log(|I_n|) \left( \sqrt{2} \eta^{-1} \sigma_0^{-1} - (1 + \kappa^2) + o(1) \right) \to -\infty
\]
if we choose e.g. \( \eta = \sigma_0^{-1} (1 + \kappa^2)^{-1} \). This proves the claim.

Proof of Theorem 3.3. Note again that \( \delta_n < \frac{1}{\kappa_n^2} \) is satisfied for free here as the right-hand side diverges or is constant and \( \delta_n \to 0 \). Furthermore in both cases the assumption \( n |I_n| \Delta_n = O(\log(|I_n|)) \) is satisfied as well. Thus we can apply Theorem 2.2.

1. Under the assumptions of this part of the theorem, we may again use (50a)-(50c) to find that (14) is satisfied if
\[
\delta_n \left( n |I_n| \frac{\Delta_n^2}{2\sigma_0^2} + n |I_n| \frac{\kappa^4}{4} + \log(|I_n|) \right) + \delta_n^2 \left( n |I_n| \frac{\Delta_n^2}{2\sigma_0^2} + n |I_n| \frac{\kappa^4}{4} \right) + o(\delta_n \varepsilon_n \log(|I_n|)) \to -\infty
\]
(53)
Here we used that \( \kappa_n^2 = o(\varepsilon_n) \). By \( \kappa_n^2 = \frac{\sigma_n^2}{\sigma_0} = \frac{\Delta_n}{\sigma_0} \frac{1}{\varepsilon_n} \), we have under (27a) with \( \delta_n := \sigma_0^{-1} \varepsilon_n \) that
\[
\delta_n \left( \frac{n |I_n| \Delta_n^2}{2\sigma_0^2} + n |I_n| \frac{\kappa^4}{4} + \log(|I_n|) \right)
= \sigma_0^{-1} \varepsilon_n \left( \frac{n |I_n| \Delta_n^2}{2\sigma_0^2} + \frac{n |I_n| \kappa^4}{4} \right) + \log(|I_n|)
= \sigma_0^{-1} \varepsilon_n \log(|I_n|) \left[ 1 - \left( \frac{2\sigma_0 - \varepsilon_n}{2\sigma_0} \right)^2 (1 + \theta_n^2) \right]
= -2 \varepsilon_n \log(|I_n|) \frac{\varepsilon_n^2}{\sigma_0} + \left( \frac{2}{\sigma_0^2} \varepsilon_n^2 \log(|I_n|) - \frac{1}{2\sigma_0^2} \varepsilon_n^2 \log(|I_n|) \right) \left( \frac{1}{2} + \theta_n^2 \right)
= \varepsilon_n^2 \log(|I_n|) \left( \frac{2}{\sigma_0^2} + o(1) \right),
\]
where we used \( \theta_n^2 = o(\varepsilon_n) \). Furthermore
\[
\delta_n^2 \left( \frac{n |I_n| \Delta_n^2}{2\sigma_0^2} + n |I_n| \frac{\kappa^4}{4} \right) + \varepsilon_n^2 \log(|I_n|) \left( \frac{2}{\sigma_0^2} + o(1) \right) + o(\delta_n \varepsilon_n \log(|I_n|))
= \sigma_0^{-2} \varepsilon_n^2 \log(|I_n|) \left( 2 - \frac{(2\sigma_0 - \varepsilon_n)^2}{2\sigma_0^2} \left( \frac{1}{2} + \theta_n^2 \right) + o(1) \right)
= \varepsilon_n^2 \log(|I_n|) (1 + o(1)) \to -\infty.
\]
Thus (53) and hence (14) is satisfied.

2. Similarly as in the proof of Theorem 3.2 we have to show that (52) is satisfied. Therefore note that with \( \delta_n := \eta \varepsilon_n \) where \( \eta > 0 \) will be chosen later we have

\[
\delta_n \left[ n |I_n| \left( \frac{\Delta_n^2}{2\sigma_0^2} - \frac{1}{2} \log (1 + \kappa^2) + \frac{\kappa^2}{2} + o(\varepsilon_n) \right) + \log (|I_n|) \right] \\
= \eta \varepsilon_n \log (|I_n|) \left[ 1 + (C - \varepsilon_n)^2 \left( \frac{1}{2} \log (1 + \kappa^2) - \frac{\kappa^2}{2} \right) + o \left( \varepsilon_n^2 \log (|I_n|) \right) \right] \\
= 2C^{-1} \eta \varepsilon_n^2 \log (|I_n|) (1 + o(1)) \\
text{by the assumption that } \Delta_n^2 = o(\varepsilon_n). \text{ Furthermore} \\
\delta_n^2 n |I_n| \left[ \frac{\Delta_n^2}{2\sigma_0^2} + \frac{\kappa^4}{4} \right] + 2C^{-1} \eta \varepsilon_n^2 \log (|I_n|) (1 + o(1)) \\
= \eta \varepsilon_n^2 \log (|I_n|) \left( 2C^{-1} \eta^{-1} - (C - \varepsilon_n)^2 \frac{\kappa^4}{4} + o(1) \right) \to -\infty \\
if we choose e.g. \( \eta^{-1} = C^3 \varepsilon_n^4 \), which proves the claim.

\(\square\)

C. Proofs of Section 4

**Proof of Theorem 4.2.** Due to Theorem 2.4 we only have to show that under the assumptions of the theorem the condition (24) is satisfied as \( n \to \infty \). The given situation is \( \kappa_n^2 = c \Delta_n / \sigma_0 (1 + o(\varepsilon_n)) \).

Inserting this into (24) we have to show that

\[
(c^2 + 2) \left( \frac{C + \varepsilon_n}{\sigma_0} \right)^2 + \frac{\Delta_n}{\sigma_0} \left( \frac{C + \varepsilon_n}{\sigma_0} \right)^2 \\
\geq 2c \Delta_n \log \left( \frac{1}{\alpha |I_n|} \right) + 2 \sqrt{\frac{n |I_n| \Delta_n^2}{\sigma_0^2} \left( 2 + c^2 \log \left( \frac{1}{\alpha |I_n|} \right) \right)} \\
+ 2 \left( 1 + \frac{c \Delta_n}{\sigma_0} \right) \sqrt{n |I_n| \Delta_n^2 \left( c^2 + 2 + \frac{2c \Delta_n}{\sigma_0} \right) \log \left( \frac{1}{\alpha} \right)}
\]

as \( n \to \infty \). Inserting (31) and dividing by \(- \log (|I_n|)\) yields that

\[
(c^2 + 2) \left( \frac{C + \varepsilon_n}{\sigma_0} \right)^2 + \frac{\Delta_n}{\sigma_0} \left( \frac{C + \varepsilon_n}{\sigma_0} \right)^2 \\
\geq \frac{2c \Delta_n}{\sigma_0} \left( 1 + \frac{\log \alpha}{\log (|I_n|)} \right) + \frac{2 \left( C + \varepsilon_n \right)^2}{\sigma_0} \left( 2 + c^2 \right) \frac{\log (\alpha) + \log (|I_n|)}{\log (|I_n|)} \\
+ 2 \left( 1 + \frac{c \Delta_n}{\sigma_0} \right) \sqrt{c^2 + 2 + \frac{2c \Delta_n}{\sigma_0} \left( C + \varepsilon_n \right)^2 \frac{\log (\alpha)}{\log (|I_n|)}}
\]

is sufficient as \( n \to \infty \). The definition of \( C \) and some straightforward calculations using \( \Delta_n = o(\varepsilon_n) \) and \( \log \alpha / \log (|I_n|) \to 0 \) show that is enough to prove

\[
4 + \frac{4 \sqrt{c^2 + 2} \varepsilon_n}{\sigma_0} + \frac{c^2 + 2 \varepsilon_n^2}{\sigma_0} > 4 + \frac{2 \sqrt{2} a_2}{\sigma_0} \varepsilon_n + 8 \frac{\log (\alpha)}{\log (|I_n|)} + \frac{4 \sqrt{2} c^2 \varepsilon_n}{\sigma_0} \log (|I_n|)
\]

which directly follows from \( \varepsilon_n \sqrt{- \log (|I_n|)} \to \infty \). \(\square\)
Proof of Theorem 4.1. 1. This follows directly from Theorem 3.2 with \( c = c_n \to 0 \).

2. Inserting the assumptions and dropping all lower-order contributions, we find with \( \kappa := \sigma / \sigma_0 \) that for (24) the following is sufficient as \( n \to \infty \):

\[
(\kappa^2 + 2) \frac{n |I_n| \Delta_n^2}{\sigma_0^2} \geq 2\kappa^2 \log \left( \frac{1}{|I_n|} \right) + 2 \sqrt{2} \frac{\sqrt{n} |I_n| \Delta_n}{\sigma_0} \sqrt{\log \left( \frac{1}{|I_n|} \right)} + 2 \sqrt{2} (1 + \kappa)^{3/2} \frac{\sqrt{n} |I_n| \Delta_n}{\sigma_0} \sqrt{\log \left( \frac{1}{\alpha |I_n|} \right)}.
\]

Inserting (30) and dividing by \(- \log (|I_n|)\), we see that it is enough to prove

\[
(\kappa^2 + 2) \left( \frac{\sqrt{2} \sigma_0 + \varepsilon_n}{\sigma_0} \right)^2 \geq 2\kappa^2 + 2 \sqrt{2} \left( \frac{\sqrt{2} \sigma_0 + \varepsilon_n}{\sigma_0} \right) + 2 \sqrt{2} (1 + \kappa)^{3/2} \frac{\sqrt{2} \sigma_0 + \varepsilon_n}{\sigma_0} \sqrt{\log \left( \frac{1}{\alpha |I_n|} \right)}.
\]

But this condition is true due to our requirements on \( \varepsilon_n \).

Proof of Theorem 4.3. Due to Theorem 2.4 we only have to show that under the assumptions of the theorem the condition (24) is satisfied as \( n \to \infty \). The given situation is \( \kappa_n^2 = \Delta_n / (\sigma_0 \theta_n) (1 + o(\varepsilon_n)) \). Inserting this into (24) we have to show that

\[
\left( 2 + \frac{1}{\theta_n^2} \right) \frac{n |I_n| \Delta_n^2}{\sigma_0^2} + \Delta_n \frac{n |I_n| \Delta_n^2}{\sigma_0} \geq 2\Delta_n \frac{n |I_n| \Delta_n^2}{\sigma_0^2} \log \left( \frac{1}{|I_n|} \right) + 2 \sqrt{2} \frac{\sqrt{n} |I_n| \Delta_n}{\sigma_0} \sqrt{\log \left( \frac{1}{|I_n|} \right)} + 2 \left( 1 + \frac{\Delta_n}{\sigma_0 \theta_n} \right) \sqrt{\frac{n |I_n| \Delta_n^2}{\sigma_0} \left( 2 + \frac{1}{\theta_n^2} + \frac{2\Delta_n}{\sigma_0 \theta_n} \right) \log \left( \frac{1}{\alpha |I_n|} \right)}
\]

as \( n \to \infty \). Inserting (33) and dividing by \(- \log (|I_n|)\) yields that

\[
\frac{(2\sigma_0 + \varepsilon_n)^2}{\sigma_0^2} (2\theta_n^2 + 1) \geq \frac{(2\sigma_0 + \varepsilon_n)^2}{\sigma_0^2} \frac{\theta_n^2 \Delta_n}{\sigma_0} \geq 2\Delta_n \frac{(2\sigma_0 + \varepsilon_n)^2}{\sigma_0^2} (2 + \frac{1}{\theta_n^2} + \frac{2\Delta_n}{\sigma_0 \theta_n} \log \left( \frac{1}{\alpha |I_n|} \right))
\]

is sufficient as \( n \to \infty \) where we used that \( \Delta_n / \theta_n \sim \sigma_n^2 = o(\varepsilon_n) \). Some straightforward calculations and skipping all terms of order \( o(\varepsilon_n) \) shows that is enough to prove

\[
4 \frac{4}{\sigma_0} \varepsilon_n > 4 \sqrt{\log \left( \frac{\alpha}{\theta_n} \right) \log \left( \frac{\log (|I_n|)}{\log (|I_n|)} \right)} + \frac{2\varepsilon_n}{\sigma_0} \sqrt{\log \left( \frac{\alpha}{\theta_n} \right) \log \left( \frac{\log (|I_n|)}{\log (|I_n|)} \right)} + 4 \sqrt{\log \left( \frac{\alpha}{\theta_n} \right) \log \left( \frac{\log (|I_n|)}{\log (|I_n|)} \right)} + 4 \frac{2\varepsilon_n}{\sigma_0} \sqrt{\log \left( \frac{\alpha}{\theta_n} \right) \log \left( \frac{\log (|I_n|)}{\log (|I_n|)} \right)}
\]

Employing \( \sqrt{a + b} \geq \sqrt{a} + \sqrt{b} \) we find the sufficient condition

\[
\frac{2}{\sigma_0} \varepsilon_n > 8 \sqrt{\log \left( \frac{\alpha}{\sigma_0} \right) \log \left( \frac{\log (|I_n|)}{\log (|I_n|)} \right)}
\]

as \( n \to \infty \), which directly follows from \( \varepsilon_n \sqrt{- \log (|I_n|)} \to \infty \).

Proof of Theorem 4.4. For \( A_n \in \mathcal{A}_n \) let us abbreviate

\[
\tilde{Y}_{A_n} := \left( n |I_n| \right)^{-1} \sum_{i \in A_n} Y_i,
\]

\[
S_n^2 := \sum_{i \in A_n} (Y_i - \tilde{Y}_{A_n})^2.
\]
In our model where the \( Y_i \)'s are independent Gaussian, it follows from Cochran’s theorem that \( S_{A_n}^2 \) obeys a \( \chi^2_{n|A_n|-1} \) distribution and \( \bar{Y}_{A_n} \) an independent \( \chi^2_1 \) distribution. Now recall that the test statistic is given by

\[
T_{n,|I_n|}^\alpha(Y) = \sup_{A_n \in \mathcal{A}_n} S(A_n),
\]

with the inner part

\[
S(A_n) := \frac{\kappa_n^2 |I_n|}{\sigma_0^2 (\kappa_n^2 + 1)^2} A_n^2 + \frac{n |I_n|}{\sigma_0^2} (\bar{Y}_{A_n})^2
\]

Using the above results, we readily see that \( S(A_n) \) obeys the following distributions:

Under \( H_0 \):

\[
S(A_n) \sim \frac{\kappa_n^2}{\kappa_n^2 + 1} \chi^2_{n|I_n|-1} (0) + \chi^2_1 (0),
\]

Under \( H_1^a \):

\[
S(I_n) \sim \frac{\kappa_n^2}{\kappa_n^2 + 1} \chi^2_{n|I_n|-1} (0) + (\kappa_n^2 + 1) \chi^2_1 \left( \frac{n |I_n| \lambda_n^2}{\sigma_0^2 (1 + \kappa_n)} \right)
\]

Here \( I_n \) in the alternative denotes the true position of the jump.

For simplicity denote by \( \chi^2_j \) a chi-squared random variable with \( j \) degrees of freedom and by \( \xi \) a standard normal variable independent of \( \chi^2_j \). Now applying Lemma (15b) with \( k = 2, b_1 = \frac{\kappa_n^2}{\kappa_n^2 + 1}, d_1 = n |I_n| - 1, b_2 = 1, d_2 = 1 \) and \( a_1 = a_2 = 0 \) we get

\[
\mathbb{P}_{H_0} \left( \sup_{A_n \in \mathcal{A}_n} S(A_n) > c_{\alpha,n} \right) \leq \frac{1}{|I_n|} \mathbb{P}_{H_0} \left( S(A_n) > c_{\alpha,n} \right)
\]

\[
= \frac{1}{|I_n|} \mathbb{P} \left( \frac{\kappa_n^2}{\kappa_n^2 + 1} \chi^2_{n|I_n|-1} + \xi^2 > c_{\alpha,n} \right)
\]

\[
\leq \alpha.
\]

Let us turn to the type II error. We will apply (15a) with \( k = 2, b_1 = \kappa_n^2, d_1 = n |I_n| - 1, b_2 = \kappa_n^2 + 1, d_2 = 1, a_1 = 0 \) and \( a_2^2 = \frac{n |I_n| \lambda_n^2}{\sigma_0^2 (1 + \kappa_n)} \). Denote

\[
y_{n,\alpha} = \kappa_n^2 n |I_n| + 1 + \frac{\Delta_n^2 n |I_n|}{\sigma_0^2} - 2 \sqrt{\left[ \kappa_n^4 n |I_n| + 2 \kappa_n^2 n + 1 + \frac{2 (1 + \kappa_n^2) \Delta_n^2 n |I_n|}{\sigma_0^2} \right] \log \frac{1}{\alpha}}
\]

We have

\[
\mathbb{P}_{H_1^a} \left( \sup_{A_n \in \mathcal{A}_n} S(A_n) < y_{n,\alpha} \right) = \sup_{B_n \in \mathcal{A}_n} \mathbb{P}_{B_n} \left( \sup_{A_n \in \mathcal{A}_n} S(A_n) < y_{n,\alpha} \right)
\]

\[
\leq \sup_{B_n \in \mathcal{A}_n} \inf_{A_n \in \mathcal{A}_n} \mathbb{P}_{B_n} \left( S(A_n) < y_{n,\alpha} \right)
\]

\[
\leq \sup_{B_n \in \mathcal{A}_n} \mathbb{P}_{B_n} \left( S(B_n) < y_{n,\alpha} \right)
\]

\[
= \mathbb{P} \left( \frac{\kappa_n^2}{\kappa_n^2 + 1} \chi^2_{n|I_n|-1} + (1 + \kappa_n^2) \left( \xi + \sqrt{\frac{\Delta_n^2 n |I_n|}{\sigma_0^2 (1 + \kappa_n^2)}} \right)^2 < y_{n,\alpha} \right)
\]

\[
= \mathbb{P} \left( \frac{\kappa_n^2}{\kappa_n^2 + 1} \chi^2_{n|I_n|-1} + (1 + \kappa_n^2) \left( \xi + \sqrt{\frac{\Delta_n^2 n |I_n|}{\sigma_0^2 (1 + \kappa_n^2)}} \right)^2 < y_{n,\alpha} \right)
\]

\[
\leq \alpha.
\]
Thus to find the detection boundary conditions we need to investigate the inequality \( y_{n,\alpha} \geq c_{n,\alpha} \):

\[
\kappa_n^2 |I_n| + 1 + \frac{\Delta_n^2 |I_n|}{\sigma_n^2} - 2 \sqrt{\left[ \kappa_n^2 |I_n| + 2\kappa_n^2 + 1 + \frac{2(1 + \kappa_n^2) \Delta_n^2 |I_n|}{\sigma_n^2} \right]} \frac{\log \frac{1}{\alpha}}{\alpha} \\
\geq \kappa_n^2 |I_n| + 1 + 2 \sqrt{\frac{\Delta_n^2 |I_n|}{\sigma_n^2} + \frac{2\kappa_n^2 + 1}{(\kappa_n^2 + 1)^2} \log \left( \frac{1}{\alpha |I_n|} \right)} - 2 \log (\alpha |I_n|) .
\]

First of all we can rewrite this inequality as follows,

\[
\frac{\kappa_n^2 |I_n|}{\kappa_n^2 + 1} + \frac{\Delta_n^2 |I_n|}{\sigma_n^2} - 2 \sqrt{\left[ \kappa_n^2 |I_n| + 2\kappa_n^2 + 1 + \frac{2(1 + \kappa_n^2) \Delta_n^2 |I_n|}{\sigma_n^2} \right]} \frac{\log \frac{1}{\alpha}}{\alpha} \\
\geq 2 \sqrt{\frac{\Delta_n^2 |I_n|}{\sigma_n^2} + \frac{2\kappa_n^2 + 1}{(\kappa_n^2 + 1)^2} \log \left( \frac{1}{\alpha |I_n|} \right)} - 2 \log (\alpha |I_n|) .
\]

Now we analyze it in the three regimes as usual:

- **Dominant signal regime**: Inserting \( \kappa_n^2 / \Delta_n = o(\varepsilon_n) \) and ignoring all \( o(\varepsilon_n) \)-terms in the following, we find that it is sufficient to prove

\[
\frac{\Delta_n^2 |I_n|}{\sigma_n^2} \geq 2 \left( \frac{1}{|I_n|} \right) + 2 \sqrt{\frac{\Delta_n^2 |I_n|}{\sigma_n^2} \log \left( \frac{1}{\alpha} \right)} .
\]

Inserting (37) and dividing by \( -\log (|I_n|) \) we find that the above is the case if

\[
2\sqrt{2}\sigma_0^{-1}\varepsilon_n \geq 4 \sqrt{\frac{\log (\alpha)}{\log (|I_n|)}}
\]

which is true by assumption.

- **Equilibrium regime**: Inserting \( \kappa_n^2 / \Delta_n = o(\varepsilon_n) \) and ignoring all \( o(\varepsilon_n) \)-terms in the following, we find that it is sufficient to prove

\[
\frac{\Delta_n^2 |I_n|}{\sigma_n^2} (1 + c^2) \\
\geq 2 \log \left( \frac{1}{|I_n|} \right) + 2 \sqrt{(c^2 + 2) \frac{\Delta_n^2 |I_n|}{\sigma_n^2} \log \left( \frac{1}{\alpha} \right)} + 2c \sqrt{\frac{\Delta_n^2 |I_n|}{\sigma_n^2} \log \left( \frac{1}{\alpha |I_n|} \right)} .
\]

Inserting (38) and dividing by \( -\log (|I_n|) \) we find that the above is the case if

\[
\frac{1 + c^2}{\sigma_n^2} C^2 + 2 \frac{1 + c^2}{\sigma_n^2} C\varepsilon_n + \frac{(1 + c^2)\varepsilon_n^2}{\sigma_n^2} \\
\geq 2 + 2C \frac{c}{\sigma_0} + \frac{2c}{\sigma_0} \varepsilon_n + \frac{2(C + \varepsilon_n)}{\sigma_0} \left( \sqrt{c^2 + 2} + c \right) \sqrt{\frac{\log (\alpha)}{\log (|I_n|)}}
\]

which is true by the definition of \( C \) and our assumption on \( \varepsilon_n \).

- **Dominant variance regime**: Inserting \( \kappa_n^2 = \Delta_n / (\sigma_0 \theta_n) \) and ignoring all \( o(\varepsilon_n) \)-terms in the following, we find that it is sufficient to prove

\[
\frac{\Delta_n^2 |I_n|}{\sigma_n^2} \left( 1 + \frac{1}{\theta_n^2} \right) \\
\geq 2 \log \left( \frac{1}{|I_n|} \right) + 2 \sqrt{\left( 2 + \frac{1}{\theta_n^2} \right) \frac{\Delta_n^2 |I_n|}{\sigma_n^2} \log \left( \frac{1}{\alpha} \right)} + \frac{2}{\theta_n^2} \sqrt{\frac{\Delta_n^2 |I_n|}{\sigma_n^2} \log \left( \frac{1}{\alpha |I_n|} \right)} .
\]
Let $C = 1 + \sqrt{3}$. Inserting (39), dividing by $-\log(|I_n|)$, and using $\theta_n^2 = o(\varepsilon_n)$ we find that the above holds true if

$$C^2 + 2 \frac{C}{\sigma_0^2} \varepsilon_n \geq 2 + 2C + (2C + 2C^2) \sqrt{\frac{\log(\alpha)}{\log(|I_n|)}}$$

which is true by the definition of $C$ and our assumption on $\varepsilon_n$.

References


