Pointwise maxima of dynamical Gaussian processes

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Abstract

Let \( Z = \{ Z_t(h); h \in \mathbb{R}^d, t \in \mathbb{R} \} \) be a Gaussian process which is stationary in the time variable \( t \). We study \( M_n(h) = \sup_{t \in [0,n]} Z_t(s_nh) \), the supremum of \( Z \) taken over \( t \in [0,n] \) and rescaled by a properly chosen sequence \( s_n \to 0 \). Under appropriate conditions on \( Z \), we show that for some normalizing sequence \( b_n \to \infty \) the process \( M^*_n(\cdot) = b_n(M_n(\cdot) - b_n) \) converges as \( n \to \infty \) to a stationary max-stable process of Brown-Resnick type. Using strong approximation, we prove an analogous result for the empirical process.

Key words: extremes of Gaussian processes, Pickands method, max-stable processes, empirical process

2000 MSC:

1. Introduction and statement of results

Let \( \{ X(h); h \in D \} \) be a sample continuous Gaussian process with zero mean and unit variance, defined on \( D \subset \mathbb{R}^d \), an open set containing the origin. Suppose that the covariance \( r^X(h_1, h_2) = \mathbb{E}[X(h_1)X(h_2)] \) satisfies the following condition for some \( \alpha \in (0, 2] \) and \( c_\alpha > 0 \):

\[
(X1) \quad r^X(h_1, h_2) = 1 - c_\alpha |h_1 - h_2|^\alpha + o(|h_1 - h_2|^\alpha) \quad \text{as} \quad h_1, h_2 \to 0.
\]

Here, \( | \cdot | \) denotes the euclidian norm on \( \mathbb{R}^d \). We need normalizing sequences

\[
s_n = \frac{1}{(2c_\alpha \log n)^{2/\alpha}}, \quad b_n = \sqrt{2 \log n} - \frac{1}{\sqrt{2 \log n}} \left( \frac{1}{2} \log \log n + \log(2\sqrt{\pi}) \right).
\]

Let \( X_i(\cdot), i \in \mathbb{N} \), be independent copies of \( X(\cdot) \). Define a spatially rescaled pointwise maximum

\[
M_n(h) = \max_{i=1, \ldots, n} X_i(s_nh).
\]

Then it follows from a more general result of [1], see Theorem 17 there, that the process \( M^*_n(\cdot) = b_n(M_n(\cdot) - b_n) \) converges as \( n \to \infty \) to a non-trivial limit \( \eta_\alpha(\cdot) \). The convergence is understood as weak convergence on the space \( C(K) \) for every compact set \( K \subset \mathbb{R}^d \). If \( X \) is the Ornstein-Uhlenbeck process on \( \mathbb{R} \)
with covariance $\rho_X(h_1, h_2) = e^{-|h_1 - h_2|}$, the above result is due to Brown and Resnick [2], the limit being $\eta_1$. Other particular cases, leading to the process $\eta_2$, were considered in [3] and [4]. A closely related result was obtained also in [5]. Applications were given in [6], [7].

The limiting process $\eta_\alpha$, which will be called Brown-Resnick process with parameter $\alpha \in (0, 2]$, can be described as follows. Let $\{U_i\}_{i=1}^\infty$ be an enumeration of the points of a Poisson point process with intensity $e^{-u}du$ on $\mathbb{R}$. Further, let $W_i(\cdot), i \in \mathbb{N}$, be independent copies of a drifted (Lévy) fractional Brownian motion $\{W(h); h \in \mathbb{R}^d\}$ with

$$\text{Cov}(W(h_1), W(h_2)) = |h_1|^\alpha + |h_2|^\alpha - |h_1 - h_2|^\alpha, \quad \mathbb{E}[W(h)] = -|h|^\alpha. \quad (2)$$

Then $\{\eta_\alpha(h); h \in \mathbb{R}^d\}$ is defined by

$$\eta_\alpha(h) = \max_{i \in \mathbb{N}}(U_i + W_i(h)). \quad (3)$$

The process $\eta_\alpha$ is stationary (although this is not evident from Eq.3), sample continuous, with unit Gumbel margins, see [1] for more properties.

In this paper, our main goal will be to study the pointwise supremum, taken over continuous time, of a dynamically evolving Gaussian process. Let us be more precise. Let $Z = \{Z_i(h); h \in D, t \in \mathbb{R}\}$ be a sample continuous Gaussian process with zero mean and unit variance. Here, $D$ is an open subset of $\mathbb{R}^d$ containing 0. We suppose that $Z$ is stationary in the time variable $t$ and that the covariance function $r_t(h_1, h_2) = \mathbb{E}[Z_0(h_1)Z_t(h_2)]$ satisfies the following conditions for some $\alpha, \beta \in (0, 2]$ and $c_\alpha, c_\beta > 0$:

(Z1) $r_t(h_1, h_2) = 1 - (c_\alpha|h_1 - h_2|^{\alpha} + c_\beta|t|^{\beta})(1 + o(1))$ as $t, h_1, h_2 \to 0$.

(Z2) $r_t(h_1, h_2) < 1$ provided that $t \neq 0, h_1, h_2 \in D$.

(Z3) $r_t(h_1, h_2) = o(1/|t|)$ as $t \to \infty$ uniformly in $h_1, h_2 \in D$.

Define

$$s_n = 1/(2c_\alpha \log n)^{1/\alpha}, \quad (4)$$

$$b_n = \sqrt{2\log n} + \frac{1}{\sqrt{2\log n}} \left( \frac{2 - \beta}{2\beta} \log \log n + \log \left( \frac{1}{(2\pi)^{2/2} c_\beta^{2/2}} \right) \right). \quad (5)$$

Here, $H_3 > 0$ is the so-called Pickands constant [8], [9], see Eqs.(22),(23) below. In our main result we are interested in the limiting behavior, as $n \to \infty$, of

$$M_n(h) = \sup_{t \in [0, n]} Z_t(s_n h). \quad (6)$$

**Theorem 1.** The process $M_n^*(\cdot) = b_n(M_n(\cdot) - b_n)$ converges as $n \to \infty$ to the Brown-Resnick process $\eta_\alpha(\cdot)$ weakly on $C(K)$ for every compact set $K \subset \mathbb{R}^d$.

For example, Theorem 1 applies to Gaussian processes with product-form covariance. More precisely, suppose that $\{X(h); h \in D\}$ satisfies (X1) and let $\{Y(t); t \in \mathbb{R}\}$ be a stationary sample continuous zero-mean Gaussian process with covariance function $r_Y(t) = \mathbb{E}[Y(0)Y(t)]$ satisfying the following three conditions for some $\beta \in (0, 2]$ and $c_\beta > 0$:
Given $X$ and $Y$, we construct a zero-mean time-stationary Gaussian process $Z = \{Z_t(h); h \in D, t \in \mathbb{R}\}$ with covariance

$$r_t(h_1, h_2) = r_X(h_1, h_2)r_Y(t).$$

(7)

The process $Z$, which may be thought of as a dynamical version of the spatial process $X$, is easily seen to satisfy Conditions (Z1)-(Z3). An example is given by the two-dimensional generalized Ornstein-Uhlenbeck process with covariance

$$r_t(h_1, h_2) = \exp\{-c_\alpha|h_1 - h_2|^\alpha - c_\beta|t|^\beta\}.$$

Let us make some remarks about Theorem 1.

1. For a fixed $h \in \mathbb{R}^d$ the limiting distribution of $M_n^*(h)$ was determined by Pickands [8], [9], who showed that

$$\lim_{n \to \infty} \mathbb{P}[M_n^*(h) \leq \tau] = \exp(-e^{-\tau}) \text{ for all } \tau \in \mathbb{R}. \quad (8)$$

2. Introducing a normalizing sequence $s_n$ into (6), which results in spatial rescaling of the process under consideration, is necessary to obtain a limit with non-trivial dependence between margins and was suggested in [2], [5]. In fact, the results of [10], [11], [12], [13], [14], [15] show that the maxima of two or more dependent stationary Gaussian processes, taken in continuous or discrete time over the interval $[0, n]$, become asymptotically independent as $n \to \infty$ under rather general conditions on the dependence between the processes (these conditions, however, do not allow the dependence to get stronger as $n \to \infty$). It can be shown that up to a multiplicative constant, the sequence $s_n$ as defined above is the only sequence for which a non-trivial limiting process for $M_n$ can be obtained.

3. The appearance of the Brown-Resnick process $\eta_\alpha$ as a limit of the discrete-time maximum $M_n$ defined in (1) has a natural explanation: it is known that 1) the properly normalized point process formed by the extremes of the sequence $X_1(0), \ldots, X_n(0)$ converges as $n \to \infty$ to the Poisson point process $\{U_i\}_{i=1}^\infty$ with intensity $e^{-u}du$; 2) the behavior of the process $X_i(\cdot)$ conditioned on the event “$X_i(0)$ is large” is described, in an appropriate sense, by the drifted fractional Brownian motion $W(\cdot)$ defined in (2). A remarkable feature of Theorem 1 is that the same process $\eta_\alpha$, constructed starting with a countable number of “extremes” $U_i$, arises as the limit of the continuous-time supremum (6). This may be explained as follows. Suppose that the process $\{Z_t(0); t \in \mathbb{R}\}$ takes only extremely large values in some small interval $I$ and let $t_0 = \arg \sup_{t \in I} Z_t(0)$. Then the “local direct sum” structure of Condition (Z1) implies that $\sup_{t \in I} Z_t(s_n h) \approx Z_{t_0}(s_n h)$ for large $n$ as long as $h$ stays bounded. Thus, $t_0$ is essentially the only point in the interval $I$ which has a chance to
contribute to the global supremum (6). In the limit $n \to \infty$ countably many such points emerge.

4. In order to keep notation under control we have stated Conditions (X1), (Z1) in a simplified form. An appropriate version of Theorem 1 still holds, the proof being the same, if (Z1) is replaced by regular variation conditions similar to that of Definition 16 from [1].

We have an interesting application of Theorem 1 to the empirical process. Let $V_1, V_2, \ldots$ be i.i.d. random variables having uniform distribution on $[0, 1]$. Recall that the empirical process $\{\alpha_n(h); h \in [0, 1], n \in \mathbb{N}\}$ is defined by

$$
\alpha_n(h) = \sqrt{n} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{V_i \leq h\}} - h \right).
$$

Fix $h_0 \in (0, 1)$. Using the notation $\log_2 n = \log \log n$, $\log_3 n = \log \log \log n$ we set

$$
\tilde{s}_n = \frac{h_0(1 - h_0)}{\log_2 n}, \quad \tilde{b}_n = \sqrt{2 \log_2 n + \frac{1}{2 \log_2 n} \left( \frac{1}{2} \log_3 n - \log(2\sqrt{\pi}) \right)}.
$$

Theorem 2. Define

$$
L_n(h) = (h_0(1 - h_0))^{-1/2} \max_{k=1, \ldots, n} \alpha_k(h_0 + \tilde{s}_n h).
$$

Then the process $L_n^*(\cdot) = \tilde{b}_n(L_n(\cdot) - \tilde{b}_n)$ converges as $n \to \infty$ to the drifted Brown-Resnick process $\{\eta_1(h) + (1 - 2h_0)h; h \in \mathbb{R}\}$ in the sense of finite-dimensional distributions.

Note that the drift $(1 - 2h_0)$ in the above theorem is positive for $h_0 < 1/2$ and negative for $h_0 > 1/2$, which agrees with the intuition that the empirical process tends to increase on $[0, 1/2]$ and to decrease on $[1/2, 1]$. Further, the convergence of finite-dimensional distributions can be straightened to weak convergence on $C(K)$ for every compact set $K \subset \mathbb{R}$ if one changes the definition of the empirical process so that it becomes continuous.

The rest of the paper is organized as follows. In Section 2 we prove Theorem 1. In Section 3 we deduce Theorem 2 from Theorem 1 using a strong invariance principle connecting the empirical process to the Kiefer process.

2. Proof of Theorem 1

2.1. Convergence of finite-dimensional distributions

Our main goal in this subsection is to prove Proposition 3 below which shows the convergence of finite-dimensional distributions in Theorem 1. We always use the notation preceding Theorem 1.

Proposition 3. Let $k \in \mathbb{N}$ and fix $h_1, \ldots, h_k \in \mathbb{R}^d$ and $\tau_1, \ldots, \tau_k \in \mathbb{R}$. Define

$$
P_n = \mathbb{P}[M_n^*(h_i) \leq \tau_i \text{ for all } i = 1, \ldots, k].
$$

(11)
Let \( \eta_\alpha \) be the Brown-Resnick process with parameter \( \alpha \). Then we have
\[
\lim_{n \to \infty} P_n = P[\eta_\alpha(h_i) \leq \tau_i \text{ for all } i = 1, \ldots, k].
\] (12)

For \( \gamma \in (0, 2) \) and \( q \in \mathbb{N} \) let \( \{B_{\gamma,q}(x); x \in \mathbb{R}^q\} \) be a drifted (Lévy) fractional Brownian motion with
\[
\text{Cov}(B_\gamma(x_1), B_\gamma(x_2)) = |x_1|^{\gamma} + |x_2|^{\gamma} - |x_1 - x_2|^{\gamma}, \quad \mathbb{E}[B_\gamma(x)] = -x |\gamma|.
\] (13)

We agree to write \( B_\alpha = B_{\alpha,1} \) and \( B_\beta = B_{\beta,1} \), and always assume \( B_\alpha \) and \( B_\beta \) to be independent. Let \( G_{h,\tau} = G_{\{h_1\}_{i=1}^k, \{\tau_i\}_{i=1}^k} \) be a constant defined by
\[
G_{h,\tau} = \mathbb{E}\exp\{ \max_{i=1,\ldots,k} (B_\alpha(h_i) - \tau_i) \}.
\]

It was shown in [1] that the finite-dimensional distributions of the process \( \eta_\alpha \) are given by
\[
\mathbb{P}[\eta_\alpha(h_i) \leq \tau_i \text{ for all } i = 1, \ldots, k] = e^{-G_{h,\tau}}.
\] (14)

So, to prove Proposition 3 we have to show that
\[
\lim_{n \to \infty} P_n = e^{-G_{h,\tau}}.
\] (15)

Our proof of (15), which will be given in a sequence of lemmas, relies strongly on the method of Pickands [8], [9]. We always refer to Chapter 12 of [16] if we need facts proved by Pickands. Sometimes, we omit technical details in order to avoid repetition.

First we fix the notation. We take some \( a > 0 \) and define \( q_n = ac_\beta^{-1/\beta}b_n^{-2/\beta} \).

We define a process \( \{Z_t^*(h); h \in s_n^{-1}D, t \in \mathbb{R}\} \) dependent also on \( n \) by
\[
Z_t^*(h) = b_n(Z_{q_n t}(s_nh) - b_n).
\]

For \( w \in \mathbb{R} \) and \( n \in \mathbb{N} \) let \( Z^{w,n}_t = \{Z^{w,n}_t(h); h \in s_n^{-1}D, t \in \mathbb{R}\} \) be the process \( \{Z_t^*(h) - Z_0^*(0); h \in s_n^{-1}D, t \in \mathbb{R}\} \) conditioned on the event \( Z_0^*(0) = w \). It will be convenient to write \( u_n = b_n + b_n^{-1}\tau_i \).

**Lemma 4.** Fix a cube \( K = [-A, A]^d \) and let \( B_\alpha \) and \( B_\beta \) be independent fractional Brownian motions as above. As \( n \to \infty \) we have
\[
\{Z^{w,n}_t(h); h \in K, t \in [0,1]\} \Rightarrow \{B_\alpha(h) + B_\beta(at); h \in K, t \in [0,1]\}
\] (16)

weakly on \( C(K \times [0,1]) \). The family of processes \( Z^{w,n} - EZ^{w,n} \), indexed by \( w \in \mathbb{R} \) and \( n \in \mathbb{N} \), is tight in \( C(K \times [0,1]) \). Finally, the family \( Z^{w,n} \) is tight as long as \( w \) stays bounded and \( n \in \mathbb{N} \).

Proof. Take \( h_1, h_2 \in K, t_1, t_2 \in [0,1] \) and let \( \Delta_h = |h_1 - h_2|, \Delta_t = |t_1 - t_2| \). Then Condition (Z1) implies
\[
r_{q_n\Delta_t}(s_nh_1, s_nh_2) = 1 - b_n^{-2}(\Delta_h^\alpha + a^2\Delta_t^\beta)(1 + o(1))
\] (17)
as $n \to \infty$, where the $o$-term is uniform in $h_1, h_2 \in K, t_1, t_2 \in [0, 1]$. The well-known formulas for the conditional Gaussian distributions show that the process $Z^{w,n}$ is Gaussian with

$$
\mathbb{E}[Z^{w,n}_t(h_1)] = -(b_n^2 + w)(1 - r_{q,t_1}(s_n h_1, 0)),
$$

(18)

$$
\text{Cov}(Z^{w,n}_t(h_1), Z^{w,n}_t(h_2)) = b_n^2(r_{q,t_1}(s_n h_1, s_n h_2) - r_{q,t_1}(s_n h_1, 0)r_{q,t_2}(0, s_n h_2)).
$$

(19)

Applying (17) yields

$$
\mathbb{E}[Z^{w,n}_{t_1}(h_1)] = -(|h_1|^\alpha + a^2|h_1|^\beta)(1 + o(1)),
$$

(20)

$$
\text{Cov}(Z^{w,n}_{t_1}(h_1), Z^{w,n}_{t_2}(h_2)) = [(|h_1|^\alpha + |h_2|^\alpha - |h_1 - h_2|^\alpha] + a^2(|t_1|^\beta + |t_2|^\beta - |t_1 - t_2|^\beta)](1 + o(1))
$$

(21)

as $n \to \infty$. This shows the convergence of finite-dimensional distributions in (16). We prove the tightness part of the lemma. It follows from (19) that the $o$-term in (21) does not depend on $w$. Thus, (21) implies that uniformly in $w \in \mathbb{R}$ and for large $n$ we have

$$
\text{Var}(Z^{w,n}_{t_1}(h_1) - Z^{w,n}_{t_2}(h_2)) = (2 + o(1))(|\Delta h|^\alpha + a^2|\Delta h|^\beta) < 3(|\Delta h|^\alpha + a^2|\Delta h|^\beta).
$$

(22)

It is well known that this, together with $Z^{w,n}_0(0) = 0$, implies that the family $Z^{w,n} = \mathbb{E}Z^{w,n}, w \in \mathbb{R}, n \in \mathbb{N}$, is tight. Further, note that (18) implies that the $o$-term in (20) is uniform in $w$ as long as $w$ stays bounded. It follows that the family of functions $(h, t) \mapsto \mathbb{E}[Z^{w,n}_t(h_1)]$, indexed by $n \in \mathbb{N}$ and $w$, is weakly precompact in $C(K \times [0, 1])$ as long as $w$ stays bounded. So, the family $Z^{w,n}$ is tight as long as $w$ stays bounded.

Let $m \in \mathbb{N}, a > 0$ be fixed. Define a constant

$$
H_\beta(m, a) = \mathbb{E}\exp\left\{ \max_{j=0,1,\ldots,m-1} B_\beta(a j) \right\}.
$$

(23)

**Lemma 5.** For the probability

$$
p_n(m, a) = \mathbb{P}\left\{ \max_{j=0,1,\ldots,m-1} Z_{j} a_n(s_n h_i) > u_n \text{ for some } i = 1, \ldots, k \right\}
$$

the following asymptotic relation holds true as $n \to \infty$:

$$
p_n(m, a) \sim G_{h, \tau} H_\beta(m, a) (\sqrt{2\pi})^{-1} b_n^{-1} e^{-b_n^2/2}.
$$

Proof. In the subsequent equations the indices $i$ and $j$ range over $1, \ldots, k$ and $0, \ldots, m - 1$ respectively. The density of the random variable $Z^*_0(0)$ is given by

$$
(2\pi)^{-1/2} b_n^{-1} e^{-(b_n^2 + b_n^{-1} w^2)/2} dw.
$$

Conditioning on $Z^*_0(0) = w$ we obtain

$$
p_n(m, a) = \mathbb{P}\left\{ \exists i, j : Z^*_j(h_i) > \tau_i \right\}
$$

$$
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbb{P}\left\{ \exists i, j : Z^*_j(h_i) > \tau_i | Z^*_0(0) = w \right\} e^{-(b_n^2 + b_n^{-1} w^2)/2} dw.
$$

$$
= \frac{1}{\sqrt{2\pi}} b_n^{-1} e^{-b_n^2/2} \int_{-\infty}^{\infty} \mathbb{P}\left\{ \exists i, j : Z^*_j^{w,n}(h_i) > \tau_i - w \right\} e^{-w^2/2} dw.
$$

$$
= \frac{1}{\sqrt{2\pi}} b_n^{-1} e^{-b_n^2/2} \int_{-\infty}^{\infty} \mathbb{P}\left\{ \exists i, j : Z^*_j^{w,n}(h_i) > \tau_i - w \right\} e^{-w^2/2\pi} dw.
$$
Applying Lemma 4 to the probability under the integral sign and omitting the standard justification of the use of the dominated convergence theorem, we obtain
\[
p_n(m, a) \sim \frac{1}{\sqrt{2\pi}} b_n^{-1} e^{-b_n^2/2} \int_{-\infty}^{\infty} P[ \max_{i=1,\ldots,k} (B_n(h_i) - \tau_i) + \max_{j=0,\ldots,m-1} B_\beta(a_j) > -w] e^{-w} dw
\]
\[
= \frac{1}{\sqrt{2\pi}} b_n^{-1} e^{-b_n^2/2} \exp\{ \max_{i=1,\ldots,k} (B_n(h_i) - \tau_i) + \max_{j=0,\ldots,m-1} B_\beta(a_j) \}
\]
\[
= \frac{1}{\sqrt{2\pi}} b_n^{-1} e^{-b_n^2/2} G_{h,\tau} H_\beta(m, a) \quad \text{as} \quad n \to \infty.
\]

The proof of Lemma 5 is finished. □

Fix \( l > 0 \) and let
\[
Q_n(a) = P[ \max_{t \in [0,l]} Z_t(s, h_i) > u_m \text{ for some } i = 1, \ldots, k].
\]

It was shown in [16], Lemmas 12.2.4, 12.2.7 and 12.2.8, that the limits in the following formula exist finitely and are strictly positive:
\[
H_\beta(a) = \lim_{m \to \infty} H_\beta(m, a)/(ma), \quad H_\beta = \lim_{a \to 0} H_\beta(a). \tag{23}
\]

**Lemma 6.** The following asymptotic equality holds as \( n \to \infty \):
\[
Q_n(a) \sim l H_\beta(a) H_\beta^{-1} G_{h,\tau} n^{-1}. \tag{24}
\]

Proof. Since the proof uses the “double sum” method of Pickands, see [16], Lemma 12.2.4, we omit some details. Define a random event \( A_n(m, a, t) \) by
\[
A_n(m, a, t) = \{ \max_{j=0,\ldots,m-1} Z_{t+j\alpha}(s, h_i) > u_m \text{ for some } i = 1, \ldots, k \}.
\]

By Bonferroni inequality we have
\[
Q_n(a) \leq S'_n(m, a), \quad Q_n(a) \geq S'_n(m, a) - S''_n(m, a), \tag{25}
\]
where
\[
S'_n(m, a) = \sum_{t \in [0,l]/m\alpha_n, Z} P[A_n(m, a, t)], \tag{26}
\]
\[
S''_n(m, a) = \sum_{t_1, t_2 \in [0,l]/m\alpha_n, Z \atop t_1 \neq t_2} P[A_n(m, a, t_1) \cap A_n(m, a, t_2)]. \tag{27}
\]

Using the first inequality in (25) and applying Lemma 5 to each term on the right-hand side of (26) yields for each \( m \in \mathbb{N} \)
\[
Q_n(a) \leq l(m\alpha_n)^{-1} G_{h,\tau} H_\beta(m, a) (\sqrt{2\pi})^{-1/2} b_n^{-1} e^{-b_n^2/2} (1 + o(1))
\]
\[
= l c_{\beta}^{1/3} G_{h,\tau} H_\beta(m, a) (\sqrt{2\pi})^{-1/2} b_n^{2/3} e^{-b_n^2/2} (1 + o(1))
\]
\[
= l(H_\beta(m, a)/(ma)) H_\beta^{-1} G_{h,\tau} n^{-1} (1 + o(1)), \quad n \to \infty. \tag{28}
\]
The last line is obtained after an easy calculation using (5). The double sum $S_n^G(m, a)$ can be estimated as in the proof of Lemma 12.2.4 of [16]:

$$
\limsup_{m \to \infty} \limsup_{n \to \infty} nS_n^G(m, a) = 0. \tag{29}
$$

The statement of the lemma follows by letting $m \to \infty$ in (25) combined with (28) and (29).

Fix $\varepsilon > 0$. For $j = 0, 1, \ldots$ let $I_j$ be the interval $[j, j + 1 - \varepsilon]$. Further, let

$$
P_n(a, \varepsilon) = \mathbb{P}\left[ \max_{t \in (u^{n-1}_j, t_j)} Z_t(s_n h_i) \leq u_{in} \text{ for all } i = 1, \ldots, k \right].
$$

Lemma 7. Denote $\rho_{a, \varepsilon} = \limsup_{n \to \infty} (P_n(a, \varepsilon) - P_n)$. Then $\lim_{a, \varepsilon \to 0} \rho_{a, \varepsilon} = 0$.

Proof. It is clear that $\rho_{a, \varepsilon} \geq 0$. For $i = 1, \ldots, k$ let

$$
\rho^{(i)}_{a, \varepsilon} = \limsup_{n \to \infty} \mathbb{P}\left[ \max_{t \in (u^{n-1}_j, t_j)} Z_t(s_n h_i) \leq u_{in} | \mathbb{P}\sup_{t \in [0, n]} Z_t(s_n h_i) \leq u_{in} \right].
$$

By Lemma 12.3.2 of [16] we have $\lim_{a, \varepsilon \to 0} \rho^{(i)}_{a, \varepsilon} = 0$. The statement of the lemma follows by noting that $\rho_{a, \varepsilon} \leq \sum_{i=1}^{k} \rho^{(i)}_{a, \varepsilon}$.

Lemma 8. We have

$$
\lim_{n \to \infty} P_n(a, \varepsilon) = \exp(- (1 - \varepsilon) H^1 (a) H^{-1} G_{h, \tau} ).
$$

Proof. For $t_1, t_2 \geq 0$ we write $t_1 \sim t_2$ if there is $j = 0, 1, \ldots$ with $t_1, t_2 \in I_j$. Otherwise, we write $t_1 \not \sim t_2$. Let $\{ U_{t,i}^{(n)} : t \in (u^{n-1}_j, t_j) \cap q_n \mathbb{Z}, i = 1, \ldots, k \}$ be a zero-mean Gaussian vector with the following covariance structure:

$$
\mathbb{E}[U_{t_1,i}^{(n)} U_{t_2,i}^{(n)}] = \begin{cases} 
 r_{t_1 - t_2}(s_n h_i, s_n h_i), & \text{if } t_1 \sim t_2, \\
 0, & \text{if } t_1 \not \sim t_2.
\end{cases}
$$

It follows from Lemma 6 that

$$
\lim_{n \to \infty} \mathbb{P}\left[ \max_{t \in (u^{n-1}_j, t_j) \cap q_n \mathbb{Z}} U_{t,i}^{(n)} \leq u_{in} \right] = \lim_{n \to \infty} (1 - \mathbb{P}\left[ \max_{t \in [0, n]} U_{t,i}^{(n)} > u_{in} \right])^n = \exp(- (1 - \varepsilon) H^1 (a) H^{-1} G_{h, \tau}). \tag{30}
$$

Recalling that $u_{in} = b_n + b_{n}^{-1} \tau$, we find a constant $C$ such that $u_{in}^2 \geq b_n^2 - C$. In the subsequent inequalities, the summation indices $t_1$ and $t_2$ take values in $(u^{n-1}_j, t_j) \cap q_n \mathbb{Z}$. Set $R(t) = \sup_{h_1, h_2 \in B, h_1 \neq h_2} r_{h_1, h_2}$, where $B$ is a fixed small
ball around the origin. By Berman Inequality, Theorem 4.2.1 in [16], we have

\[
|P_n(a, \varepsilon) - P[\forall i : \sup_{t \in \cup_{i=1}^{n-1} I_i} t^{(n)}_{i,i} \leq u_{in}| \\
\leq K \sum_{t_1 = t_2} r_{t_1-t_2}(s_{n}h_{i_1}, s_{n}h_{i_2}) \exp \left( -\frac{(u_{i_1}^2 + u_{i_2}^2)/2}{1 + r_{t_1-t_2}(s_{n}h_{i_1}, s_{n}h_{i_2})} \right)
\]

\[
\leq K \sum_{t_1 = t_2} R(t_1 - t_2) \exp \left( -\frac{b_n^2 - C}{1 + R(t_1 - t_2)} \right)
\]

\[
\leq K e^{Ct_2} \sum_{t_1 = t_2} R(t_1 - t_2) \exp \left( -\frac{b_n^2}{1 + R(t_1 - t_2)} \right).
\]

Condition (Z3) implies that \( R(t) = o(1/\log t) \) as \( t \to +\infty \). Further, Condition (Z2) implies that there is \( \delta > 0 \) such that for \( t_1 \approx t_2 \) we have \( R(t_1 - t_2) < 1 - \delta \). These two facts allow to use Lemma 12.3.1 of [16] to show that the sum on the right-hand side of (31) converges to 0 as \( n \to \infty \). To finish the proof recall (30). \( \square \)

Now we can finish the proof of Proposition 3. By letting \( a, \varepsilon \to 0 \) in Lemmas 7 and 8 and recalling that \( \lim_{n \to 0} H_3(a) = H_3 \) we obtain \( \lim_{n \to -\infty} P_n = e^{-G_{n, \tau}} \). This proves (15), which in combination with (14) gives Proposition 3. \( \square \)

2.2. Tightness

In this subsection we finish the proof of Theorem 1 by showing that the sequence of processes \( M_n^*(\cdot) = b_n(M_n(\cdot) - b_n) \) is tight on \( C(K) \), where \( K = [-A, A]^d \) is a fixed cube. We will use some ideas from the proof of Theorem 17 in [1]. However, the main difficulty of our proof, namely handling extremes in continuous time, is not present in [1]. To start with, note that by (8) the sequence of random variables \( M_n^*(\cdot) \) is tight. The continuity modulus of a function \( f \in C(K) \) is defined by

\[
\omega_\delta(f) = \sup_{h_1, h_2 \in K, |h_1 - h_2| \leq \delta} |f(h_1) - f(h_2)|, \quad \delta > 0.
\]

To prove the tightness we have to show that for all \( \varepsilon > 0 \) and \( \varrho > 0 \) there exist \( \delta > 0 \) such that for all sufficiently large \( n \)

\[
P[|\omega_\delta(M_n^*(\cdot))| > \varrho] < 7\varepsilon.
\]

(32)

Let \( q_n = b_n^{-2/\beta} \). We set \( Z^*_n(h) = b_n(Z(s_nh) - b_n) \) (note that in the previous subsection we have used a slightly different notation). Further, for \( w \in \mathbb{R} \) and \( n \in \mathbb{N} \) we define \( Z_n^{w,n} = \{Z_n^{w,n}(h); h \in s_n^{-1}D, \theta \in \mathbb{R} \} \) to be the process \( \{Z_n^{w,n}(h) - Z_n^0(0); h \in s_n^{-1}D, \theta \in \mathbb{R} \} \) conditioned on the event \( Z_n^0(0) = w \). (So, \( Z_n^{w,n} \) is the same as in the previous subsection.

**Lemma 9.** Let \( E_n(C) \) be the random event \( \{\inf_{h \in K} M_n^*(h) \leq -C\} \). Then we can find a sufficiently large \( C_1 > 0 \) such that \( P[E_n(C_1)] < 2\varepsilon \).
Proof. The proof of the analogous statement in [1], see the proof of Theorem 17 there, does not apply in our situation, so we have to use a different method. For $0 < c < C < \infty$ we define the auxiliary random events
\[
E'_n(C) = \{ \inf_{h \in K} \max_{t \in [0,n] \cap q_n \mathbb{Z}} Z^*_t(h) \leq -C \}, \quad E''_n(c) = \{ \max_{t \in [0,n] \cap q_n \mathbb{Z}} Z^*_t(0) \in [-c, c] \}.
\]
It is implicit in Chapter 12 of [16] that $\max_{t \in [0,n] \cap q_n \mathbb{Z}} Z^*_t(0)$ has limiting (non-unit) Gumbel distribution (or see [17], Theorem 2, for an explicit statement).
Thus, we may choose $c$ so large that $\mathbb{P}[E''_n(c)] \geq 1 - \varepsilon$ for all $n$. We have
\[
\begin{align*}
\mathbb{P}[E_n(C)] &\leq \mathbb{P}[E'_n(C)] \\
&\leq \mathbb{P}[E'_n(C) \cap E''_n(c)] + (1 - \mathbb{P}[E''_n(c)]) \\
&\leq \mathbb{P}[E'_n(C) \cap E''_n(c)] + \varepsilon.
\end{align*}
\]
Further, it is clear that $E'_n(C) \cap E''_n(c) \subset \cup_{t \in [0,n] \cap q_n \mathbb{Z}} A_{t,n}$, where $A_{t,n}$ is a random event defined by
\[
A_{t,n} = \{ Z^*_t(0) \in [-c, c] \text{ and } \inf_{h \in K} (Z^*_t(h) - Z^*_t(0)) < -(C - c) \}.
\]
Using this and the stationarity of $Z$ in $t$, we obtain
\[
\mathbb{P}[E'_n(C) \cap E''_n(c)] \leq [n/q_n] \mathbb{P}[A_{0,n}].
\]
So, we have to show that
\[
\mathbb{P}[A_{0,n}] < \varepsilon q_n/n \quad \text{(33)}
\]
if $C = C_1$ is large enough. By the tightness part of Lemma 4, we may choose for every $\Delta > 0$ a sufficiently large $C$ such that uniformly in $w \in [-c, c]$, $n \in \mathbb{N}$
\[
\mathbb{P}[\inf_{h \in K} Z^{w,n}_0(h) < -(C - c)] < \Delta.
\]
(34)
Conditioning on the event $Z^{0}_0(0) = w$ we obtain
\[
\begin{align*}
\mathbb{P}[A_{0,n}] &= \frac{1}{\sqrt{2\pi}} \int_{-c}^c \mathbb{P}[\inf_{h \in K} (Z^{0}_0(h) - Z^{0}_0(0)) < -(C - c) | Z^{0}_0(0) = w] \\
&\quad \times e^{-(b_n + b_n^{-1}w)^2/2b_n^{-1}} dw \\
&= \frac{1}{\sqrt{2\pi}} b_n^{-1} e^{-b_n^2/2} \int_{-c}^c \mathbb{P}[\inf_{h \in K} Z^{w,n}_0(h) < -(C - c)] e^{-w^2/(2\pi)} dw.
\end{align*}
\]
By (5) we have $b_n^{-1} e^{-b_n^2/2} = O(q_n/n)$. Using (34) we obtain $\mathbb{P}[A_{0,n}] \leq K\Delta q_n/n$, where $K = K(c)$ is some constant. We may choose $C = C_1$ so that $\Delta < \varepsilon/K$, which yields (33). The proof is finished.

Next we would like to show that with high probability only those times $t$ for which $Z^*_t(0) \in [-C, C]$, $C$ large, contribute to the supremum $M^*_n(h) = \sup_{t \in [0,n]} Z^*_t(h)$. To this end, we define, for $C > 0$,
\[
U^{(n)}_t(h) = \sup_{\theta \in [0,q_n]} Z^*_{t+\theta}(h),
\]
\[
M^*_n(h) = \max\{U^{(n)}_t(h) : t \in [0,n] \cap q_n \mathbb{Z}, Z^*_t(0) \in [-C, C] \}.
\]

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Lemma 10. Let $F_n(C)$ be the random event $\{ \exists h \in K : M^*_n(h) \neq M^*_{n,C}(h) \}$. Then for a sufficiently large $C_2$ we have $\mathbb{P}[F_n(C_2)] \leq 4\varepsilon$.

Proof. Let $C_1$ be given by Lemma 9 and take $C > C_1$. For $t \in [0, n] \cap q_n \mathbb{Z}$ define a random event $B_{t,n}$ by

$$B_{t,n} = \{ Z^*_t(0) \leq -C, \sup_{h \in K} \mathbb{P}[U^{(n)}_t(h)] > -C_1 \}.$$ 

Then

$$\mathbb{P}[F_n(C)] \leq \mathbb{P}[F_n(C_1)] + \mathbb{P}[M^*_n(0) > C] + \mathbb{P}[[\cup_{t \in [0,n)} \cap q_n \mathbb{Z}] B_{t,n}].$$

By Lemma 9 we have $\mathbb{P}[F_n(C_1)] < 2\varepsilon$. Further, by (8) we have $\mathbb{P}[M^*_n(0) > C] < \varepsilon$ if $C$ is sufficiently large. So, we concentrate on estimating the third summand on the right-hand side of (37). We use a method from [18], Section D. It follows from (17) and (18) that for sufficiently large $n$ and $h \in K$, $\theta \in [0, 1]$ we have $E[Z^*_w,n(h)] < |w|/2$ if $|w|$ is large. Conditioning on $Z^*_0(0) = -w$ we obtain

$$\mathbb{P}[B_{0,n}] = \frac{1}{\sqrt{2\pi}} \int_C \sup_{h \in K} \sup_{\theta \in [0,q_n]} Z^*_\theta(h) > -C_1 |Z^*_0(0) = -w|$$

$$\times e^{-(b_n - b_n^{1/2}w^2)/2b_n^{-1}} dw$$

$$= \frac{1}{\sqrt{2\pi}} b_n^{-1} e^{-b_n^{1/2}/2} \int_C \mathbb{P}[\sup_{h \in K} \sup_{\theta \in [0,1]} Z^*_\theta(h) > w - C_1] e^{w} e^{-\frac{w^2}{2b_n}} dw$$

$$\leq O(q_n/n) \int_C \mathbb{P}[\sup_{h \in K} \sup_{\theta \in [0,1]} (Z^*_\theta(h) - \mathbb{E}[Z^*_w,n(h)]) > \frac{w}{2} - C_1]$$

$$\times e^{w} dw.$$  

(38)

Since the family of processes $Z^*_w,n - \mathbb{E}Z^*_w,n$, $w \in \mathbb{R}$, $n \in \mathbb{N}$, is tight on $C(K \times [0,1])$ by Lemma 4, we can find a number $c_1$ such that

$$\mathbb{P}[\sup_{h \in K} \sup_{\theta \in [0,1]} (Z^*_\theta(h) - \mathbb{E}[Z^*_w,n(h)]) > c_1] < 1/2.$$ 

Using Borell inequality as in [18], Section D, we obtain, uniformly in $w \in \mathbb{R}$,

$$\mathbb{P}[\sup_{h \in K} \sup_{\theta \in [0,1]} (Z^*_\theta(h) - \mathbb{E}[Z^*_w,n(h)]) > \frac{w}{2} - C_1] \leq e^{-c_2 w^2}.$$ 

Recalling (38) yields

$$\mathbb{P}[B_{0,n}] \leq O(q_n/n) \int_C e^{-c_1 w^2} e^{w} dw.$$ 

It follows that for a sufficiently large $C = C_2$ we have $\mathbb{P}[B_{t,n}] < \varepsilon q_n/n$. Recalling (37) we obtain $\mathbb{P}[F_n(C_2)] \leq 4\varepsilon$. Now we can finish the proof of (32). Let $D_{t,n}$ be the event defined by

$$D_{t,n} = \{ \omega \mathbb{P}(U^{(n)}_t(\cdot)) > g, Z^*_t(0) \in [-C_2,C_2] \}.$$  

(39)
Then, using Lemma 10, we obtain
\[
\mathbb{P}[\omega_R(M^*_n(\cdot)) > \varrho] \leq \mathbb{P}[F_n(C_2)] + \mathbb{P}[\omega_R(M^*_{n,C_2}(\cdot)) > \varrho] \\
\leq 4\varepsilon + \lfloor n/q_n \rfloor \mathbb{P}[D_{0,n}].
\] (40)

We estimate \( \mathbb{P}[D_{0,n}] \). To this end, we will use the approximate direct sum structure of the field \( Z_{w,n} \) given in Lemma 4. More precisely, we would like to use the fact that, conditioned on \( Z_0^*(0) = w \), we have
\[
Z_0^*(h) \approx Z_0^*(0) + (Z_0^*(0) - Z_0^*(0)) + (Z_0^*(h) - Z_0^*(0)).
\]
Thus, we set
\[
V_0^*(h) = Z_0^*(h) - Z_0^*(0) - Z_0^*(0) + Z_0^*(0)
\] (41)
and
\[
V^{(n)}_\sup = \sup_{h \in K} \sup_{\theta \in [0,q_n]} V_0^*(h), \quad V^{(n)}_\inf = \inf_{h \in K} \inf_{\theta \in [0,q_n]} V_0^*(h).
\]
Later we will show that \( V^{(n)}_\sup \) and \( V^{(n)}_\inf \) are in some sense small. By (35) we have
\[
U_0^{(n)}(h) = \sup_{\theta \in [0,q_n]} (V_0^*(h) + Z_0^*(0) + Z_0^*(h) - Z_0^*(0)) \\
\leq Z_0^*(h) - Z_0^*(0) + V^{(n)}_\sup + \sup_{\theta \in [0,q_n]} Z_0^*(0).
\]
In the same way we obtain the lower bound
\[
U_0^{(n)}(h) \geq Z_0^*(h) - Z_0^*(0) + V^{(n)}_\inf + \sup_{\theta \in [0,q_n]} Z_0^*(0).
\]
Using both bounds, we obtain for any \( h_1, h_2 \in K \)
\[
|U_0^{(n)}(h_1) - U_0^{(n)}(h_2)| \leq \|Z_0^*(h_1) - Z_0^*(h_2)| + V^{(n)}_\sup - V^{(n)}_\inf.
\]
Consequently, we may estimate the continuity modulus of \( U_0^{(n)} \) by
\[
\omega_R(U_0^{(n)}(\cdot)) \leq \omega_R(Z_0^*(\cdot)) + V^{(n)}_\sup - V^{(n)}_\inf.
\]
Using the above inequality and recalling (39) we obtain that the random event \( D_{0,n} \) is contained in
\[
\{V^{(n)}_\sup > \varrho/3\} \cup \{V^{(n)}_\inf < -\varrho/3\} \cup \{\omega_R(Z_0^*(\cdot)) > \varrho/3\} \cap \{Z_0^*(0) \in [-C_2,C_2]\}.
\]
Thus, Lemmas 11 and 12 below imply that \( \mathbb{P}[D_{0,n}] \leq 3\varepsilon q_n/n \). Taking into account (40), we obtain (32). This finishes the proof of tightness. □

**Lemma 11.** If \( n \) is sufficiently large then
\[
\mathbb{P}[V^{(n)}_\sup > \varrho/3, Z_0^*(0) \in [-C_2,C_2]] \leq \varepsilon q_n/n,
\] (42)
\[
\mathbb{P}[V^{(n)}_\inf < -\varrho/3, Z_0^*(0) \in [-C_2,C_2]] \leq \varepsilon q_n/n.
\] (43)
Proof. Let \( V^{w,n} = \{ V^{w,n}_\theta(h); h \in K, \theta \in [0,1] \} \) be the distribution of the process \( \{ V^{*,q}_\theta(h); h \in K, \theta \in [0,1] \} \) conditioned on \( Z^*_0(0) = w \). So, recalling (41), we have an equality in distribution

\[
V^{w,n} \overset{D}{=} \{ Z^{w,n}_\theta(h) - Z^{w,n}_0(h) - Z^{w,n}_0(0); h \in K, \theta \in [0,1] \}.
\]

Now, the asymptotic direct sum structure of the process \( Z^{w,n} \) given in Lemma 4 implies that \( V^{w,n} \) converges to 0 weakly on \( C(K \times [0,1]) \), the convergence being uniform as long as \( w \) stays bounded. It follows that for every \( \Delta > 0 \) there is \( N(\Delta) \) such that for all \( n > N(\Delta) \) and \( w \in [-C_2, C_2] \) we have

\[
P[\sup_{h \in K} \sup_{\theta \in [0,1]} V^{w,n}_\theta(h) > \varrho/3] < \Delta.
\]

Now, the probability on the left-hand side of (42) may be written as

\[
\text{LHS of (42)} = \frac{1}{\sqrt{2\pi}} \int_{-C_2}^{C_2} P[\sup_{h \in K} \sup_{\theta \in [0,1]} V^{w,n}_\theta(h) > \varrho/3] e^{-(b_n + b^{-1}_n w)^2/2b^{-1}_n} dw
\]

\[
\leq O(q_n/n) \int_{-C_2}^{C_2} e^{w^2/2\pi} dw
\]

\[
\leq O(q_n/n) \Delta.
\]

Choosing \( \Delta \) small enough and \( n > N(\Delta) \), we obtain (42). The proof of (43) is analogous. \( \square \)

**Lemma 12.** If \( \delta > 0 \) is sufficiently small then

\[
P[\omega_\delta(Z^*_0(\cdot)) > \varrho/3, Z^*_0(0) \in [-C_2, C_2]] < \varepsilon q_n/n. \tag{44}
\]

Proof. By Lemma 4 the family of stochastic processes \( Z^{w,n}_0(\cdot) \) is tight on \( C(K) \) as long as \( w \in [-C_2, C_2] \) and \( n \in \mathbb{N} \). Thus, for every \( \Delta > 0 \) we can choose \( \delta \) so small that

\[
P[\omega_\delta(Z^{w,n}_0(\cdot)) > \varrho/3] < \Delta.
\]

Conditioning on \( Z^*_0(0) = w \), we obtain

\[
\text{LHS of (44)} = \frac{1}{\sqrt{2\pi}} \int_{-C_2}^{C_2} P[\omega_\delta(Z^{w,n}_0(\cdot)) > \varrho/3] e^{-(b_n + b^{-1}_n w)^2/2b^{-1}_n} dw
\]

\[
\leq O(q_n/n) \int_{-C_2}^{C_2} e^{w^2/2\pi} dw
\]

\[
\leq O(n/q_n) \Delta.
\]

Choosing \( \Delta \) and then \( \delta \) small enough, we obtain (44). \( \square \)
3. Proof of Theorem 2

In this section we prove Theorem 2. The idea is to approximate the empirical process \( \alpha_n(\cdot) \) by a Kiefer process \( K_n(\cdot) \) and then to apply Theorem 1 to a time change of \( K_n(\cdot) \). We use the notation \( \sigma^2(h) = h(1-h) \). Let \( \{X(h); h \in (0,1)\} \) be a zero-mean Gaussian process with covariance function

\[
r_X(h_1, h_2) = (\sigma(h_1)\sigma(h_2))^{-1}(\min(h_1, h_2) - h_1h_2),
\]

and let \( \{Y(t); t \in \mathbb{R}\} \) be a stationary zero-mean Gaussian process with covariance function \( r_Y(t) = e^{-|t|/2} \). That is, \( X \) is the Brownian bridge normalized by the square-root of its own variance and \( Y \) is an Ornstein-Uhlenbeck process. An easy calculation shows that the translated process \( (X_1) \) with \( \beta = 1 \), \( c_\alpha = 1/(2\sigma^2(h_0)) \), and \( Y \) satisfies Conditions (Y1)-(Y3) with \( \beta = 1 \), \( c_\beta = 1/2 \). Define \( s_n \) and \( b_n \) by (4) and (5) and recall that \( \tilde{s}_n \) and \( \tilde{b}_n \) were defined in (9). Since the Pickands constant \( H_1 \) equals 1, see Section 12.2 of [16], we have \( \tilde{b}_n = b_{\log n} \) and \( \tilde{s}_n = s_{\log n} \).

Define a zero-mean Gaussian process \( \{Z_t(h); h \in (0,1), t \in \mathbb{R}\} \) by requiring its covariance function \( r_t(h_1, h_2) \) to be of the product form (7).

\[
M_n(h) = \sup_{t \in [0,n]} Z_t(h_0 + s_nh).
\]

Then, applying Theorem 1, we obtain that the process \( b_n(M_n(\cdot) - b_n) \) converges as \( n \to \infty \) to the Brown-Resnick process \( \eta_1 \). Now, the process \( K = \{K_t(h); h \in (0,1), t \geq 0\} \) defined by

\[
K_t(h) = \sigma(h)t^{1/2}Z_{\log t}(h)
\]

has the covariance function

\[
E[K_{t_1}(h_1), K_{t_2}(h_2)] = \min(t_1, t_2)(\min(h_1, h_2) - h_1h_2).
\]

That is, \( K \) is the well-known Kiefer process. By a strong approximation theorem of Komlós-Major-Tusnády, see e.g. Theorem 4.4.3 in [19], we can construct on some probability space an empirical process and a Kiefer process such that

\[
\sup_{h \in [0,1]} |\alpha_n(h) - n^{-1/2}K_n(h)| = O(\log^2 n/\sqrt{n}) \text{ a.s. as } n \to \infty.
\]

With the notation

\[
\tilde{M}_n = \sigma(h_0)^{-1} \sup_{t \in [0,1]} (K_t(h_0 + \tilde{s}_nh)/\sqrt{t})
\]

and after recalling the definition of \( L_n \) in (10) it follows that

\[
|L_n - \tilde{M}_n| = O(\log^2 n/\sqrt{n}) \text{ a.s. as } n \to \infty.
\]

So, to prove Theorem 2 it suffices to show the convergence of finite-dimensional distributions

\[
\{\tilde{b}_n(\tilde{M}_n(h) - \tilde{b}_n); h \in \tilde{s}_n^{-1}\min(h_0, 1-h_0)\} \Rightarrow \{\eta_1(h) + (1 - 2h_0)h; h \in \mathbb{R}\}. \quad (45)
\]
Denoting $\varsigma_n(h) = \sigma(h_0 + \tilde{s}_n h) \sigma(h_0)^{-1}$, we have, as $n \to \infty$,

$$
\tilde{b}_n(\tilde{M}_n(h) - \tilde{b}_n) = \tilde{b}_n(\varsigma_n(h) M_{\log n}(h) - \tilde{b}_n) \\
= \varsigma_n(h) \tilde{b}_n(\log n(h) - \tilde{b}_n) + \tilde{b}_n^2(\varsigma_n(h) - 1) \\
= (1 + o(1)) \tilde{b}_n(\log n(h) - b_n) + \tilde{b}_n^2(\varsigma_n(h) - 1).
$$

An easy calculation shows that

$$
\tilde{b}_n^2(\varsigma_n(h) - 1) = (1 - 2h_0)h + o(1) \text{ as } n \to \infty.
$$

To finish the proof of (45) recall that $b_n(M_n(\cdot) - b_n)$ converges as $n \to \infty$ to the Brown-Resnick process $\eta_1$. □

**Acknowledgement.** The author is grateful to M.Schlather for useful comments.

**References**


