On the uniqueness of the maximum of the paths of random walks

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Abstract

Let \( \xi \) be i.i.d. symmetric random variables with \( P(\xi = 1) = 1/2 \) and denote by \( S_n \) their partial sums. We derive the distribution of the difference in the position of the last and first maximum of the random walk \( S_n \) up to time \( t \).

1 Introduction

Let \( \xi, \xi_1, \xi_2, ... \) be independent, identically distributed random variables with distribution \( P(\xi = 1) = P(\xi = -1) = 1/2 \). Then

\[
S_0 = 0, \quad S_n = \sum_{i=1}^{n} \xi_i
\]

forms a symmetric random walk on \( \mathbb{Z} \) and counts the difference in the occurrence of 1 compared to -1 in a random string of these two numbers.

For every \( n \geq 1 \) we are interested in the probability that the maximum of the function \( k \mapsto S_k \ (k = 0, 1, 2, ..., n) \) is unique. In fact, we show a little bit more:

Fix \( t \geq 1 \). Let

\[
m_t = m = \min \{ l \geq 0 : S_l = \max_{0 \leq j \leq t} S_j \}
\]

and

\[
M_t = M = \max \{ l \geq 0 : S_l = \max_{0 \leq j \leq t} S_j \}
\]

denote the first and the last maximum up to time \( t \). Then the random variable

\[
D_t = D = M - m
\]  

(1)

describes the 'time' difference between the last and the first occurrence of the maximum of a path until time \( t \). Our original question is to determine \( P(D = 0) \), so we need to determine the distribution of \( D \).
2 Some results from fluctuation theory

The following results are known in the literature ([1], [4], [5], [6]).

Proposition 2.1 For every $t = 1, 2, 3, ...$

\[
\sum_{k=0}^{t} \binom{2k}{k} \binom{2(t-k)}{t-k} 2^{-2t} = 1.
\]

Proposition 2.2 For any $0 \leq \alpha \leq \beta \leq 1$,

\[
\lim_{t \to \infty} \sum_{t \leq k \leq \beta t} \binom{2k}{k} \binom{2(t-k)}{t-k} 2^{-2t} = \frac{1}{\pi} \int_{\alpha}^{\beta} \frac{1}{\sqrt{x(1-x)}} dx.
\]

Proposition 2.3 For every integers $t \geq 1$ and $0 \leq r \leq t$ such that $t, r$ are both even or both odd,

\[
P(S_1 > 0, S_2 > 0, ..., S_{t-1} > 0, S_t = r) = \frac{r}{t} \left( \frac{t}{t+r} \right)^{2-t} \left[ \left( \frac{t-1}{\frac{t+r-2}{2}} \right) - \left( \frac{t-1}{\frac{t+r}{2}} \right) \right] 2^{-t}.
\]

Proposition 2.4 For every integer $t \geq 1$

\[
P(S_1 \geq 0, S_2 \geq 0, ..., S_t \geq 0) = \left( \frac{2([t+1]/2)}{[t+1]/2} \right) 2^{-2([t+1]/2)},
\]

where $[z]$ denotes the Gauß bracket.

Proposition 2.5 For every integer $t \geq 1$

\[
P(S_1 > 0, S_2 > 0, ..., S_t > 0) = \left( \frac{2[t/2]}{[t/2]} \right) 2^{-2[t/2]-1},
\]
The next result we were not able to find in the standard references on fluctuation theory.

**Proposition 2.6** For every integers $t \geq 1$ and $0 \leq r \leq t$ such that $t, r$ are both even or both odd,

$$P(S_1 \geq 0, S_2 \geq 0, ..., S_{t-1} \geq 0, S_t = r) = \frac{r + 1}{t + 1} \left( \frac{t + 1}{r + 2} \right)^{2-t}$$

$$= \left[ \left( \frac{t}{r+2} \right) - \left( \frac{t}{r+2} \right)^2 \right]^{2-t}.$$

**Proof.** Since $S_1 \geq 0$ means that $\xi_1 = 1$,

$$\{S_1 > 0, S_2 > 0, ..., S_t > 0, S_{t+1} = r + 1\} = \{\xi_1 = 1, \xi_2 = S_2 - 1 \geq 0, \xi_2 + \xi_3 = S_3 - 1 \geq 0, ..., \xi_2 + ... + \xi_t = S_{t+1} - 1 = r\} = \{\xi_1 = 1\} \cap \{\xi_2 \geq 0, \xi_2 + \xi_3 \geq 0, ..., \xi_2 + ... + \xi_t \geq 0, \xi_2 + ... + \xi_{t+1} = r\},$$

by Proposition 2.3 and stationarity we obtain

$$P(S_1 \geq 0, S_2 \geq 0, ..., S_{t-1} \geq 0, S_t = r) = 2P(S_1 > 0, S_2 > 0, ..., S_t > 0, S_{t+1} = r + 1)$$

$$= 2^{r + 1} \left( \frac{t + 1}{r + 2} \right)^{2-t}.$$ 

\[\square\]

### 3 The distribution of $(m, M)$

**Theorem 3.1** For every integer $t \geq 1$ and for any integers $0 \leq k \leq K \leq t$ such that $K - k$ is even

$$P(m = k, M = K) =$$

$$\frac{1}{K - k + 1} \left( \frac{K - k + 1}{K - k + 2} \right) \left( \frac{2[k/2]}{[k/2]} \right) \left( \frac{2[(t - K)/2]}{[(t - K)/2]} \right) 2^{-u(k, K)};$$

where

$$u(k, K) = 2[(t - K)/2] + 2[j/2] + K - k + 2 - \delta_{0,k} - \delta_{t,K},$$

and where $\delta_{\cdot\cdot}$ denotes the Kronecker symbol.

In case $K - k$ is odd

$$P(m = k, M = K) = 0.$$

**Remark 3.2** Note that $-2[(t - K)/2] - 2[k/2] + k - K - 2$ equals $-t - 2$ if $t, k, K$ are even, equals $-t$ if $t$ is even and $k, K$ are odd and equals $-t - 1$ if $t$ is odd (for $k, K$ even and for $k, K$ odd, respectively).
Proof. The event $A(k, K) = \{m = k, M = K\}$ can be written as

$$A(k, K) = A_1(k) \cap A_2(k, K) \cap A_3(K),$$

where

- $A_1(k) := \{S_k - S_{k-1} > 0, S_k - S_{k-2} > 0, ..., S_k - S_0 > 0\} = \{\xi_k > 0, \xi_k + \xi_{k-1} > 0, ..., \xi_k + + \xi_1 > 0\}$ for $k \geq 1$ and $A_1(0)$ a set of full measure,

- $A_2(k, K) = \{S_k - S_{k+1} > 0, S_k - S_{k+2} > 0, ..., S_k - S_K = 0\} = \{-\xi_{k+1} \geq 0, -\xi_{k+1} - \xi_{k+2} \geq 0, ..., -\xi_{k+1} - ... - \xi_{K-1} \geq 0, -\xi_{k+1} - ... - \xi_K = 0\},$

- $A_3(K) = \{S_K - S_{K+1} > 0, S_K - S_{K+2} > 0, ..., S_K - S_t > 0\} = \{-\xi_{K+1} > 0, -\xi_{K+1} - \xi_{K+2} > 0, ..., -\xi_{K+1} - ... - \xi_t > 0\}$ for $K \leq t - 1$ and $A_3(t)$ a set of full measure.

One notices by this representation that the three sets $A_1(k), A_2(k, K)$ and $A_3(K)$ are independent, hence

$$P(A(k, K)) = P(A_1(k))P(A_2(k, K))P(A_3(K)).$$

By Proposition 2.5, if $k \neq 0$ and $K \neq t$,

$$P(A_1(k)) = \left(\frac{2^{[k/2]}}{[k/2]}\right)2^{-2[k/2]-1},$$

and

$$P(A_3(K)) = \left(\frac{2^{[(t - K)/2]}}{[(t - K)/2]}\right)2^{-2[(t-K)/2]-1}.$$ 

In case $k = 0$ we have $P(A_1(0)) = 1$ and if $K = t$ $P(A_3(t)) = 1$. By Proposition 2.6 (and by symmetry and stationarity)

$$P(A_2(k, K)) = P(S_1 \geq 0, S_2 \geq 0, ..., S_{K-k-1} \geq 0, S_{K-k} = 0)$$

$$= \frac{1}{K-k+1} \left(\frac{K-k+1}{2} \right)^{2k-K}.$$ 

The theorem follows from these equalities. \qed

4 The distribution of $D$

Theorem 4.1 For every $d = 0, 1, ..., [t/2]$

$$P(D = 2d) = \frac{1}{d+1} \left(\frac{2d}{d}\right) \sum_{k=0}^{t-2d} \left(\frac{2^{[k/2]}}{[k/2]}\right) \left(\frac{2^{[(t - 2d - k)/2]}}{[(t - 2d - k)/2]}\right) 2^{-u(k,k+2d)}$$

$$= \frac{1}{d+1} \left(\frac{2d}{d}\right) \left(2^{-2d-1} + \left(\frac{2^{[(t - 2d)/2]}}{[(t - 2d)/2]}\right) 2^{-2[t/2]-1}\right).$$
Proof. Let $t = 2s$ be even. Consider the case $s = d$ first. We have by Theorem 3.1

$$P(D = 2s) = P(m = 0, M = 2s) = \frac{1}{d+1} \binom{2d}{d} 2^{-t} = \sum_{j=0}^{s-d} \frac{1}{2d+1} \binom{2d+1}{d+1} \binom{2j}{j} \binom{2(s-d-j)}{s-d-j} 2^{-t} 2^{2(s-d-j+2s)}$$

Next consider the cases $s - d > 0$. For $k = 2j$ or $k = 2j + 1$ we have $[k/2] = j$ and for $k = 2j$ or $k = 2j - 1$ we have $[(t - K)/2] = s - d - j$, since $K = k + 2d$. Therefore, using Theorem 3.1, Remark 3.2 and Proposition 2.1 for $s - d > 0$

$$P(D = 2d) = P(\exists k \in \{0, \ldots, t - 2d\} \text{ s. th. } m = k, M = k + 2d) = \frac{1}{d+1} \binom{2d}{d} \sum_{j=0}^{s-d} \frac{1}{2d+1} \binom{2d+1}{d+1} \binom{2j}{j} \binom{2(s-d-j)}{s-d-j} 2^{-t} 2^{2(s-2d-j)}$$

Now let $t = 2s + 1$ be odd. Then for $k = 2j$ or $k = 2j + 1$ we have $[k/2] = j$ and for $k = 2j$ or $k = 2j + 1$ we have $[(t - K)/2] = s - d - j$. Therefore, using Theorem 3.1, Remark 3.2 and Proposition 2.1 again

$$P(D = 2d) = \frac{1}{d+1} \binom{2d}{d} \sum_{j=0}^{s-d} \frac{1}{2d+1} \binom{2d+1}{d+1} \binom{2j}{j} \binom{2(s-d-j)}{s-d-j} 2^{2s-2}$$
Remark 4.2  Note that, for all $s \in \mathbb{N}$ and for all $d \in \{0, \ldots, s\}$,

\[ P(D_{2s} = 2d) = P(D_{2s+1} = 2d). \]

Here, $D_t$ is defined as in (1).

Corollary 4.3

\[ P(\text{maximum is unique}) = \frac{1}{2} + \left( \frac{2[t/2]}{[t/2]} \right) 2^{-2[t/2]-1}. \]

Corollary 4.4  For $n \in \mathbb{N}$,

\[ \sum_{k=0}^{n} \frac{1}{k+1} \left( \frac{2k}{k} \right) 2^{-2k-1} \left( 1 + \left( \frac{2(n-k)}{n-k} \right) 2^{-(2n-2k)} \right) = 1. \]

Corollary 4.5  For every $s \in \mathbb{N}$ let $D'_s := \frac{D_{2s}}{2}$. (Here, $D_{2s}$ is defined as in (1), for all $s \in \mathbb{N}$.) Then

\[ E(D'_s) = E\left( \frac{D_{2s}}{2} \right) = \sum_{k=1}^{s} \left( \frac{2k}{k} \right) 2^{-2k-1}. \]

Proof. It is immediately checked that for $s = 1$

\[ E(D'_1) = E\left( \frac{D_{2}}{2} \right) = 2^{-1} \cdot E(D_{2}) = 2^{-1} \left( 2 \cdot \frac{1}{4} \right) = \sum_{k=1}^{1} \left( \frac{2k}{k} \right) 2^{-2k-1}. \]

Moreover, for every $t \geq 3$,

\[ E\left( \frac{D_{t}}{2} + 1 \right) = \sum_{d=0}^{[t/2]} (d+1)P(D_t = 2d) \]

\[ = \sum_{d=0}^{[t/2]} \left( \frac{2d}{d} \right) \left( 2^{-2d-1} + \left( \frac{2[(t-2d)/2]}{[(t-2d)/2]} \right) 2^{-2[(t/2)-1]} \right) \]

\[ = \frac{1}{4} \sum_{d=1}^{[t/2]} (4(d-1)+2) \left( \frac{1}{d-1} + \left( \frac{2(d-1)}{d-1} \right) \left( 2^{-2(d-1)-1} + \left( \frac{2([t/2]-1-(d-1))}{[t/2]-1-(d-1)} \right) 2^{-2([t/2]-1)-1} \right) \right) \]

\[ + 2^{-1} + \left( \frac{2[t/2]}{[t/2]} \right) 2^{-2[t/2]-1} \]

\[ = \frac{1}{4} \left( 4E\left( \frac{D_{t-2}}{2} \right) + 2 \right) + \frac{1}{2} + \left( \frac{2[t/2]}{[t/2]} \right) 2^{-2[t/2]-1} \]

\[ = E\left( \frac{D_{t-2}}{2} \right) + 1 + \left( \frac{2[t/2]}{[t/2]} \right) 2^{-2[t/2]-1}. \]

Hereby

\[ E(D'_s) = E\left( \frac{D_{2s}}{2} \right) = E\left( \frac{D_{2s-2}}{2} \right) + \left( \frac{2s}{s} \right) 2^{-2s-1} = E(D'_{s-1}) + \left( \frac{2s}{s} \right) 2^{-2s-1}. \]
Remark 4.6  By Remark 4.2 and the foregoing Corollary, for all \( s \in \mathbb{N} \),
\[
E(D_{2s+1}) = 2E(D'_s).
\]

Corollary 4.7  For each \( D_s = D \) as in (1), \( s = 1, 2, \ldots \),
\[
ED^2_s = \frac{1}{2} \left( s + \sum_{k=1}^{s} (s - k - 2) \binom{2k}{k} 2^{-2k-1} \right).
\]

Proof. Similarly as before one shows that
\[
E(D^2_s + D_s) = E(D^2_{s-1}) + 3 \frac{1}{4} E(D_{s-1}) + \frac{1}{2}.
\]

Using Corollary 4.5 and \( E(D^2_1) = \frac{1}{4} \) one finds by induction
\[
E(D^2_s) = \frac{1}{4} + \frac{s - 1}{2} + \frac{1}{2} \sum_{l=1}^{s-1} \sum_{k=1}^{l} \binom{2k}{k} 2^{-2k-1} - \sum_{j=2}^{s} \binom{2j}{j} 2^{-2j-1}.
\]

5  The asymptotic distribution of \( D \)

Corollary 5.1  (Local limit theorem) Let \( d_t \) be a sequence of even integers satisfying
\[
\lim_{t \to \infty} d_t/t = x > 0.
\]
Then
\[
\lim_{t \to \infty} \sqrt{t^3} P(D_t = d_t) = \sqrt{\frac{2}{\pi x^3}}.
\]

Proof. It is well known that
\[
\binom{2n}{n} 2^{-2n} \sim \frac{1}{\sqrt{\pi n}}.
\]

Therefore, with \( 2d = d_t \sim tx \),
\[
P(D_t = 2d) = \frac{1}{d+1} \binom{2d}{d} 2^{-2d-1} + \frac{1}{d+1} \binom{2d}{d} (2\left\lfloor (t-2d)/2 \right\rfloor) 2^{-2\left\lfloor t/2 \right\rfloor - 1}
\]
\[
\sim \frac{1}{2\sqrt{\pi d^3}} + \frac{1}{2\sqrt{\pi d^3}} \frac{1}{t-2d/2}
\]
\[
\sim \frac{1}{t^{3/2} \sqrt{\pi x^3}} + \frac{2}{t^2 \pi \sqrt{x^3(1-x)}}.
\]
Corollary 5.2 (Distributional limit theorem) For any $0 < \alpha \leq \beta \leq 1$

$$\lim_{t \to \infty} \sqrt{t}P\left(\frac{1}{t}D_t \in [\alpha, \beta]\right) = \int_{\alpha}^{\beta} \sqrt{\frac{2}{\pi x^3}} dx = \sqrt{\frac{2}{\pi}}(\alpha^{-1/2} - \beta^{-1/2}).$$

Proof. This follows from Corollary 5.1, because the convergence there is uniform on intervals $[a, 1]$ for $a > 0$. □

Corollary 5.3 There exists a constant $C$ such that for every $N \geq 1$

$$\lim_{t \to \infty} P(D_t \geq N) \leq CN^{-1/2}.$$  

Proof. Using Stirling’s formula, there is a constant $C$ such that for any integer $d \geq 1$

$$P(D_t = d) \leq Cd^{-3/2}.$$  

Summing over $d \geq N$ shows the result. □

Corollary 5.4 For every integer $d \geq 0$

$$\lim_{t \to \infty} P(D_t = 2d) = \frac{1}{d+1} \binom{2d}{d} 2^{-2d-1}.$$  

In particular,

$$\sum_{d=0}^{\infty} \frac{1}{d+1} \binom{2d}{d} 2^{-2d-1} = 1,$$

hence the sequence $(D_t)_{t \in \mathbb{N}}$ converges weakly.

6 Lower and upper limit of $D_t$

In this section we study the lower limit of $D_t$ and the upper limit of $D_t/t$ for $t \to \infty$.

Proposition 6.1 We have

$$\liminf_{t \to \infty} D_t = 0 \ P\text{-a.s.}$$  

Proof. By $P(\limsup_{n \to \infty} S_n = \infty) = 1$ there exists, for almost every $\omega$ in the sample space, a sequence $(n_k)_{k \in \mathbb{N}} := (n_k(\omega))_{k \in \mathbb{N}}$ in $\mathbb{N}$, satisfying $S_{n_k}(\omega) = \max_{n \leq n_k} S_n(\omega)$ and $n_k \to \infty$. This implies $D_{n_k}(\omega) = 0$ for all $k \in \mathbb{N}$. □
Fig. 2: Realizations of $D_t/t$ for a path of the random walk up to time 100.

**Proposition 6.2** We have

$$\limsup_{t \to \infty} \frac{D_t}{t} > 0 \quad P\text{-a.s.}$$

**Proof.** We define a sequence $(X_n)_{n \in \mathbb{N}}$ of random variables by

$$X_1 := \min\{k > 0 : S_k = 1\},$$

and, for every $n \geq 2$,

$$X_n := \min\{k - X_{n-1} : S_k = n\}.$$

The $(X_n)_{n \in \mathbb{N}}$ are independent and identically distributed. By Proposition 2.6 and symmetry we obtain, for $t \in \mathbb{N}$,

$$P(X_1 = t) = P(S_t = 1|S_{t-1} = 0, S_{t-2} \leq 0, \ldots, S_1 \leq 0) \cdot P(S_{t-1} = 0, S_{t-2} \leq 0, \ldots, S_1 \leq 0)$$

$$= \left\{ \begin{array}{ll} 0 & : t \text{ even} \\ \frac{1}{t} \left( \frac{t}{2} \right)^{2-t} & : t \text{ odd.} \end{array} \right.$$
where \( Q_2(s) := \frac{Q_1(s)}{Q_1(s) + Q_3(s)} \) tends to one for \( s \to \infty \). Now
\[
\left( 1 + \frac{1}{s} \right)^{s - 1} \left( 1 - \frac{1}{s} \right)^{s - 1} = \left( 1 - \frac{1}{s^2} \right)^s \cdot Q_3(s),
\]
where \( Q_3(s) \) tends to 1 for \( s \to \infty \). Denoting the product of \( Q_2 \) and \( Q_3 \) by \( Q_4 \), we obtain
\[
P(X_1 \geq t) = \sqrt{\frac{2}{\pi}} \sum_{s \geq t, s \text{ odd}} \frac{1}{\sqrt{s^3 - s}} \cdot \left( 1 - \frac{1}{s^2} \right)^s \cdot Q_4(s)
\]
Using the binomial formula for \( (1 - s^{-2})^{s/2} \), we obtain
\[
(1 - s^{-2})^{s/2} = \sqrt{1 - s^{-2}} \cdot (1 - s^{-2})^{s/2} = \sqrt{1 - s^{-2}} \cdot (1 - \frac{s}{2s^2} + O(s^{-2})).
\]
Since \( Q_5(s) := \sqrt{1 - s^{-2}} \cdot (1 - (2s) - 1 + O(s^{-2})) \cdot Q_4(s) \) tends to 1 for \( s \to \infty \),
\[
P(X_1 \geq t) = \sqrt{\frac{2}{\pi}} \sum_{p \geq (t-1)/2} \frac{1}{\sqrt{(2p + 1)^3 - (2p + 1)}} \cdot Q_5(2p + 1)
\]
\[
= \sqrt{\frac{1}{2\pi}} \sum_{p \geq (t-1)/2} \frac{1}{\sqrt{2p^3 + 3p^2 + p}} \cdot Q_5(2p + 1).
\]
\[
= \sqrt{\frac{1}{4\pi}} \sum_{p \geq (t-1)/2} \frac{1}{\sqrt{p^3}} \cdot Q_6(2p + 1)
\]
\[
= \sqrt{\frac{1}{8\pi}} (t - 1)^{-\frac{3}{2}} Q_7(t)
\]
where the \( Q_l(p) \) \( (l = 5, 6, 7) \) tend to 1 for \( p \to \infty \). Therefore
\[
\lim_{x \to \infty} \frac{P(X_1 \geq tx)}{P(X_1 \geq x)} = \lim_{x \to \infty} \frac{Q_7(tx) \sqrt{x - 1}}{Q_7(x) \sqrt{tx - 1}} = t^{-1/2},
\]
and the distribution function of \( X_1 \) is regularly varying with index 1/2.

By Theorem 5.1 of Darling [3] (see [2], Theorem 3 for the converse statement) the random variables
\[
\max\{X_1, ..., X_n\}
\]
\[
X_1 + ... + X_n
\]
converge weakly to some non-degenerate distribution. It follows that there are constants \( a, \delta > 0 \) such that the sets
\[
A_n = \left\{ \frac{\max\{X_1, ..., X_n\}}{X_1 + ... + X_n} > a \right\}
\]
satisfy \( P(A_n) \geq \delta \) for all sufficiently large \( n \). Consequently a.e. point belongs to infinitely many sets \( A_n \).
Fix \( n \geq 1 \). Let \( X_j = \max\{X_1, \ldots, X_n\} \) and \( t_j = X_1 + \ldots + X_j \). Then at time \( t_j - 1 \) a last maxima up to time \( t_j - 1 \) occurs, while the first of these maxima occurs at time \( X_1 + \ldots + X_{j-1} \), hence

\[
\frac{D_{t_j-1}}{t_j - 1} = \frac{X_j - 1}{t_j - 1} \geq \frac{\max\{X_1, \ldots, X_n\}}{X_1 + \ldots + X_n}.
\]

The proposition follows from this estimate observing that it is \( > a \) for infinitely many \( n \) and that the associated \( j \) tends to infinity as does \( \max\{X_1, \ldots, X_n\} \). \( \square \)

References


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