Markov partitions for fibre expanding systems

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Abstract

Fibre expanding systems have been introduced in [6]. Here we show the existence of a finite partition for such systems which is fibrewise a Markov partition. Such partitions have direct applications to the Abramov-Rokhlin formula for relative entropy and certain polynomial endomorphisms of $\mathbb{C}^2$.

1 Fibre expanding systems

Let Y be a compact metric space and $T : Y \to Y$ be a continuous surjective map. Consider a fibred situation of a dynamical system (Y, T), where the map T is foliated over a continuous map $S : X \to X$ on some compact metric space X, with a continuous surjective factor map $\pi : Y \to X$ which semi-conjugates T to S:

$$\begin{array}{ccc} Y & \xrightarrow{T} & Y \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow{S} & X \end{array}$$

X is called the *base space* and S the *base transformation*. According to [6] T is said to be *fibre expanding* if there exist $a > 0$ and $\lambda \in (0, 1)$ such that

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the following holds:
If $u, v' \in Y$, $\pi(T(u)) = \pi(v')$ and $d(T(u), v') < 2a$, there exists a unique $v \in Y$ such that $\pi(v) = \pi(u)$, $\quad T(v) = v'$ and $d(u, v) < 2a$. Furthermore,

$$d(u, v) \leq \lambda d(T(u), T(v)).$$

The situation was examined from a purely topological point of view by Roy ([17]), who discussed the relations to expansiveness and openness of fibre maps.

When extending the thermodynamical formalism to the fibred situation, the first question is to generalize the notion of Gibbs measures to the relative case. A Gibbs measure is defined by the usual property that the Jacobian (of the disintegration measures) under fibre maps has a prescribed fibrewise (Hölder) continuous version. Problems of this type in the relativized context of fibred systems have been considered in the literature. In the work of Ferrero and Schmitt ([8]) and later of Bogenschütz and Gundlach ([2], [4]), this problem has been considered when the base transformation S is an invertible measure-preserving map of some probability space. The case of non-invertible transformations and fibrewise expanding system was considered in [6], while a Gibbs family for certain fibrewise expansive systems appears in [18].

In the present situation the classical Frobenius-Perron theory or spectral theory is not applicable, since the transfer operators act between different function spaces (see [7]). However, the construction of equilibrium measures has been accomplished in some cases (e.g. [12], [13], [11], [7]). The associated pressure (in the relative setting) is defined in [16] and a variational formula is proved. Bogenschütz in [2] and [4], and Kifer in [14] studied pressure functions for random bundle maps and their relative variational formulas. A new type of relative variational formula using the Abramov-Rohklin relative entropy formula has been derived in [7], where the maximum of the sum of conditional entropy and expectation over potentials is described by the integral of some generalized eigenvalue function over the module of functions constant on fibres.

The existence of relative generators has been studied in the invertible case by Kifer and Weiss (cf. [15]), independently by Danilenko and Park ([5]). In the non-invertible case the problem is more delicate (as in the non-relativised case). Some results in this direction are contained in [20].

The theory has been developed without the use of Markov partitions. This is because the existence of such a partition does not seem to be known. In this note we prove that there is a partition which, restricted to each fibre, is a generator and has the Markov property with respect to fibre maps. When the base space is reduced to a one point space, this partition will be a Markov partition in the usual sense (cf. [19]). It should be noted that our proof (when reduced to this particular case) gives a new and direct proof for the existence of a Markov partition for expanding and open maps.

2 Markov partitions

Let (Y,T) denote a fibred system which is fibrewise expanding over (X,S) as defined in section 1. The fibres over X will be denoted by $Y_x = \pi^{-1}\{x\}$. $T_x : Y_x \to Y_{S(x)}$ denotes the map T restricted to the fibre over $x \in X$. We shall prove the following theorem:

Theorem 1 *There exists a finite partition γ of Y such that*

(A) $T_x(G \cap Y_x) = \bigcup_{g \in \gamma; g \cap T(G) \neq \emptyset} g \cap Y_{S(x)}$ *for all* $G \in \gamma$ *and* $x \in X$.

(B) *There is a constant C such that*

$$\sup_{x \in X} \sup_{G \in \gamma_0^n} diam(G \cap Y_x) \leq C\lambda^n.$$

Proof. Let a and λ be as in the definition of the fibrewise expanding property. Choose δ so small that $\frac{\delta}{1-\lambda} < a$. Let $\mathcal{U}_0$ be a finite open cover of Y of sets of diameter $\leq \delta$.

For $U \in \mathcal{U}_0$ define

$$\mathcal{U}(U) = \{V \in \mathcal{U}_0 : V \cap T(U) \neq \emptyset\}.$$

Recursively, we let

$$\Psi_0(U) = U$$
$$\Psi_n(U) = \left\{ y \in \pi^{-1}(\pi(U)) : \begin{array}{l} T(y) \in \Psi_{n-1}(V) \text{ for some } V \in \mathcal{U}(U) \\ \text{and } d(y,U) < a \end{array} \right\}.$$

We first claim that

(a) $U \subset \Psi_{n-1}(U) \subset \Psi_n(U) \subset B(U, (\lambda + \ldots + \lambda^n)\delta)$

(b) $\pi(\Psi_n(U)) = \pi(U)$

(c) $T_x(\Psi_n(U)\cap Y_x) = \bigcup_{V\in\mathcal{U}(U)} \Psi_{n-1}(V)\cap Y_{S(x)}$ for all $x\in\pi(U)$.

This is proved by induction over n. For $n=1$ we obtain:

(a): $y\in U$, $T(y)\in V=\Psi_0(V)\in\mathcal{U}(U)$ implies $y\in\Psi_1(U)$, hence $U\subset\Psi_1(U)$. If $y\in\Psi_1(U)\cap Y_x$, there exist $V\in\mathcal{U}(U)$ and $z\in U\cap Y_x$ such that $T(y),T(z)\in V\cap Y_{S(x)}$. Therefore $d(T(z),T(y))<\delta$, and by the expanding property $d(z,y)<\lambda\delta$, i.e. $y\in B(U,\lambda\delta)$.

(b): Let $y\in\Psi_1(U)$. Then $\pi(y)\in\pi(U)$ by definition, so $\pi(\Psi_1(U))\subset\pi(U)$. The converse follows from $U\subset\Psi_1(U)$.

(c): Let $x\in\pi(U)$ and $y\in\Psi_1(U)\cap Y_x$. Then there exists $V\in\mathcal{U}(U)$ such that $T_x(y)=T(y)\in V$. Therefore

$$T_x(\Psi_1(U)\cap Y_x)\subset\bigcup_{V\in\mathcal{U}(U)} V\cap Y_{S(x)}.$$

Conversely, if $z\in V\cap Y_{S(x)}$, where $V\in\mathcal{U}(U)$, there exists $y\in T(U)$ with $d(z,T(y))<\delta<2a$. By the expanding property there exists $z'\in B(y,2a)\cap Y_x$ such that $T(z')=z$ and $d(z',y)<\lambda\delta<a$, whence $z'\in\Psi_1(U)$.

Assume that (a)–(c) hold for $n-1$.

(a): From the induction hypothesis we have that

$$U\subset\Psi_{n-2}(U)\subset\Psi_{n-1}(U)\subset B(U,(\lambda+\ldots+\lambda^{n-1})\delta).$$

Let $y\in\Psi_{n-1}(U)$. Then

- (i) $T(y)\in\Psi_{n-2}(V)\subset\Psi_{n-1}(V)$ for some $V\in\mathcal{U}(U)$.
- (ii) $y\in\pi^{-1}(\pi(U))$
- (iii) $d(y,U)<a$,

hence $y\in\Psi_n(U)$ and $U\subset\Psi_{n-1}(U)\subset\Psi_n(U)$.

Let now $y\in\Psi_n(U)$. Then there exists $V\in\mathcal{U}(U)$ such that $T(y)\in\Psi_{n-1}(V)$. Choose $z\in V$ such that $d(z,T(y))=d(V,T(y))$ and $z'\in U$ such that $T(z')\in V$. Then

$$d(T(y),T(z'))\le d(T(y),z)+d(z,T(z'))\le(\lambda+\ldots+\lambda^{n-1})\delta+\delta<2a,$$

hence by the expanding property,

$$d(y,z')\le\lambda(1+\lambda+\ldots+\lambda^{n-1})\delta=(\lambda+\ldots+\lambda^n)\delta.$$

(b): Let $y \in \Psi_n(U)$. Then $\pi(y) \in \pi(U)$ by definition, so $\pi(\Psi_n(U)) \subset \pi(U)$. The converse follows from $U \subset \Psi_n(U)$ (the induction hypothesis, resp. (a) as proved above).

(c): Let $x \in \pi(U)$ and $y \in \Psi_n(U) \cap Y_x$. Then there exists $V \in \mathcal{U}(U)$ such that $T_x(y) = T(y) \in \Psi_{n-1}(V)$. Therefore

$$T_x(\Psi_n(U) \cap Y_x) \subset \bigcup_{V \in \mathcal{U}(U)} \Psi_{n-1}(V) \cap Y_{S(x)}.$$

Conversely, if $z \in \Psi_{n-1}(V) \cap Y_{S(x)}$, where $V \in \mathcal{U}(U)$, there exist $z_1 \in V$ and $T(y) = z_2 \in T(U)$ with $d(z, z_1) \leq (\lambda + ... + \lambda^{n-1})\delta$ and $d(z_1, z_2) < \delta < 2a$. Therefore $d(z, z_2) < 2a$ and by the expanding property there exists $z' \in B(y, 2a) \cap Y_x$ such that $T(z') = z$ and $d(z', y) < \lambda\delta < a$, whence $z' \in \Psi_n(U) \cap Y_x$.

The theorem follows from (a)–(c) in a canonical way. Define

$$\Psi(U) = \lim_{n\to\infty} \Psi_n(U).$$

Then, with $\Lambda = \frac{\lambda\delta}{1-\lambda}$, we have that

(a) $U \subset \Psi(U) \subset B(U, \Lambda\delta)$,

(b) $\pi(\Psi(U)) = \pi(U)$,

(c) For $(x \in \pi(U))$, $T_x(\Psi(U) \cap Y_x) = \lim_{n\to\infty} T_x(\Psi_n(U) \cap Y_x)$
$= \lim_{n\to\infty} \bigcup_{V\in\mathcal{U}(U)} \Psi_{n-1}(V) \cap Y_{S(x)} = \bigcup_{V\in\mathcal{U}(U)} \Psi(V) \cap Y_{S(x)}$

Now we construct the partition γ. Write $\mathcal{U}_0 = \{U_1, ..., U_s\}$ for some $s \geq 1$ and define the atoms of γ by

$$G = \bigcap_{j \in I(G)} \Psi(U_j) \cap \bigcap_{j \notin I(G)} \Psi(U_j)^c,$$

where $I(G)$ is any (nonempty) subset of $\{1, ..., s\}$. If

$$H = \bigcap_{j \in I} \Psi(U_j)$$

for some $I \subset \{1, ..., s\}$, then by invertibility of T_x on sets of diameter $< 2a$

$$\begin{aligned} T_x(H \cap Y_x) &= \bigcap_{j \in I} T_x(\Psi(U_j)) \cap \cap Y_{S(x)} \\ &= \bigcap_{j \in I} \bigcup_{V \in \mathcal{U}(U_j)} \Psi(V) \cap \cap Y_{S(x)} \\ &= \bigcup_{V_j \in \mathcal{U}(U_j); j \in I} \bigcap_{j \in I} \Psi(V_j) \cap Y_{S(x)}, \end{aligned}$$

hence $T_x(H \cap Y_x)$ is a union of elements in $\gamma \cap Y_{S(x)}$. This proves (A) by taking differences of appropriate sets.

It is left to show (B). Clearly, since $\mathrm{diam}(G) < \delta$ for every $G \in \gamma$, we have that

$$\mathrm{diam}\left(\bigcap_{j=0}^{n-1} T^{-j}(G_{i_j}) \cap Y_x\right) = \mathrm{diam}\left(T^{-n+1}(G_{i_{n-1}}) \cap Y_x\right) \leq \lambda^{n-1}\delta.$$

$\square$

3 Conditional entropy

Let (Y, T) be a dynamical system which is fibred over the base (X, S). We fix a T-invariant measure μ on Y and denote the conditional entropy of measurable partitions ξ and η by $H(\xi|\eta)$. We denote by ϵ_Y (resp. ϵ_X) the partitions of Y (resp. X) into points. Let $\{\mu_x : x \in X\}$ denote the disintegration of μ with respect to $\pi^{-1}\epsilon_X$ and let $\nu = \mu \circ \pi^{-1}$.

The relative entropy $h(T|S)$ of the endomorphism T with respect to its factor S is defined by the expression

$$h(T|S) = \sup\{h(T|S, \xi) : \xi \text{ meas. partition of } Y \text{ s.t. } H(\xi|\pi^{-1}\epsilon_X) < \infty\},$$

where

$$h(T|S, \xi) = \lim_{n\to\infty} H(T^{-n}\xi|T^{-(n+1)}\xi^- \vee \pi^{-1}\epsilon_X)$$

is called the entropy of ξ relative to $T|S$, and where $\xi^- = \bigvee_{n=1}^{\infty} T^{-n}\xi$. It is known ([7]) that

$$h(T|S, \xi) = \lim_{n\to\infty} \frac{1}{n} H(\xi^{(n)}|\pi^{-1}\epsilon_X \vee T^{-n}\xi^-)$$

and

$$h(T|S,\xi) = \lim_{n\to\infty} \frac{1}{n} H(\xi^{(n)}|\pi^{-1}\epsilon_X). \tag{1}$$

Corollary: *Let (Y,T) be fibre expanding. For every T–invariant measure μ on Y with disintegration μ_x on Y_x we have*

$$h_\nu(T|S) = \lim_{n\to\infty} \frac{1}{n} \int_X H_{\mu_x}(\gamma_0^{n-1} \cap Y_x)\nu(dx),$$

where γ denotes the Markov partition of section 2.

Proof. Since

$$\pi^{-1}(\epsilon_X) \vee \bigvee_{n\geq 0} T^{-n}\gamma = \epsilon_Y,$$

γ is a unilateral relative generator for T and S. By Proposition 3.10 in [7]

$$h(T|S,\gamma) = h(T|S) = \lim_{n\to\infty} H(T^{-n}\epsilon_Y | T^{-(n+1)}\epsilon_Y \vee \pi^{-1}\epsilon_X).$$

The corollary follows from (1). □

From the corollary we immediately obtain the following version of the Abramov-Rokhlin formula for the entropies $h(T)$ of the transformation T with respect to the invariant measure μ and the entropy $h(S)$ of S with respect to the image measure ν:

$$h(T) - h(S) = \lim_{n\to\infty} \frac{1}{n} \int_X H_{\mu_x}(\gamma_0^{n-1} \cap Y_x)\nu(dx).$$

4 Polynomial endomorphisms of $\mathbb{C}^2$

Let $\widehat{T}$ denote a polynomial mapping of $\mathbb{C}^2$. Such a mapping can be written in the form

$$\widehat{T}(x,y) = (p(x,y), q(x,y)),$$

where p and q are polynomials in $x, y \in \mathbb{C}$. It is called (d,d')-regular where $d \in \mathbb{Q}$ and $d' \in \mathbb{N}$, if there are constants $k_1, k_2 > 0$ and $r \geq 0$ such that for every $z \in \mathbb{C}^2$, $\|z\| \geq r$

$$k_1\ \|z\|^d \leq \|\widehat{T}(z)\| \leq k_2\ \|z\|^{d'}.$$

In case that $d = d'$, $\widehat{T}$ is called strict. A special case are skew products when p does not depend on y. Then $\pi \circ \widehat{T} = p \circ \pi$ where $\pi : \mathbb{C}^2 \to \mathbb{C}$ denotes the projection map onto the first factor.

A point $z \in \mathbb{C}^2$ is called weakly normal if there exists an open neighborhood V of z and a family $\{\mathcal{K}_x : x \in V\}$ of at least one-dimensional complex analytic sets $\mathcal{K}_x$ such that $x \in \mathcal{K}_x$ and the family $\{\widehat{T}^n|_{\mathcal{K}_x} : n \geq 0\}$ is normal in x. The complement of the set of normal points is called the Julia set of $\widehat{T}$ and is denoted by $J(\widehat{T})$. It is shown in [9] that for regular polynomial mappings $J(\widehat{T})$ is compact and fully invariant. In particular, it follows that a regular skew product restricted to $J(\widehat{T})$ is a fibred system, but not a skew product in general. It is worth mentioning that $J_x = \pi^{-1}(x) \cap J(\widehat{T})$ is the fibre over x, and (for certain maps $\widehat{T}$ at least) is the Julia set of $\widehat{T}_x^n$ in case x is periodic with period n.

Let $\widehat{T} : \mathbb{C}^2 \to \mathbb{C}^2$ be a skew product and $T = \widehat{T}|_{J(\hat{T})}$ be its restriction to the Julia set $J = J(\widehat{T})$. Denote by J_x^* the Julia set for the family of maps

$$q_{p^n(x)} \circ q_{p^{n-1}(x)} \circ \cdots \circ q_x \qquad (n \geq 0).$$

Then $J = \bigcup_{x \in J(p)} \{x\} \times J_x^*$ if each q_x for $x \in J(p)$ is hyperbolic (see [10]). If in addition, p is a hyperbolic polynomial, then T is hyperbolic as well (on J) (see [9] and theorem 2.3.1 in [10]). In particular, these maps are fibrewise expanding. Hence we obtain from theorem that it has a fibrewise Markov partition.

The same result can be proved for hyperbolic rational semigroups (see e.g. [21] for a definition).

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