Selected Results of Ulrich Krengel

1. The Stochastic Ergodic Theorem

The following theoren is proved in #8 of my list of publications:

Let T be a linear operator of norm ≤ 1 in a space L_1. Then the Cesàro-averages of the sequence f, Tf, T^2f, ...converge stochastically for every f in L_1.

The point is that pointwise convergence and norm convergence fail unless additional conditions are added.

The theorem is treated for example in the books of Edgar and Sucheston, and of Eisner, Farkas,, Haase and Nagel.

2. Entropy of Conservative Transformations

A classical problem of ergodic theory is to decide which automorphisms of a probability space are isomorphic. (An automorphism of a measure space

(Ω, B,m) is an invertible measure preserving transformation of Ω onto Ω.) In 1958, Kolmogorov introduced a new invariant, now called entropy, which permitted showing that there are nonisomorphic Bernoulli shifts. (All nontrivial Bernoulli shifts are spectrally isomorphic, and only spectral invariants had been used before.)

Automorphisms of σ-finite measure spaces split into a conservative and a dissipative part.. The isomorphy problem is trivial in the dissipative case. An automorphism T is conservative if for all subsets E the orbit ω, Tω, $T^2ω$,...of almost all points ω in E returns to E. Let T_E ω be the point of the first return of ω to E. In #10 of my list of publications, I used the induced map T_E to extend the definition of entropy to conservative transformations.

Some properties of entropy carry over from finite measure spaces to the infinite case. But for automorphisms with entropy 0 there are sigificant differences. Some problems I posed are only partially answered by now.

In 1969, W. Parry gave an alternative definition of entropy in the infinite case. So far, his entropy can only be computed when it agrees with the Krengel entropy.

When my paper appeared, it was not known, if all automorphisms which are spectrally isomorphic and have the same entropy, are isomorphic. I gave , in infinite measure spaces, examples which are spectrally isomorphic and have the same entropy, but which are not isomorphic.

3. Representation of Flows and Semiflows

In 1941, W.Ambrose gave an interesting representation of ergodic flows $\{T_t: -\infty < t < \infty\}$ in a measure space (Ω, F, μ).. Each T_t is an invertible measure preserving transformation, and $T_{t+s} = T_t T_s$ holds for all t and s. He showed that ergodic flows are isomorphic to „flows under a function". This means that there exists a measure space (Ω_0, μ_0) such that Ω is isomorhic to

$\Omega^* = \{ (x,t) : x \text{ in } \Omega_0, 0 \leq t < f(x)\}$

for a suitable strictly positive function f on Ω_0. Under the isomorphism, the images of the points in Ω move with unit speed vertically

$T^*_u (x,t) = (x,t+u)$

as long as $t+u < f(x)$. And when $t+u = f(x)$, then

$T^*_u (x,t) = (Sx,0)$

where S is a measure preserving map of Ω_0. In 1942, Ambrose and Kakutani extended this to nonergodic flows, excluding periodic flows.

In 1968 and 1969 , I gave an analoguous representation, first for flows in infinite measure spaces and then for nonsingular flows and semiflows . A transformation T is called nonsingular if the measure $T\mu$, defined by $T\mu(A) = \mu(T^{-1}A)$ and the measure μ have the same sets of measure 0. For a semiflow, T_t is only defined for $0 \leq t < \infty$ and the maps T_t need not be invertible. For semiflows, the representation is more complicated. The space Ω^* is replaced by a space with different layers. Izumi Kubo independently gave the representation for nonsingular flows.

Our results found important applications in the theory of operator algebras. The representation theorem for flows has been called Theorem of Ambrose- Kakutani-Krengel-Kubo. For example, it is discussed in the english version of Wikipedia. (headword: Ergodic Flow), and in a monograph by M.Takesaki on operator algebras.

In 1975, Daniel Rudolph sharpened the result of Ambrose-Kakutani by showing that f can be chosen to assume only two values p, q > 0 with p/q irrational. I extended this to the nonsingular case, and I showed that the space (Ω_0, μ_0) and the function f can be chosen identical for all aperiodic flows in a Lebesgue probability space. (# 29 of my list of publications).

4. Deterministic and Completely Nondeterministic Stationary Processes

I used a variant of my results on semiflows to construct stationary processes with special properties. If $X= \{...X_{-2}, X_{-1}, X_0, X_1, X_2, X_3,...\}$ is a stochastic process, let $F(-\infty,n)$ denote the smallest σ-algebra with respect to which the X_k with $k\leq n$ are measurable, and let $F(-\infty)$ be the intersection of the σ-algebras $F(-\infty,n)$. The process X is called (forward) deterministic, if $F(-\infty) = F(-\infty, +\infty)$. X is called (forward) completely nondeterministic, if $F(-\infty)$ is trivial. Backward means that the time is reversed.

For discrete time stationary processes with finitely many states, it is known that they are forward deterministic if and only if they are backward deterministic. Using the representation in the case of K-flows, I showed: There are stationary processes with continous time parameter, with two states with only finitely many jumps in any finite time interval, which are forward deterministic and backward completely nondeterministic.

5. Generators in Ergodic Theory

In 1970, Wolfgang Krieger showed that any ergodic invertible measure preserving transformation T in a Lebesgue probability space (Ω, $\mathcal{F}$, μ) with entropy $h(T) < \log_2 (n)$ has a generator with n sets A_1, A_2, ...A_n.. A generator is a partition of Ω into n measurable sets, for which $\mathcal{F}$ is generated by the sets $T^i A_k$ with $-\infty < i < \infty$, $1 \le k \le n$. (The existence of such a generator is equivalent to the isomorphy of T with a stationary process with n states.)

Trying to find an analoguos result in infinite measure spaces, I noticed that a much stronger result holds in the infinite case. My generator theorem asserts: If T is a nonsingular invertible transformation in a separable σ-finite measure space (Ω, $\mathcal{F}$, μ) and there exists no nonzero, finite T-invariant measure $\mu_0 \ll \mu$, then the system of sets A ,for which A together with its complement is a strong generator, is dense in $\mathcal{F}$. (For a strong generator, the negative powers of T are not needed.

These strongly generating sets A are even dense in any σ-algebra $\mathcal{G} \subset \mathcal{F}$ with the following properties

$T^{-1} \mathcal{G} \subset \mathcal{G}$ and $T^k \mathcal{G}$ tends to $\mathcal{F}$ as k tends to ∞. The greatest difficulty appears if a nonzero finite invarant measure exists on $\mathcal{G}$. To show that this can happen, I started the study of nonsingular Bernoulli shifts. This has led to very deep and interesting work by T. Hamachi, Z. Kosloff and others.

Amy Kuntz,, a student of Wolfgang Krieger and me,, extended my results on generators to groups of nonsingular transformations.

I worked with Christian Grillenberger on sharpening the result of Krieger. He had shown that the generators can be constructed such that the finite dimensional marginals of the generated process can be prescribed up to a small error. Grillenberger and I showed that they can be prescribed exactly. This has been called Grillenberger-Krengel-Theorem.(See: Wikipedia, headword: Grillenberger.)

6. Weakly Wandering Vectors and Spectral Theory

Let T denote an isometry in a Hilbert space H. A vector f ≠ 0 in H is called weakly wandering if there exists an infinite subset I of the integers such that all vectors $T^i f$ with i in I are orthogonal. I showed: T has continous spectrum (no eigenvectors) if and only if the the weakly wandering vectors span H. In this case, they are even dense in H. T has discrete spectrum if and only if there are no weakly wandering vectors.

This result has been called „classical theorem of Krengel". It has been extended to various groups of unitary operators, and the structure of I has been studied by

V. Graham, V. Bergelson, I. Kornfeld, B. Mitryagin, V. Müller, Y. Tomilov and others.

7. Exhaustive Weakly Wandering Sets

Let T be an invertible nonsingular transformation of a probability space (Ω, B, μ).

It is simple to see that Ω splits into a set F and its complement G with the following properties: F and G are invariant, and there exists a finite T-invariant measure ν on F which is equivalent to the restriction of μ to F, and there is no nonzero finite T-invariant measure on G ,absolutely continous with respect to μ.

A set W in B is called weakly wandering if μ(W) > 0, and there exists an infinite subset I of the nonnegative integers, such that the sets T^{-i} W are disjoint for i in I. Hajian and Kakutani showed in 1964: Ω = F (there exists a finite invariant measure equivalent to μ) if and only if there exists no weakly wanderind set.

W is called exhaustive weakly wandering set, if the sets T^{-i} W with i in I partition all of Ω, i.e. their union is Ω and they are disjoint. In 1970, Hajian and Kakutani constructed an example of a transformation T with Ω = G, for which there exists an exhaustive weakly wandering W.

In 1974, Lee K. Jones and I showed that all T with Ω = G have this property. (For example, the set of integers is the disjoint union of translates of an infinite subset of the integers.)

8. The Prophet Inequality

In 1976, L.Sucheston visited Göttingen. We studied „Amarts" (asymptotic martingales). In the search for examples we proved an inequality for sequences of independent nonnegative random variables, which touched off a large number of subsequent publications.

Let $X_1, X_2, \ldots, X_n$ be independent nonnegative random variables. Let **T** be the set of stopping times for this sequence, and

V= sup {$E(X_t)$ with t in **T}.**

Then there exists a constant K with $2 \le K \le 4$ such that

$E(\max(X_1, X_2, \ldots, X_n)) \le K V$.

The left side is the expected gain of a prophet with complete foresight. He stops when he gets the maximum. The number V can be interpreted as the expected gain of a gambler who knows the distributions of the random variables and uses the optimal stopping rule. Before we could complete the search for the optimal value of K, Garling showed that the inequality holds with K = 2. Our result has been named prophet inequality of Krengel, Sucheston and Garling.

We proved an analoguous result for processes of the form

$Y_k = c_k (X_1+X_2+....+X_k)$.

In this case, we arrived at $K= 2(1+\sqrt{3})$.

Soon, a large number of similar results and new proofs were published by Hill, Kertz, Samuel-Cahn, Wittmann and others. A book with the title „Prophet Theory“ was published by F. Harten, A. Meyerthole and N. Schmitz.

9. Differentiation

M.Akcoglu and I lifted the classical local ergodic theorems to a new level which includes differentiation.

Let $\{T_t, t>0\}$ be a strongly continous semigroup of positive contractions in

L_1 of a σ-finite measure space (Ω, B, μ). A family $\{F_t, t > 0\}$ of integrable functions is called an additive process if $F_{t+s} = F_t + T_t F_s$ for all t,s > 0. Examples are

$F_t = \int T_s f\, ds$

(integral from 0 to t)., and

$F_t = (I - T_t) f$

where I is the identity and f is intergrable. We studied the convergence a.e. of $(1/t)F_t$ as $t \to 0+0$. Our principal result is: If $\{T_t\}$ and $\{F_t\}$ are as above, and

$\sup\{ (1/t) \|F_t\| : 0 < \leq 1\} < \infty$,

then $\lim (1/t) F_t$ exists a.e. (One must select suitable representatives in L_1 or take the limit along countably many t.)

For Markovian semigroups $\{T_t\}$ we have a similar result for superadditive processes.

Our results yield new insights even for real valued functions on the real line.We define:

$\| f \|_{ess.sup.t.v.} = \lim (1/t) \int | f(x+t) - f(x) | \, dx.$

(as t tends to 0+0.). This turns out to be the minimal total variation of all functions which agree with f a.e. with respect to Lebesgue measure.

In a subsequent paper we also studied semigroups in L_p with $1<\leq p<\infty$.

10. The Ergodic Theorem for Multiparameter Superadditive Processes

Subadditive and superadditive processes, introduced in 1965 by Hammersley and Welch have evolved as an important tool in the study of stochastic processes. In 1968, J. F. C. Kingman proved the pointwise ergodic theorem for 1-parameter subadditive processes. Smythe and Nguyen proved pointwise ergodic theorems for multiparameter subadditive processes, but , in addition to the usual subadditivity condition they required an additional strong subadditivity condition.

In 1981, Akcoglu and I proved the multiparameter subadditive ergodic theorem with out extra condition.We gave a proof which was new even in the 1-parameter case via a new maximal inequality inspired by the work of N.Wiener. The proofs of Kingman, Smythe and Nguyen used the existence of „exact dominants“ of the process. We showed that in the multiparameter case, exact dominants need not exist if the strong subadditivity condition fails.

Our results have found applications in percolation theory, in the study of networks, Schroedinger operators and Hamilton-Jacobi-equations.

11. Stopping Rules and Tactics for Processes indexed by a Directed Set

In a joint publication with L.Suchesto (# 46), we develop important foundations for the theory of optimal stopping for processes with a directed index set. When I arrived in Columbus in the fall of 1979, Louis showed me an article by R.Cairoli and J.-P. Gabriel „ Arrêt de certaines suites multiples...“ They studied processes $\{X_j : j \text{ in } J\}$ with $\mathbf{J} = \mathbb{N}^{\mathbf{d}}$ and $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. The X_j are measurable with respect to σ-algebras F_j defining a filtration. They introduced „increasing pathes“ in J, but they did not require that the decision of going from a point j to its direct successor was governed by F_j. We used F_j -measurable partitions telling us to which direct successor of j one should go next after arriving in j. As usual, a stopping rule (or stopping point) is a map t from Ω to J such that $\{t \leq j\}$ belongs to F_j. For all j.. A stopping rule given by a tactic is a stopping rule in which the point, in which one stops, is reached by an increasing path determined by the above partitions.

In 1996, R.Cairoli and R. C. Dalang published a monograph „Sequential Stochastic Optimization“. Basic concepts in this monograph have been introduced in our paper with different terminology. E.g., the pathes constructed by us with the help of partitions are called predictable increasing pathes in the book, and the stopping rules given by a tactic are called predictable stopping points.

We showed that in general stopping rules need not be predictable, and we identified the condition CQI of conditional qualitative independence which guaranties that all stopping rules are predictable.

Our main result was an „embedding theorem“ which can be used to extend results on optimal stopping for 1-parameter sequences of i.i.d. variables to i.i.d. processes with locally finite index set and predictable stopping rules. An extension of this to exchangeable random variable is the main result of chapter 4 of the book of Cairoli and Dalang. One of our applications has been the solution of a problem asked by Cairoli and Gabriel for $J = \mathbb{N}^d$

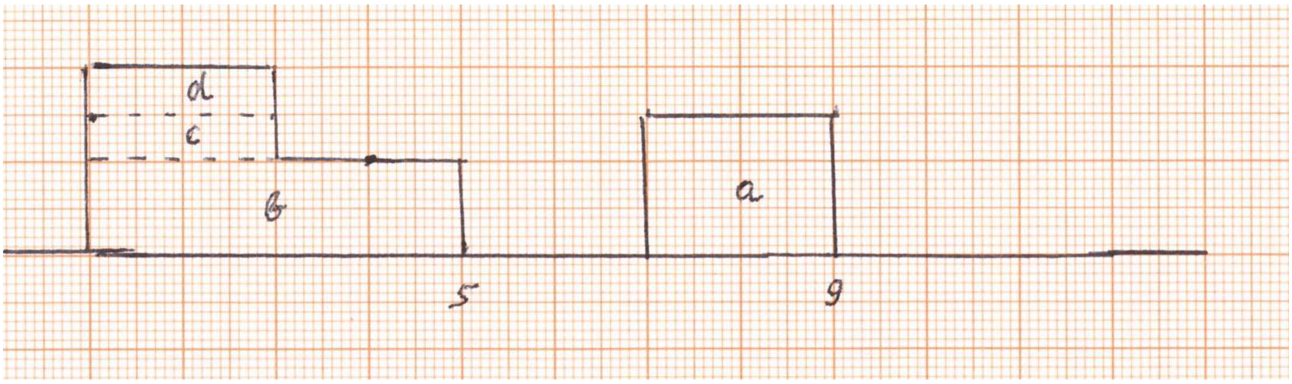
We also explored when stopping rules can be replaced by equally good stopping rules with totally ordered range.

12. Speed Limit Operators

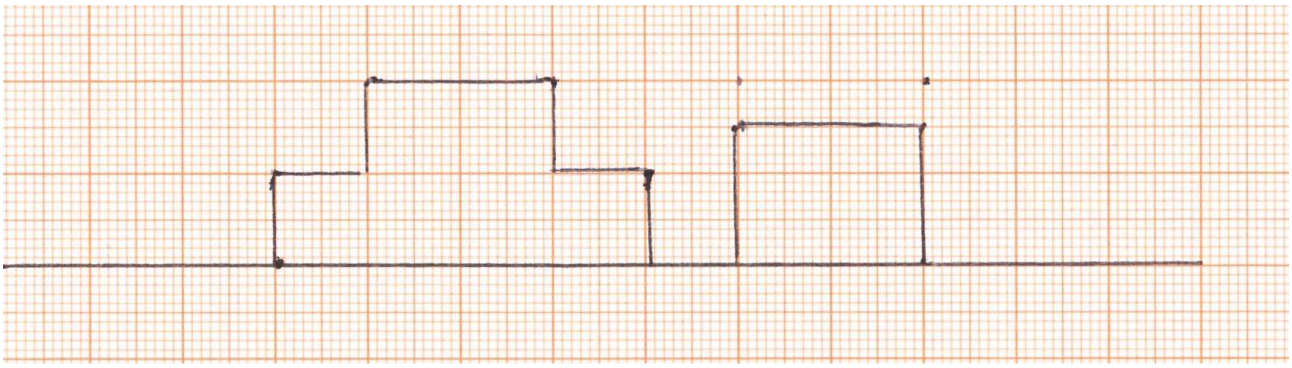
During my visit at Be'er Sheva, I worked with Michael Lin on order preserving nonexpansive operators in L_1. (T is called nonexpansive if $\|Tf - Tg\| \le \| f-g \|$ holds for all f and all g). T need not be linear.

We showed: If T contracts also the L_∞- norm, the sequence $T^n f$ converges in distribution, and the Cesàro-averages of the sequence $T^k f$ converge weakly in L_p with 1 < p ∞. and also in L_1, if the measure is finite. While this is quite nontrivial, the main novelty of the paper may be a new type of operators in L_1, which we constructed in order to show that strong convergence need not hold.

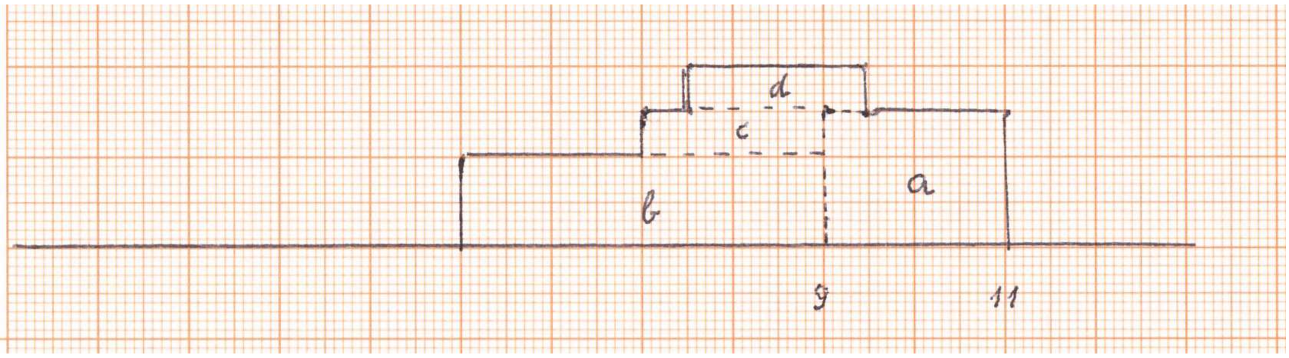
A nonincreasing, nonnegative function v on the real line is interpreted as a speed limit. We explain the action of the speed limit operator by first looking at functions with finitely many positive values in finitely many finite intervals like the one pictured below:



The function describes a cloud of microscopic cars driving from left to right, bumper to bumper, and stictly obeying the speed limit. The cars in bloc a all move with the speed limit assigned to the rightmost cars in the bloc. The cars on the right side of bloc c may be subject to a higher speed limit than those on the right side of bloc b. Therfore bloc c slides on top of b a bit forward. Also the gap between a nd b narrows.



Some time later the picture looks as in the third picture:



Due to the nonexpansiveness, the operator is extended to all nonnegative intergrable functions. We might let negative mass disappear rightaway by putting

$T_t f = T_t f^+$.

I have used an infinite dimensional version of this idea to show that pointwise ergodic theorems fail in the classical setting if the linearity is dropped. R.Wittmann obtained positive results on pointwise convergence by replacing Cesàro averages by certain recursively defined averages.

My student Bernd Fernow and I also studied speed limit operators on the circle. Then also increases of the speed limit occur. We obtained convergence to a constant with exponential speed.

13. Nonlinear Markov Chains

The transition matrix $(p_{i,k})$ of a Markov chain with finite state space {1, 2, …,m} induces a linear operator T. For $f = (f_i)$ the k-th coordinate of Tf is $\sum f_i \, p_{i,k}$. If we interprete the action of T as transport of a distribution $f = (f_i)$ of mass, the movement of the mass coming from state i is not affected by the distribution of the mass coming from the remaining states. Models with interaction can be described by a nonlinear operator S in $\mathbb{H} = \mathbb{R}^m$ which is nonexpansive with respect to the ℓ_1-norm. Akcoglu and I proved: If the orbit $S^k f$ is bounded, there exists a period $p \geq 1$, such that the sequence S^{pk} f converges when k tends to ∞. This is analoguous to the corresponding result for T. But the convergence for T is exponentially fast, and that for S can be arbitrarily slow.

Assuming S0 = 0 and an aperiodicity condition , we obtain the convergence of S^kf.

A key to our proof is a topological result. A sequence {a(k), k = 0, 1, 2, …} of elements of a complete metric space (E, ϱ) is called rigid if its closure is compact and ϱ(a(n+k), a(n)) = ϱ(a(k), a(0)) holds for all n≥ 0 and all k≥ 1. A rigid set is the closure of a rigid sequence. We proved: In E= $\mathbb{R}^m$ with the ℓ_1- norm any rigid set is finite.

R. Sine showed that this result is not restricted to the ℓ_1-norm. It holds whenever the unit ball has only finitely many extreme points. Scheutzow, Nussbaum and others obtained interesting upper bounds for the number of points of a rigid set.

14. Ergodic Theorem for Subsequences

After Friedman and Ornstein had given an example of a strongly mixing measure preserving transformation for which the individual ergodic theorem fails along suitable subsequences, Nathaniel Friedman asked in his book, if the individual ergodic theorem holds along all subsequences for Bernoulli shifts. In 1971, I gave a negative answer by constructing a subsequence of the integers along which the individual ergodic theorem fails for all aperiodic measure preserving transformations. Apparently the existence of such a „universally bad sequence“ came as a surprize and touched of considerable research by several groups of mathematicians, in particular by Alexandra Bellow and her collaborators..

15. A Problem of Eberhard Hopf

In 1936, in the Jahresbericht of the DMV, Eberhard Hopf studied a limiting problem for a cloud of particles in a box with the help of a mean ergodic theorem. In 1991, in a paper of A .Bellow and U. Krengel [65], his question, whether almost everywhere convergence holds in his setting was answered in the negative.. In a subsequent note written jointly with A. Calderon [70], we showed by a different method, the „strong sweeping out property“ for the Hopf operators, which means, that almost everywhere convergence fails in the worst possible way.